Lectures on Homotopy Theory

http://uwo.ca/math/faculty/jardine/courses/homth/homotopy_theory.html

Basic References

[1] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition

[2] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999

[3] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998

Lecture 01: Homological algebra

Contents

1 Chain complexes

$R =$ commutative ring with 1 (eg. $\mathbb{Z}$, a field k)

R-modules: basic definitions and facts

- $f : M \to N$ an R-module homomorphism:

  The *kernel* $\ker(f)$ of f is defined by

  $$\ker(f) = \{\text{all } x \in M \text{ such that } f(x) = 0\}.$$

  $\ker(f) \subset M$ is a submodule.

  The *image* $\operatorname{im}(f) \subset N$ of f is defined by

  $$\operatorname{im}(f) = \{f(x) \mid x \in M\}.$$

  The *cokernel* $\operatorname{cok}(f)$ of f is the quotient

  $$\operatorname{cok}(f) = N/\operatorname{im}(f).$$

- A sequence

$$M \xrightarrow{f} M' \xrightarrow{g} M''$$

is *exact* if $\ker(g) = \mathrm{im}(f)$. Equivalently, $g \cdot f = 0$ and $im(f) \subset \ker(g)$ is surjective.

The sequence $M_1 \to M_2 \to \cdots \to M_n$ is *exact* if $\ker = \mathrm{im}$ everywhere.

Examples: 1) The sequence

$$0 \to \ker(f) \to M \xrightarrow{f} N \to \mathrm{cok}(f) \to 0$$

is exact.

2) The sequence

$$0 \to M \xrightarrow{f} N$$

is exact if and only if f is a monomorphism (monic, injective)

3) The sequence

$$M \xrightarrow{f} N \to 0$$

is exact if and only if f is an epimorphism (epi, surjective).

Lemma 1.1 (Snake Lemma)**.** *Given a commutative diagram of R-module homomorphisms*

$$\begin{tikzcd} & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r, "p"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "i"'] & B_2 \arrow[r] & B_3 & \end{tikzcd}$$

in which the horizontal sequences are exact. There is an induced exact sequence

$$\ker(f_1) \to \ker(f_2) \to \ker(f_3) \xrightarrow{\partial} \operatorname{cok}(f_1) \to \operatorname{cok}(f_2) \to \operatorname{cok}(f_3).$$

$\partial(y) = [z]$ for $y \in \ker(f_3)$, where $y = p(x)$, and $f_2(x) = i(z)$.

Lemma 1.2 ((3×3)-Lemma)**.** *Given a commutative diagram of R-module maps*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "f"] \arrow[d] & B_2 \arrow[r, "g"] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

With exact columns.

1) If either the top two or bottom two rows are exact, then so is the third.

2) If the top and bottom rows are exact, and $g \cdot f = 0$, then the middle row is exact.

Lemma 1.3 (5-Lemma)**.** *Given a commutative diagram of R-module homomorphisms*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r] \arrow[d, "h_2"] & A_3 \arrow[r] \arrow[d, "h_3"] & A_4 \arrow[r, "g_1"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r, "f_2"'] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r, "g_2"'] & B_5 \end{tikzcd}$$

with exact rows, such that h_1, h_2, h_4, h_5 are isomorphisms. Then h_3 is an isomorphism.

The Snake Lemma is proved with an element chase. The (3×3)-Lemma and 5-Lemma are consequences.

e.g. Prove the 5-Lemma with the induced diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{cok}(f_1) \arrow[r] \arrow[d, "\cong"] & A_3 \arrow[r] \arrow[d, "h_3"] & \ker(g_1) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \operatorname{cok}(f_2) \arrow[r] & B_3 \arrow[r] & \ker(g_2) \arrow[r] & 0 \end{tikzcd}$$

Chain complexes

A *chain complex* C in R-modules is a sequence of R-module homomorphisms

$$\dots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \dots$$

such that $\partial^2 = 0$ (or that $\mathrm{im}(\partial) \subset \ker(\partial)$) everywhere. C_n is the module of *n-chains*.

