#### **Lectures on Homotopy Theory**

http://uwo.ca/math/faculty/jardine/courses/homth/homotopy\_theory.html

#### **Basic References**

- [1] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition
- [2] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999
- [3] Saunders Mac Lane. Categories for the Working Mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998

# Lecture 01: Homological algebra

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## 1 Chain complexes

 $R = \text{commutative ring with 1 (eg. } \mathbb{Z}, \text{ a field } k)$ 

## R-modules: basic definitions and facts

•  $f: M \to N$  an *R*-module homomorphism:

The  $kernel \ker(f)$  of f is defined by

$$\ker(f) = \{ \text{all } x \in M \text{ such that } f(x) = 0 \}.$$

 $\ker(f) \subset M$  is a submodule.

The *image*  $\operatorname{im}(f) \subset N$  of f is defined by

$$\operatorname{im}(f) = \{ f(x) \mid x \in M \}.$$

The cokernel cok(f) of f is the quotient

$$\operatorname{cok}(f) = N/\operatorname{im}(f)$$
.

# • A sequence

$$M \xrightarrow{f} M' \xrightarrow{g} M''$$

is *exact* if  $\ker(g) = \operatorname{im}(f)$ . Equivalently,  $g \cdot f = 0$  and  $\operatorname{im}(f) \subset \ker(g)$  is surjective.

The sequence  $M_1 \to M_2 \to \cdots \to M_n$  is *exact* if  $\ker = \operatorname{im} \operatorname{everywhere}$ .

**Examples**: 1) The sequence

$$0 \to \ker(f) \to M \xrightarrow{f} N \to \operatorname{cok}(f) \to 0$$

is exact.

2) The sequence

$$0 \to M \xrightarrow{f} N$$

is exact if and only if f is a monomorphism (monic, injective)

3) The sequence

$$M \xrightarrow{f} N \to 0$$

is exact if and only if f is an epimorphism (epi, surjective).

**Lemma 1.1** (Snake Lemma). *Given a commutative diagram of R-module homomorphisms* 

$$A_{1} \longrightarrow A_{2} \xrightarrow{p} A_{3} \longrightarrow 0$$

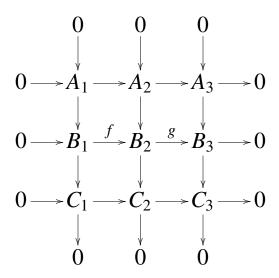
$$\downarrow f_{1} \qquad \downarrow f_{2} \qquad \downarrow f_{3}$$

$$0 \longrightarrow B_{1} \xrightarrow{i} B_{2} \longrightarrow B_{3}$$

in which the horizontal sequences are exact. There is an induced exact sequence

$$\ker(f_1) \to \ker(f_2) \to \ker(f_3) \xrightarrow{\partial} \operatorname{cok}(f_1) \to \operatorname{cok}(f_2) \to \operatorname{cok}(f_3).$$
  
 $\partial(y) = [z] \text{ for } y \in \ker(f_3), \text{ where } y = p(x), \text{ and } f_2(x) = i(z).$ 

**Lemma 1.2**  $((3 \times 3)$ -Lemma). *Given a commutative diagram of R-module maps* 



With exact columns.

- 1) If either the top two or bottom two rows are exact, then so is the third.
- 2) If the top and bottom rows are exact, and  $g \cdot f = 0$ , then the middle row is exact.

**Lemma 1.3** (5-Lemma). *Given a commutative diagram of R-module homomorphisms* 

$$A_{1} \xrightarrow{f_{1}} A_{2} \xrightarrow{} A_{3} \xrightarrow{} A_{4} \xrightarrow{g_{1}} A_{5}$$

$$\downarrow h_{1} \qquad \downarrow h_{2} \qquad \downarrow h_{3} \qquad \downarrow h_{4} \qquad \downarrow h_{5}$$

$$B_{1} \xrightarrow{f_{2}} B_{2} \xrightarrow{} B_{3} \xrightarrow{} B_{4} \xrightarrow{g_{2}} B_{5}$$

with exact rows, such that  $h_1, h_2, h_4, h_5$  are isomorphisms. Then  $h_3$  is an isomorphism.

