Lecture 03: Homotopical algebra

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6 Example: Chain homotopy

$C =$ an ordinary chain complex. We have two constructions:

1) C^I is the complex with

$$C^I_n = C_n \oplus C_n \oplus C_{n+1}$$

for $n > 0$, and with

$C^I_0 = \{(x,y,z) \in C_0 \oplus C_0 \oplus C_1 \mid (x-y) + \partial(z) = 0\}$.

The boundary map $\partial : C^I_n \to C^I_{n-1}$ is defined by

$$\partial(x,y,z) = (\partial(x), \partial(y), (-1)^n(x-y) + \partial(z)).$$

2) $\tilde{C}$ is the chain complex with

$$\tilde{C}_n = C_n \oplus C_{n+1}$$

for $n > 0$ and

$$\tilde{C}_0 = \{(x,z) \in C_0 \oplus C_1 \mid x + \partial(z) = 0\}.$$

The boundary $\partial : \tilde{C}_n \to \tilde{C}_{n-1}$ of $\tilde{C}$ is defined by

$$\partial(x,z) = (\partial(x), (-1)^n x + \partial(z)).$$

Lemma 6.1. *The complex $\tilde{C}$ is acyclic.*

Proof. If $\partial(x,z) = 0$ then $\partial(x) = 0$ and $\partial(z) = (-1)^{n+1}x$. It follows that

$$\partial((-1)^{n+1}z, 0) = (x,z)$$

if (x,z) is a cycle, so (x,z) is a boundary. □

There is a pullback diagram

$$\begin{tikzcd} C^I \arrow[r, "\alpha"] \arrow[d, "p"'] & \tilde{C} \arrow[d, "p'"] \\ C \oplus C \arrow[r, "\beta"'] & C \end{tikzcd}$$

in which p and p' are projections defined in each degree by $p(x,y,z) = (x,y)$ and $p'(x,z) = x$. The map α is defined by $\alpha(x,y.z) = (x-y,z)$, while $\beta(x,y) = x-y$.

p' is a fibration, and fibrations are closed under pullback, so p is also a fibration. The maps α and β are surjective in all degrees, and the diagram above expands to a comparison

$$\begin{tikzcd} & & C^I \arrow[r, "\alpha"] \arrow[d, "p"] & \tilde{C} \arrow[r] \arrow[d, "p'"] & 0 \\ 0 \arrow[r] & C \arrow[r, "\Delta"'] \arrow[ur, "s"] & C \oplus C \arrow[r, "\beta"'] & C \arrow[r] & 0 \end{tikzcd}$$

where Δ is the diagonal map.

Lemma 6.1 and a long exact sequence argument imply that the map s is a weak equivalence.

We have a functorial diagram

$$\begin{tikzcd} & C^I \arrow[d, "p"] \\ C \arrow[ur, "s"] \arrow[r, "\Delta"'] & C \oplus C \end{tikzcd} \tag{1}$$

in which p is a fibration and s is a weak equivalence. This is a *path object*.

A commutative diagram of chain maps

$$\begin{tikzcd} & C^I \arrow[d, "p"] \\ D \arrow[ur, "h"] \arrow[r, "(f,g)"'] & C \oplus C \end{tikzcd} \tag{2}$$

is a *right homotopy* between the chain maps $f,g : D \to C$

The map h, if it exists, is defined by

$$h(x) = (f(x), g(x), s(x))$$

for a collection of R-module maps $s : D_n \to C_{n+1}$. The fact that h is a chain map forces

$$s(\partial(x)) = (-1)^n(f(x) - g(x)) + \partial(s(x))$$

for $x \in D_n$. Thus

$$(-1)^n s(\partial(x)) = (f(x) - g(x)) + \partial((-1)^n s(x)),$$

so

$$(-1)^n s(\partial(x)) + \partial((-1)^{n+1} s(x) = f(x) - g(x).$$

The maps $x \mapsto (-1)^{n+1}s(x)$, $x \in D_n$, arising from the right homotopy h define a *chain homotopy* between the chain maps f and g.