A *morphism* $f : C \to D$ of chain complexes consists of R-module maps $f_n : C_n \to D_n$, $n \in \mathbb{Z}$ such that there are comm. diagrams

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial"'] & D_n \arrow[d, "\partial"] \\ C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1} \end{tikzcd}$$

The chain complexes and their morphisms form a category, denoted by $Ch(R)$.

- If C is a chain complex such that $C_n = 0$ for $n < 0$, then C is an *ordinary* chain complex. We usually drop all the 0 objects, and write

  $$\to C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

  $Ch_+(R)$ is the full subcategory of ordinary chain complexes in $Ch(R)$.

- Chain complexes indexed by the integers are often called *unbounded* complexes.

  Slogan: Ordinary chain complexes are spaces, and unbounded complexes are spectra.

- Chain complexes of the form

  $$\cdots \to 0 \to C_0 \to C_{-1} \to \dots$$

  are *cochain complexes*, written (classically) as

  $$C^0 \to C^1 \to C^2 \to \dots.$$

  Both notations are in common (confusing) use.

Morphisms of chain complexes have kernels and cokernels, defined degreewise.

A sequence of chain complex morphisms

$$C \to D \to E$$

is *exact* if all sequences of morphisms

$$C_n \to D_n \to E_n$$

are exact.

Homology

Given a chain complex C :

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \dots$$

Write

$Z_n = Z_n(C) = \ker(\partial : C_n \to C_{n-1})$, ($n$-cycles), and
$B_n = B_n(C) = \mathrm{im}(\partial : C_{n+1} \to C_n)$ (n-boundaries).

$\partial^2 = 0$, so $B_n(C) \subset Z_n(C)$.

The n^{th} *homology group* $H_n(C)$ of C is defined by

$$H_n(C) = Z_n(C)/B_n(C).$$

A chain map $f : C \to D$ induces R-module maps

$$f_* : H_n(C) \to H_n(D),\ n \in \mathbb{Z}.$$

$f : C \to D$ is a *homology isomorphism* (resp. *quasi-isomorphism*, *acyclic map*, *weak equivalence*) if all induced maps $f_* : H_n(C) \to H_n(D),\ n \in \mathbb{Z}$ are isomorphisms.

A complex C is *acyclic* if the map $0 \to C$ is a homology isomorphism, or if $H_n(C) \cong 0$ for all n, or if the sequence

$$\dots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \dots$$

is exact.

Lemma 1.4. *A short exact sequence*

$$0 \to C \to D \to E \to 0$$

induces a natural long exact sequence

$$\ldots \xrightarrow{\partial} H_n(C) \to H_n(D) \to H_n(E) \xrightarrow{\partial} H_{n-1}(C) \to \ldots$$

Proof. The short exact sequence induces comparisons of exact sequences

$$\begin{tikzcd} & C_n/B_n(C) \arrow[r] \arrow[d, "\partial_*"] & D_n/B_n(D) \arrow[r] \arrow[d, "\partial_*"] & E_n/B_n(E) \arrow[r] \arrow[d, "\partial_*"] & 0 \\ 0 \arrow[r] & Z_{n-1}(C) \arrow[r] & Z_{n-1}(D) \arrow[r] & Z_{n-1}(E) & \end{tikzcd}$$

Use the natural exact sequence

$$0 \to H_n(C) \to C_n/B_n(C) \xrightarrow{\partial_*} Z_{n-1}(C) \to H_{n-1}(C) \to 0$$

Apply the Snake Lemma. □

2 Ordinary chain complexes

A map $f : C \to D$ in $Ch_+(R)$ is a

- *weak equivalence* if f is a homology isomorphism,
- *fibration* if $f : C_n \to D_n$ is surjective for $n > 0$,
- *cofibration* if f has the left lifting property (LLP) with respect to all morphisms of $Ch_+(R)$ which

are simultaneously fibrations and weak equivalences.

A *trivial fibration* is a map which is both a fibration and a weak equivalence. A *trivial cofibration* is both a cofibration and a weak equivalence.

f has the *left lifting property* with respect to all trivial fibrations (ie. f is a cofibration) if given any solid arrow commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] & X \arrow[d, "p"] \\ D \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

in $Ch_+(R)$ with p a trivial fibration, then the dotted arrow exists making the diagram commute.