The Snake Lemma is proved with an element chase. The  $(3 \times 3)$ -Lemma and 5-Lemma are consequences.

e.g. Prove the 5-Lemma with the induced diagram

$$0 \longrightarrow \operatorname{cok}(f_1) \longrightarrow A_3 \longrightarrow \ker(g_1) \longrightarrow 0$$

$$\downarrow \cong \qquad \qquad \downarrow h_3 \qquad \downarrow \cong$$

$$0 \longrightarrow \operatorname{cok}(f_2) \longrightarrow B_3 \longrightarrow \ker(g_2) \longrightarrow 0$$

# Chain complexes

A *chain complex C* in *R*-modules is a sequence of *R*-module homomorphisms

$$\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \ldots$$

such that  $\partial^2 = 0$  (or that  $im(\partial) \subset ker(\partial)$ ) everywhere.  $C_n$  is the module of *n*-chains.

A *morphism*  $f: C \to D$  of chain complexes consists of R-module maps  $f_n: C_n \to D_n$ ,  $n \in \mathbb{Z}$  such that there are comm. diagrams

$$C_{n} \xrightarrow{f_{n}} D_{n}$$

$$\downarrow \partial$$

$$C_{n-1} \xrightarrow{f_{n-1}} D_{n-1}$$

The chain complexes and their morphisms form a category, denoted by Ch(R).

If C is a chain complex such that C<sub>n</sub> = 0 for n < 0, then C is an *ordinary* chain complex.
We usually drop all the 0 objects, and write

$$\rightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

 $Ch_+(R)$  is the full subcategory of ordinary chain complexes in Ch(R).

• Chain complexes indexed by the integers are often called *unbounded* complexes.

*Slogan*: Ordinary chain complexes are spaces, and unbounded complexes are spectra.

• Chain complexes of the form

$$\cdots \rightarrow 0 \rightarrow C_0 \rightarrow C_{-1} \rightarrow \cdots$$

are cochain complexes, written (classically) as

$$C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \dots$$

Both notations are in common (confusing) use.

Morphisms of chain complexes have kernels and cokernels, defined degreewise.

A sequence of chain complex morphisms

$$C \rightarrow D \rightarrow E$$

is exact if all sequences of morphisms

$$C_n \to D_n \to E_n$$

are exact.

# Homology

Given a chain complex C:

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$$

Write

$$Z_n = Z_n(C) = \ker(\partial : C_n \to C_{n-1})$$
, (*n*-cycles), and  $B_n = B_n(C) = \operatorname{im}(\partial : C_{n+1} \to C_n)$  (*n*-boundaries).

$$\partial^2 = 0$$
, so  $B_n(C) \subset Z_n(C)$ .

The  $n^{th}$  homology group  $H_n(C)$  of C is defined by

$$H_n(C) = Z_n(C)/B_n(C).$$

A chain map  $f: C \rightarrow D$  induces R-module maps

$$f_*: H_n(C) \to H_n(D), n \in \mathbb{Z}.$$

 $f: C \to D$  is a homology isomorphism (resp. quasi-isomorphism, acyclic map, weak equivalence) if all induced maps  $f_*: H_n(C) \to H_n(D), n \in \mathbb{Z}$  are isomorphisms.

A complex C is *acyclic* if the map  $0 \to C$  is a homology isomorphism, or if  $H_n(C) \cong 0$  for all n, or if the sequence

$$\dots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \dots$$

is exact.

# Lemma 1.4. A short exact sequence

$$0 \rightarrow C \rightarrow D \rightarrow E \rightarrow 0$$

induces a natural long exact sequence

$$\dots \xrightarrow{\partial} H_n(C) \to H_n(D) \to H_n(E) \xrightarrow{\partial} H_{n-1}(C) \to \dots$$

*Proof.* The short exact sequence induces comparisons of exact sequences

$$C_n/B_n(C) \longrightarrow D_n/B_n(D) \longrightarrow E_n/B_n(E) \longrightarrow 0$$

$$\downarrow \partial_* \qquad \qquad \downarrow \partial_* \qquad \qquad \downarrow \partial_*$$

$$0 \longrightarrow Z_{n-1}(C) \longrightarrow Z_{n-1}(D) \longrightarrow Z_{n-1}(E)$$

Use the natural exact sequence

$$0 \to H_n(C) \to C_n/B_n(C) \xrightarrow{\partial_*} Z_{n-1}(C) \to H_{n-1}(C) \to 0$$
  
Apply the Snake Lemma.