All chain homotopies arise in this way.

Exercise: Show that there is a functorial diagram of the form (1) for unbounded chain complexes C, such that the corresponding right homotopies (2) define chain homotopies between maps $f, g : D \to C$ of unbounded chain complexes.

7 Homotopical algebra

A *closed model category* is a category $\mathscr{M}$ equipped with weak equivalences, fibrations and cofibrations, such that the following hold:

CM1 The category $\mathscr{M}$ has all (finite) limits and colimits.

CM2 Given a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

in $\mathscr{M}$, if any two of f, g and h are weak equivalences, then so is the third.

CM3 The classes of cofibrations, fibrations and weak equivalences are closed under retraction.

CM4 Given a commutative solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

such that i is a cofibration and p is a fibration. Then the lift exists if either i or p is a weak equivalence.

CM5 Every morphism $f : X \to Y$ has factorizations

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where p is a fibration and i is a trivial cofibration, and q is a trivial fibration and j is a cofibration.

Here's the meaning of the word "closed":

Lemma 7.1. *1) $i : A \to B$ is a cofibration if and only if it has the LLP wrt all trivial fibrations.*

2) $i : A \to B$ is a trivial cofibration if and only if it has the LLP wrt all fibrations.

3) $p : X \to Y$ is a fibration if and only if it has the RLP wrt all trivial cofibrations.

4) p is a trivial fibration if and only if it has the RLP wrt all cofibrations.

Proof. I'll prove statement 2). The rest are similar.

If i is a trivial cofibration, then it has the LLP wrt all fibrations by **CM4**.

Suppose i has the LLP wrt all fibrations. i has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & X \arrow[d, "p"] \\ & B \end{tikzcd}$$

where j is a trivial cofibration and p is a fibration. Then the lifting exists in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "1"'] \arrow[ur, dotted] & B \end{tikzcd}$$

Then i is a retract of j and is therefore a trivial cofibration by **CM3**. $\square$

Corollary 7.2. *1) The classes of cofibrations and trivial cofibrations are closed under compositions and pushout. Any isomorphism is a trivial cofibration.*

2) The classes of fibrations and trivial fibrations are closed under composition and pullback. Any isomorphism is a trivial fibration.

Remark: Lemma 7.1 implies that, in order to describe a closed model structure, one needs only specify the weak equivalences and either the cofibrations or fibrations.

We saw this in the descriptions of the model structures for the chain complex categories and for spaces.

Homotopies

1) A *path object* for $Y \in \mathscr{M}$ is a commutative diagram

$$\begin{tikzcd} & Y^I \arrow[d, "p"] \\ Y \arrow[ur, "s"] \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

such that Δ is the diagonal map, s is a weak equivalence and p is a fibration.

2) A *right homotopy* between maps $f, g : X \to Y$ is a commutative diagram

$$\begin{tikzcd} & Y^I \arrow[d, "p"] & \\ X \arrow[ur, "h"] \arrow[r, "(f,g)"'] & Y \times Y & Y \arrow[ul, dotted, "s"'] \arrow[l, dotted, "\Delta"] \end{tikzcd}$$

where p is the fibration for *some* (displayed) path object for Y.

f is *right homotopic* to g if such a right homotopy exists. Write $f \sim_r g$.

Examples: 1) Path objects abound in nature, since the diagonal map $\Delta : Y \to Y \times Y$ factorizes as a fibration following a trivial cofibration, by **CM5**.

2) Chain homotopy is a type of right homotopy in both $Ch_+(R)$ and $Ch(R)$.