Special chain complexes and chain maps:

- $R(n)$ [$= R[-n]$ in "shift notation"] consists of a copy of the free R-module R, concentrated in degree n:

  $$\dots \to 0 \to 0 \to \overset{n}{R} \to 0 \to 0 \to \dots$$

  There is a natural R-module isomorphism

  $$\hom_{Ch_+(R)}(R(n), C) \cong Z_n(C).$$

- $R\langle n+1 \rangle$ is the complex

  $$\dots \to 0 \to \overset{n+1}{R} \xrightarrow{1} \overset{n}{R} \to 0 \to \dots$$

- There is a natural R-module isomorphism

$$\hom_{Ch_+(R)}(R\langle n+1\rangle, C) \cong C_{n+1}.$$

- There is a chain $\alpha : R(n) \to R\langle n+1\rangle$

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & R \arrow[r, "1"'] & R \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

α *classifies* the cycle $1 \in R\langle n+1\rangle_n$.

Lemma 2.1. *Suppose that $p : A \to B$ is a fibration and that $i : K \to A$ is the inclusion of the kernel of p. Then there is a long exact sequence*

$$\dots \xrightarrow{p_*} H_{n+1}(B) \xrightarrow{\partial} H_n(K) \xrightarrow{i_*} H_n(A) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} \dots$$
$$\dots \xrightarrow{\partial} H_0(K) \xrightarrow{i_*} H_0(A) \xrightarrow{p_*} H_0(B).$$

Proof. $j : \mathrm{im}(p) \subset B$, and write $\pi : A \to \mathrm{im}(p)$ for the induced epimorphism. Then $H_n(\mathrm{im}(p)) = H_n(B)$ for $n > 0$, and there is a diagram

$$\begin{tikzcd}
H_0(A) \arrow[rr, "p_*"] \arrow[dr, "\pi_*"'] & & H_0(B) \\
& H_0(\mathrm{im}(p)) \arrow[ur, "i_*"'] &
\end{tikzcd}$$

in which π_* is an epimorphism and i_* is a monomorphism (exercise). The long exact sequence is constructed from the long exact sequence in homology for the short exact sequence

$$0 \to K \xrightarrow{i} A \xrightarrow{\pi} \mathrm{im}(p) \to 0,$$

with the monic $i_* : H_0(\mathrm{im}(p)) \to H_0(B)$. □

Lemma 2.2. *$p : A \to B$ is a fibration if and only if p has the RLP wrt. all maps $0 \to R\langle n+1\rangle$, $n \geq 0$.*

Proof. The lift exists in all solid arrow diagrams

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ R\langle n+1\rangle \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

for $n \geq 0$. □

Corollary 2.3. *$0 \to R\langle n+1\rangle$ is a cofibration for all $n \geq 0$.*

Proof. This map has the LLP wrt all fibrations, hence wrt all trivial fibrations. □

Lemma 2.4. *The map $0 \to R(n)$ is a cofibration.*

Proof. The trivial fibration $p : A \to B$ induces an epimorphism $Z_n(A) \to Z_n(B)$ for all $n \geq 0$:

$$\begin{tikzcd} A_{n+1} \arrow[r, two heads] \arrow[d] & B_n(A) \arrow[r] \arrow[d] & Z_n(A) \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d, "\cong"] & 0 \\ B_{n+1} \arrow[r, two heads] & B_n(B) \arrow[r] & Z_n(B) \arrow[r] & H_n(B) \arrow[r] & 0 \end{tikzcd}$$

□

A chain complex A is *cofibrant* if the map $0 \to A$ is a cofibration.

eg. $R\langle n+1\rangle$ and $R(n)$ are cofibrant.

All chain complexes C are *fibrant*, because all chain maps $C \to 0$ are fibrations.

Proposition 2.5. *$p : A \to B$ is a trivial fibration and if and only if*

1) $p : A_0 \to B_0$ is a surjection, and

2) p has the RLP wrt all $\alpha : R(n) \to R\langle n+1\rangle$.

Corollary 2.6. *$\alpha : R(n) \to R\langle n+1\rangle$ is a cofibration.*

Proof of Proposition 2.5. 1) Suppose that $p : A \to B$ is a trivial fibration with kernel K.