# 2 Ordinary chain complexes

A map  $f: C \to D$  in  $Ch_+(R)$  is a

- weak equivalence if f is a homology isomorphism,
- *fibration* if  $f: C_n \to D_n$  is surjective for n > 0,
- *cofibration* if f has the left lifting property (LLP) with respect to all morphisms of  $Ch_+(R)$  which

are simultaneously fibrations and weak equivalences.

A *trivial fibration* is a map which is both a fibration and a weak equivalence. A *trivial cofibration* is both a cofibration and a weak equivalence.

f has the *left lifting property* with respect to all trivial fibrations (ie. f is a cofibration) if given any solid arrow commutative diagram

$$\begin{array}{c}
C \longrightarrow X \\
f \downarrow \qquad \qquad \downarrow p \\
D \longrightarrow Y
\end{array}$$

in  $Ch_+(R)$  with p a trivial fibration, then the dotted arrow exists making the diagram commute.

Special chain complexes and chain maps:

• R(n) [= R[-n] in "shift notation"] consists of a copy of the free R-module R, concentrated in degree n:

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow \stackrel{n}{R} \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

There is a natural *R*-module isomorphism

$$hom_{Ch_+(R)}(R(n),C) \cong Z_n(C).$$

•  $R\langle n+1\rangle$  is the complex

$$\cdots \to 0 \to \stackrel{n+1}{R} \xrightarrow{1} \stackrel{n}{R} \to 0 \to \cdots$$

• There is a natural *R*-module isomorphism

$$\hom_{Ch_+(R)}(R\langle n+1\rangle,C)\cong C_{n+1}.$$

• There is a chain  $\alpha : R(n) \to R\langle n+1 \rangle$ 

 $\alpha$  classifies the cycle  $1 \in R\langle n+1 \rangle_n$ .

**Lemma 2.1.** Suppose that  $p: A \rightarrow B$  is a fibration and that  $i: K \rightarrow A$  is the inclusion of the kernel of p. Then there is a long exact sequence

$$\dots \xrightarrow{p_*} H_{n+1}(B) \xrightarrow{\partial} H_n(K) \xrightarrow{i_*} H_n(A) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} \dots$$
$$\dots \xrightarrow{\partial} H_0(K) \xrightarrow{i_*} H_0(A) \xrightarrow{p_*} H_0(B).$$

*Proof.*  $j : \text{im}(p) \subset B$ , and write  $\pi : A \to \text{im}(p)$  for the induced epimorphism. Then  $H_n(\text{im}(p)) = H_n(B)$  for n > 0, and there is a diagram

$$H_0(A) \xrightarrow{p_*} H_0(B)$$
 $H_0(\operatorname{im}(p))$ 

in which  $\pi_*$  is an epimorphism and  $i_*$  is a monomorphism (exercise). The long exact sequence is con-

structed from the long exact sequence in homology for the short exact sequence

$$0 \to K \xrightarrow{i} A \xrightarrow{\pi} \operatorname{im}(p) \to 0,$$

with the monic  $i_*: H_0(\operatorname{im}(p)) \to H_0(B)$ .

**Lemma 2.2.**  $p: A \to B$  is a fibration if and only if p has the RLP wrt. all maps  $0 \to R\langle n+1 \rangle$ ,  $n \ge 0$ . *Proof.* The lift exists in all solid arrow diagrams

$$\begin{array}{c}
0 \longrightarrow A \\
\downarrow \qquad \qquad \downarrow p \\
R\langle n+1 \rangle \longrightarrow B
\end{array}$$

for 
$$n \ge 0$$
.

**Corollary 2.3.**  $0 \to R\langle n+1 \rangle$  is a cofibration for all  $n \ge 0$ .

*Proof.* This map has the LLP wrt all fibrations, hence wrt all trivial fibrations.

**Lemma 2.4.** The map  $0 \rightarrow R(n)$  is a cofibration.

*Proof.* The trivial fibration  $p: A \to B$  induces an epimorphism  $Z_n(A) \to Z_n(B)$  for all  $n \ge 0$ :

$$A_{n+1} \longrightarrow B_n(A) \longrightarrow Z_n(A) \longrightarrow H_n(A) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \cong$$

$$B_{n+1} \longrightarrow B_n(B) \longrightarrow Z_n(B) \longrightarrow H_n(B) \longrightarrow 0$$

A chain complex A is *cofibrant* if the map  $0 \rightarrow A$  is a cofibration.

eg. R(n+1) and R(n) are cofibrant.

All chain complexes C are *fibrant*, because all chain maps  $C \rightarrow 0$  are fibrations.

**Proposition 2.5.**  $p: A \rightarrow B$  is a trivial fibration and if and only if

- 1)  $p: A_0 \rightarrow B_0$  is a surjection, and
- 2) p has the RLP wrt all  $\alpha : R(n) \rightarrow R\langle n+1 \rangle$ .

**Corollary 2.6.**  $\alpha : R(n) \rightarrow R\langle n+1 \rangle$  is a cofibration.

*Proof of Proposition 2.5.* 1) Suppose that  $p: A \rightarrow B$  is a trivial fibration with kernel K.

Use Snake Lemma with the comparison

$$A_{1} \xrightarrow{\partial} A_{0} \longrightarrow H_{0}(A) \longrightarrow 0$$

$$\downarrow p \qquad \qquad \downarrow \cong$$

$$B_{1} \xrightarrow{\partial} B_{0} \longrightarrow H_{0}(B) \longrightarrow 0$$

to show that  $p: A_0 \to B_0$  is surjective.

Suppose given a diagram

$$R(n) \xrightarrow{x} A$$

$$\alpha \downarrow \qquad \qquad \downarrow p$$

$$R\langle n+1 \rangle \xrightarrow{y} B$$

Choose  $z \in A_{n+1}$  such that p(z) = y. Then  $x - \partial(z)$  is a cycle of K, and K is acyclic (exercise) so there is a  $v \in K_{n+1}$  such that  $\partial(v) = x - \partial(z)$ .  $\partial(z+v) = x$  and  $\partial(z+v) = y$ , so  $\partial(z+v) = x$  is the desired lift.

2) Suppose that  $p: A_0 \to B_0$  is surjective and that p has the right lifting property with respect to all  $R(n) \to R\langle n+1 \rangle$ .

The solutions of the lifting problems

$$R(n) \xrightarrow{0} A$$

$$\downarrow \qquad \qquad \downarrow p$$

$$R\langle n+1 \rangle \xrightarrow{x} B$$

show that p is surjective on all cycles, while the solutions of the lifting problems

$$R(n) \xrightarrow{x} A$$

$$\downarrow \qquad \qquad \downarrow p$$

$$R\langle n+1 \rangle \xrightarrow{y} B$$

show that p induces a monomorphism in all homology groups. It follows that p is a weak equivalence.

We have the diagram

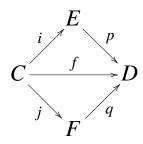
$$Z_{n+1}(A) \longrightarrow A_{n+1} \stackrel{\partial}{\longrightarrow} Z_n(A) \longrightarrow H_n(A) \longrightarrow 0$$

$$\downarrow^p \qquad \downarrow^p \qquad \downarrow^p \qquad p \downarrow \cong$$

$$Z_{n+1}(B) \longrightarrow B_{n+1} \stackrel{\partial}{\longrightarrow} Z_n(B) \longrightarrow H_n(B) \longrightarrow 0$$

Then  $p: B_n(A) \to B_n(B)$  is epi, so  $p: A_{n+1} \to B_{n+1}$  is epi, for all  $n \ge 0$ .

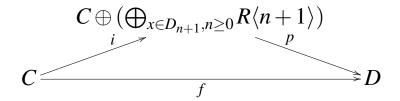
**Proposition 2.7.** Every chain map  $f: C \rightarrow D$  has two factorizations



where

- 1) p is a fibration. i is a monomorphism, a weak equivalence and has the LLP wrt all fibrations.
- 2) q is a trivial fibration and j is a monomorphism and a cofibration.

## *Proof.* 1) Form the factorization



p is the sum of f and all classifying maps for chains x in all non-zero degrees. It is therefore surjective in non-zero degrees, hence a fibration.