Use Snake Lemma with the comparison

$$\begin{tikzcd} A_1 \arrow[r, "\partial"] \arrow[d, "p"'] & A_0 \arrow[r] \arrow[d, "p"'] & H_0(A) \arrow[r] \arrow[d, "\cong"] & 0 \\ B_1 \arrow[r, "\partial"'] & B_0 \arrow[r] & H_0(B) \arrow[r] & 0 \end{tikzcd}$$

to show that $p : A_0 \to B_0$ is surjective.

Suppose given a diagram

$$\begin{tikzcd} R(n) \arrow[r, "x"] \arrow[d, "\alpha"'] & A \arrow[d, "p"] \\ R\langle n+1\rangle \arrow[r, "y"'] & B \end{tikzcd}$$

Choose $z \in A_{n+1}$ such that $p(z) = y$. Then $x - \partial(z)$ is a cycle of K, and K is acyclic (exercise) so there is a $v \in K_{n+1}$ such that $\partial(v) = x - \partial(z)$. $\partial(z+v) = x$ and $p(z+v) = p(v) = y$, so $v+z$ is the desired lift.

2) Suppose that $p : A_0 \to B_0$ is surjective and that p has the right lifting property with respect to all $R(n) \to R\langle n+1 \rangle$.

The solutions of the lifting problems

$$\begin{tikzcd} R(n) \arrow[r, "0"] \arrow[d] & A \arrow[d, "p"] \\ R\langle n+1 \rangle \arrow[r, "x"'] \arrow[ur, dotted] & B \end{tikzcd}$$

show that p is surjective on all cycles, while the solutions of the lifting problems

$$\begin{tikzcd} R(n) \arrow[r, "x"] \arrow[d] & A \arrow[d, "p"] \\ R\langle n+1 \rangle \arrow[r, "y"'] \arrow[ur, dotted] & B \end{tikzcd}$$

show that p induces a monomorphism in all homology groups. It follows that p is a weak equivalence.

We have the diagram

$$\begin{tikzcd} Z_{n+1}(A) \arrow[r] \arrow[d, "p"] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "p"] & Z_n(A) \arrow[r] \arrow[d, "p"] & H_n(A) \arrow[r] \arrow[d, "p"', "\cong"] & 0 \\ Z_{n+1}(B) \arrow[r] & B_{n+1} \arrow[r, "\partial"'] & Z_n(B) \arrow[r] & H_n(B) \arrow[r] & 0 \end{tikzcd}$$

Then $p : B_n(A) \to B_n(B)$ is epi, so $p : A_{n+1} \to B_{n+1}$ is epi, for all $n \geq 0$. $\square$

Proposition 2.7. *Every chain map $f : C \to D$ has two factorizations*

$$\begin{tikzcd} & E \arrow[dr, "p"] & \\ C \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & D \\ & F \arrow[ur, "q"'] & \end{tikzcd}$$

where

1) *p is a fibration. i is a monomorphism, a weak equivalence and has the LLP wrt all fibrations.*

2) *q is a trivial fibration and j is a monomorphism and a cofibration.*

Proof. 1) Form the factorization

$$\begin{tikzcd}[column sep=small] & C \oplus (\bigoplus_{x \in D_{n+1}, n \geq 0} R\langle n+1 \rangle) \arrow[dr, "p"] & \\ C \arrow[ur, "i"] \arrow[rr, "f"'] & & D \end{tikzcd}$$

p is the sum of f and all classifying maps for chains x in all non-zero degrees. It is therefore surjective in non-zero degrees, hence a fibration.

i is the inclusion of a direct summand with acyclic cokernel, and is thus a monomorphism and a weak equivalence. i is a direct sum of maps which have the LLP wrt all fibrations, and thus has the same lifting property.

2) Recall that $A \to B$ is a trivial fibration if and only if it has the RLP wrt all cofibrations $R(n) \to R\langle n+1 \rangle$, $n \geq -1$.

Notation: $R(-1) \to R\langle 0 \rangle$ is the map $0 \to R(0)$.