*i* is the inclusion of a direct summand with acyclic cokernel, and is thus a monomorphism and a weak equivalence. *i* is a direct sum of maps which have the LLP wrt all fibrations, and thus has the same lifting property.

2) Recall that  $A \to B$  is a trivial fibration if and only if it has the RLP wrt all cofibrations  $R(n) \to R\langle n+1\rangle$ ,  $n \ge -1$ .

Notation:  $R(-1) \rightarrow R(0)$  is the map  $0 \rightarrow R(0)$ .

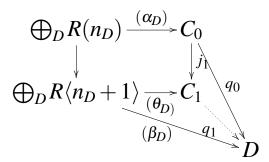
Consider the set of all diagrams

$$D: R(n_D) \xrightarrow{\alpha_D} C$$

$$\downarrow \qquad \qquad \downarrow f = q_0$$

$$R\langle n_D + 1 \rangle \xrightarrow{\beta_D} D$$

and form the pushout



where  $C = C_0$ . Then  $j_1$  is a monomorphism and a cofibration, because the collection of all such maps is closed under direct sum and pushout.

Every lifting problem D as above is solved in  $C_1$ :

$$R(n_D) \xrightarrow{\alpha_D} C_0 \xrightarrow{j_1} C_1$$

$$\downarrow \qquad \qquad \downarrow q_1$$

$$R\langle n_D + 1 \rangle \xrightarrow{\beta_D} D$$

commutes.

Repeat this process inductively for the maps  $q_i$  to produce a string of factorizations

$$C_0 \xrightarrow{j_1} C_1 \xrightarrow{j_2} C_2 \xrightarrow{j_3} \dots$$
 $q_0 \downarrow q_1 \qquad q_2 \qquad \dots$ 

Let  $F = \underset{i}{\underline{\lim}} C_i$ . Then f has a factorization



Then j is a cofibration and a monomorphism, because all  $j_k$  have these properties and the family of such maps is closed under (infinite) composition.

Finally, given a diagram

$$R(n) \xrightarrow{\alpha} F$$

$$\downarrow \qquad \qquad \downarrow q$$

$$R\langle n+1 \rangle \xrightarrow{\beta} D$$

The map  $\alpha$  factors through some finite stage of the filtered colimit defining F, so that  $\alpha$  is a composite

$$R(n) \xrightarrow{\alpha'} C_k \to F$$

for some k. The lifting problem

$$R(n) \xrightarrow{\alpha'} C_k \downarrow q_k \\ \downarrow R\langle n+1 \rangle \xrightarrow{\beta} D$$

is solved in  $C_{k+1}$ , hence in F.

Remark: This proof is a small object argument.

The R(n) are *small* (or compact): hom(R(n), ) commutes with filtered colimits.

**Corollary 2.8.** 1) Every cofibration is a monomorphism.

2) Suppose that  $j: C \to D$  is a cofibration and a weak equivalence. Then j has the LLP wrt all fibrations.

*Proof.* 2) The map j has a factorization



where i has the left lifting property with respect to all fibrations and is a weak equivalence, and p is a fibration. Then p is a trivial fibration, so the lifting exists in the diagram

$$\begin{array}{c}
C \xrightarrow{i} F \\
j \downarrow \qquad \downarrow p \\
D \xrightarrow{1} D
\end{array}$$

since j is a cofibration. Then j is a retract of a map (namely i) which has the LLP wrt all fibrations, and so j has the same property.

1) is an exercise. 
$$\Box$$

#### **Resolutions**

Suppose that P is a chain complex. Proposition 2.7 says that  $0 \rightarrow P$  has a factorization

$$0 \xrightarrow{j} F \downarrow q \\ P$$

where j is a cofibration (so that F is cofibrant) and q is a trivial fibration, hence a weak equivalence.

The proof of Proposition 2.7 implies that each Rmodule  $F_n$  is free, so F is a *free resolution* of P.

If the complex *P* is cofibrant, then the lift exists in



All modules  $P_n$  are direct summands of free modules and are therefore projective.

This observation has a converse:

**Lemma 2.9.** A chain complex P is cofibrant if and only if all modules  $P_n$  are projective.

*Proof.* Suppose that *P* is a complex of projectives, and  $p: A \rightarrow B$  is a trivial fibration.