Consider the set of all diagrams

$$D: \quad \begin{tikzcd} R(n_D) \arrow[r, "\alpha_D"] \arrow[d] & C \arrow[d, "f=q_0"] \\ R\langle n_D+1 \rangle \arrow[r, "\beta_D"'] & D \end{tikzcd}$$

and form the pushout

$$\begin{tikzcd}
\bigoplus_D R(n_D) \arrow[r, "(\alpha_D)"] \arrow[d] & C_0 \arrow[d, "j_1"] \arrow[ddr, "q_0"] & \\
\bigoplus_D R\langle n_D+1\rangle \arrow[r, "(\theta_D)"'] \arrow[drr, "(\beta_D)"'] & C_1 \arrow[dr, dotted, "q_1"] & \\
& & D
\end{tikzcd}$$

where $C = C_0$. Then j_1 is a monomorphism and a cofibration, because the collection of all such maps is closed under direct sum and pushout.

Every lifting problem D as above is solved in C_1:

$$\begin{tikzcd}
R(n_D) \arrow[r, "\alpha_D"] \arrow[d] & C_0 \arrow[r, "j_1"] & C_1 \arrow[d, "q_1"] \\
R\langle n_D+1\rangle \arrow[urr, "\theta_D"'] \arrow[rr, "\beta_D"'] & & D
\end{tikzcd}$$

commutes.

Repeat this process inductively for the maps q_i to produce a string of factorizations

$$\begin{tikzcd}
C_0 \arrow[r, "j_1"] \arrow[d, "q_0"'] & C_1 \arrow[r, "j_2"] \arrow[dl, "q_1"'] & C_2 \arrow[r, "j_3"] \arrow[dll, "q_2"] & \dots \\
D & & &
\end{tikzcd}$$

Let $F = \varinjlim_i C_i$. Then f has a factorization

$$\begin{tikzcd}
C \arrow[r, "j"] \arrow[dr, "f"'] & F \arrow[d, "q"] \\
& D
\end{tikzcd}$$

Then j is a cofibration and a monomorphism, because all j_k have these properties and the family of such maps is closed under (infinite) composition.

Finally, given a diagram

$$\begin{tikzcd} R(n) \arrow[r, "\alpha"] \arrow[d] & F \arrow[d, "q"] \\ R\langle n+1\rangle \arrow[r, "\beta"'] & D \end{tikzcd}$$

The map α factors through some finite stage of the filtered colimit defining F, so that α is a composite

$$R(n) \xrightarrow{\alpha'} C_k \to F$$

for some k. The lifting problem

$$\begin{tikzcd} R(n) \arrow[r, "\alpha'"] \arrow[d] & C_k \arrow[d, "q_k"] \\ R\langle n+1\rangle \arrow[r, "\beta"'] & D \end{tikzcd}$$

is solved in C_{k+1}, hence in F. $\square$

Remark: This proof is a *small object argument*.

The $R(n)$ are *small* (or compact): $\hom(R(n),\ )$ commutes with filtered colimits.

Corollary 2.8. *1) Every cofibration is a monomorphism.*

2) Suppose that $j : C \to D$ is a cofibration and a weak equivalence. Then j has the LLP wrt all fibrations.

Proof. 2) The map j has a factorization

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[dr, "j"'] & F \arrow[d, "p"] \\ & D \end{tikzcd}$$

where i has the left lifting property with respect to all fibrations and is a weak equivalence, and p is a fibration. Then p is a trivial fibration, so the lifting exists in the diagram

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & F \arrow[d, "p"] \\ D \arrow[r, "1"'] \arrow[ur, dotted] & D \end{tikzcd}$$

since j is a cofibration. Then j is a retract of a map (namely i) which has the LLP wrt all fibrations, and so j has the same property.

1) is an exercise. □

Resolutions

Suppose that P is a chain complex. Proposition 2.7 says that $0 \to P$ has a factorization

$$\begin{tikzcd} 0 \arrow[r, "j"] \arrow[dr] & F \arrow[d, "q"] \\ & P \end{tikzcd}$$

where j is a cofibration (so that F is cofibrant) and q is a trivial fibration, hence a weak equivalence.

The proof of Proposition 2.7 implies that each R-module F_n is free, so F is a *free resolution* of P.