Then  $p: A_n \to B_n$  is surjective for all  $n \ge 0$  and has acyclic kernel  $i: K \to A$ .

Suppose given a lifting problem

$$\begin{array}{ccc}
0 \longrightarrow A \\
\downarrow \theta & \downarrow p \\
P \longrightarrow B
\end{array}$$

There is a map  $\theta_0: P_0 \to A_0$  which lifts  $f_0$ :

$$P_0 \xrightarrow{f_0} B_0$$

Suppose given a lift up to degree n, ie. homomorphisms  $\theta_i: P_i \to A_i$  for  $i \le n$  such that  $p_i\theta_i = f_i$  for  $i \le n$  and  $\partial \theta_i = \theta_{i-1}\partial$  for  $1 \le i \le n$ 

There is a map  $\theta'_{n+1}: P_{n+1} \to A_{n+1}$  such that  $p_{n+1}\theta'_{n+1} = f_{n+1}$ .

Then

$$p_n(\partial \theta'_{n+1} - \theta_n \partial) = \partial p_{n+1} \theta'_{n+1} - f_n \partial = \partial f_{n+1} - f_n \partial = 0$$

so there is a  $v: P_{n+1} \to K_n$  such that

$$i_n v = \partial \theta'_{n+1} - \theta_n \partial$$
.

Also

$$\partial(\partial\theta'_{n+1}-\theta_n\partial)=0$$

and K is acyclic, so there is a  $w: P_{n+1} \to K_{n+1}$  such that

$$i_n \partial w = \partial \theta'_{n+1} - \theta_n \partial.$$

Then

$$\partial(\theta'_{n+1}-i_{n+1}w)=\theta_n\partial$$

and

$$p_{n+1}(\theta'_{n+1}-i_{n+1}w)=p_{n+1}\theta'_{n+1}=f_{n+1}.$$

### **Remarks**:

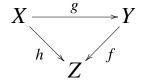
- 1) Every chain complex C has a *cofibrant model*, i.e. a weak equivalence  $p: P \rightarrow C$  with P cofibrant (aka. complex of projectives).
- 2) M = an R-module. A cofibrant model  $P \rightarrow M(0)$  is a projective resolution of M in the usual sense.
- 3) Cofibrant models  $P \rightarrow C$  are also (commonly) constructed with Eilenberg-Cartan resolutions.

#### 3 Closed model categories

A *closed model category* is a category  $\mathcal{M}$  equipped with three classes of maps, namely weak equivalences, fibrations and cofibrations, such that the following conditions are satisfied:

**CM1** The category  $\mathcal{M}$  has all finite limits and colimits.

CM2 Given a commutative triangle



of morphisms in  $\mathcal{M}$ , if any two of f,g and h are weak equivalences, then so is the third.

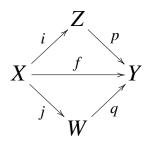
**CM3** The classes of cofibrations, fibrations and weak equivalences are closed under retraction.

CM4 Given a commutative solid arrow diagram

$$\begin{array}{ccc}
A \longrightarrow X \\
\downarrow & & \downarrow p \\
B \longrightarrow Y
\end{array}$$

such that i is a cofibration and p is a fibration. Then the lift exists making the diagram commute if either i or p is a weak equivalence.

**CM5** Every morphism  $f: X \to Y$  has factorizations



where p is a fibration and i is a trivial cofibration, and q is a trivial fibration and j is a cofibration.

**Theorem 3.1.** With the definition of weak equivalence, fibration and cofibration given above,  $Ch_+(R)$  satisfies the axioms for a closed model category.

*Proof.* CM1, CM2 and CM3 are exercises. CM5 is Proposition 2.7, and CM4 is Corollary 2.8.  $\Box$ 

**Exercise**: A map  $f: C \to D$  of Ch(R) (unbounded chain complexes) is a *weak equivalence* if it is a homology isomorphism.

f is a *fibration* if all maps  $f: C_n \to D_n$ ,  $n \in \mathbb{Z}$  are surjective.

A map of is a *cofibration* if and only if it has the left lifting property with respect to all trivial fibrations.

Show that, with these definitions, Ch(R) has the structure of a closed model category.