If the complex P is cofibrant, then the lift exists in

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F \arrow[d, "q"] \\ P \arrow[r, "1"'] \arrow[ur, dotted] & P \end{tikzcd}$$

All modules P_n are direct summands of free modules and are therefore projective.

This observation has a converse:

Lemma 2.9. *A chain complex P is cofibrant if and only if all modules P_n are projective.*

Proof. Suppose that P is a complex of projectives, and $p : A \to B$ is a trivial fibration.

Then $p : A_n \to B_n$ is surjective for all $n \geq 0$ and has acyclic kernel $i : K \to A$.

Suppose given a lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ P \arrow[r, "f"'] \arrow[ur, dotted, "\theta"] & B \end{tikzcd}$$

There is a map $\theta_0 : P_0 \to A_0$ which lifts f_0:

$$\begin{tikzcd} & A_0 \arrow[d, "p_0"] \\ P_0 \arrow[r, "f_0"'] \arrow[ur, "\theta_0"] & B_0 \end{tikzcd}$$

Suppose given a lift up to degree n, ie. homomorphisms $\theta_i : P_i \to A_i$ for $i \leq n$ such that $p_i\theta_i = f_i$ for $i \leq n$ and $\partial\theta_i = \theta_{i-1}\partial$ for $1 \leq i \leq n$

There is a map $\theta'_{n+1} : P_{n+1} \to A_{n+1}$ such that $p_{n+1}\theta'_{n+1} = f_{n+1}$.

Then

$$p_n(\partial\theta'_{n+1} - \theta_n\partial) = \partial p_{n+1}\theta'_{n+1} - f_n\partial = \partial f_{n+1} - f_n\partial = 0$$

so there is a $v : P_{n+1} \to K_n$ such that

$$i_n v = \partial\theta'_{n+1} - \theta_n\partial.$$

Also

$$\partial(\partial\theta'_{n+1} - \theta_n\partial) = 0$$

and K is acyclic, so there is a $w : P_{n+1} \to K_{n+1}$ such that

$$i_n \partial w = \partial \theta'_{n+1} - \theta_n \partial.$$

Then

$$\partial(\theta'_{n+1} - i_{n+1} w) = \theta_n \partial$$

and

$$p_{n+1}(\theta'_{n+1} - i_{n+1} w) = p_{n+1} \theta'_{n+1} = f_{n+1}.$$

$\square$

Remarks:

1) Every chain complex C has a *cofibrant model*, i.e. a weak equivalence $p : P \to C$ with P cofibrant (aka. complex of projectives).
2) $M =$ an R-module. A cofibrant model $P \to M(0)$ is a projective resolution of M in the usual sense.
3) Cofibrant models $P \to C$ are also (commonly) constructed with Eilenberg-Cartan resolutions.

3 Closed model categories

A *closed model category* is a category $\mathscr{M}$ equipped with three classes of maps, namely weak equivalences, fibrations and cofibrations, such that the following conditions are satisfied:

CM1 The category $\mathscr{M}$ has all finite limits and colimits.

CM2 Given a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

of morphisms in $\mathscr{M}$, if any two of f, g and h are weak equivalences, then so is the third.

CM3 The classes of cofibrations, fibrations and weak equivalences are closed under retraction.

CM4 Given a commutative solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

such that i is a cofibration and p is a fibration. Then the lift exists making the diagram commute if either i or p is a weak equivalence.

CM5 Every morphism $f : X \to Y$ has factorizations

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where p is a fibration and i is a trivial cofibration, and q is a trivial fibration and j is a cofibration.

Theorem 3.1. *With the definition of weak equivalence, fibration and cofibration given above, $Ch_+(R)$ satisfies the axioms for a closed model category.*

Proof. **CM1**, **CM2** and **CM3** are exercises. **CM5** is Proposition 2.7, and **CM4** is Corollary 2.8. □

Exercise: A map $f : C \to D$ of $Ch(R)$ (unbounded chain complexes) is a *weak equivalence* if it is a homology isomorphism.

f is a *fibration* if all maps $f : C_n \to D_n$, $n \in \mathbb{Z}$ are surjective.

A map of is a *cofibration* if and only if it has the left lifting property with respect to all trivial fibrations.

Show that, with these definitions, $Ch(R)$ has the structure of a closed model category.