3) For ordinary spaces X, there is a space X^I, whose elements are the paths $I \to X$ in X. Restricting to the two ends of the paths defines a map $d : X^I \to X \times X$, which is a Serre fibration (exercise). There is a constant path map $s : X \to X^I$, and a commu-

tative diagram

$$\begin{tikzcd} & X^I \arrow[d, "d"] \\ X \arrow[ur, "s"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

The composite $X^I \xrightarrow{d} X \times X \xrightarrow{pr_L} X$ is a trivial fibration (exercise), so s is a weak equivalence.

The traditional path space defines a path object construction. Right homotopies $X \to Y^I$ are traditional homotopies $X \times I \to Y$ by adjointness.

Here's the dual cluster of definitions:

1) A *cylinder object* for an object $X \in \mathscr{M}$ is a commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "i"'] & X \\ X \otimes I \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where ∇ is the "fold" map, i is a cofibration and σ is a weak equivalence.

2) A *left homotopy* between maps $f,g : X \to Y$ is a commutative diagram

$$\begin{tikzcd} X & X \sqcup X \arrow[l, dotted, "\nabla"'] \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & Y \\ & X \otimes I \arrow[ul, dotted, "\sigma"] \arrow[ur, "h"'] & \end{tikzcd}$$

where i is the cofibration appearing in *some* cylinder object for X.

Say f is *left homotopic* to g if such a left homotopy exists. Write $f \sim_l g$.

Examples: 1) Suppose X is a CW-complex and I is the unit interval. The standard picture

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "i"'] & X \\ X \times I \arrow[ur, "pr"'] & \end{tikzcd}$$

is a cylinder object for X. The space $X \times I$ is obtained from $X \sqcup X$ by attaching cells, so i is a cofibration.

2) There are lots of cylinder objects: the map $\nabla : X \sqcup X \to X$ has a factorization as a cofibration followed by a trivial fibration, by **CM5**.

Duality

Here is what I mean by "dual":

Lemma 7.3. *$\mathscr{M}$ = a closed model category.*

Say a morphism $f^{op} : Y \to X$ of the opposite category $\mathscr{M}^{op}$ is a fibration (resp. cofibration, weak equivalence) if and only if the corresponding map $f : X \to Y$ is a cofibration (resp. fibration, weak equivalence) of $\mathscr{M}$.

Then with these definitions, $\mathscr{M}^{op}$ satisfies the axioms for a closed model category.

Proof. Exercise. $\square$

Reversing the arrows in a cylinder object gives a path object, and vice versa. All homotopical facts about a model category $\mathscr{M}$ have equivalent dual assertions in $\mathscr{M}^{op}$.

Examples: In Lemma 7.1, statement 3) is the dual of statement 1), and statement 4) is the dual of statement 2).

Lemma 7.4. *Right homotopy of maps $X \to Y$ is an equivalence relation if Y is fibrant.*

The dual of Lemma 7.4 is the following:

Lemma 7.5. *Left homotopy of maps $X \to Y$ is an equivalence relation if X is cofibrant.*

Proof. Lemma 7.5 is equivalent to Lemma 7.4 in $\mathscr{M}^{op}$. $\square$

Proof of Lemma 7.4. If Y if fibrant then any projection $X \times Y \to X$ is a fibration (exercise).

Thus, if

$$\begin{tikzcd} & Y^I \arrow[d, "(p_0,p_1)"] \\ Y \arrow[ur, "s"] \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

is a path object for a fibrant object Y, then the maps p_0 and p_1 are trivial fibrations.

Suppose given right homotopies

$$\begin{tikzcd} & Y^I \arrow[d, "{(p_0,p_1)}"] \\ X \arrow[ur, "h_1"] \arrow[r, "{(f_1,f_2)}"'] & Y \times Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y^J \arrow[d, "{(q_0,q_1)}"] \\ X \arrow[ur, "h_2"] \arrow[r, "{(f_2,f_3)}"'] & Y \times Y \end{tikzcd}$$

Form the pullback

$$\begin{tikzcd} Y^I \times_Y Y^J \arrow[r, "p_*"] \arrow[d, "q_*"'] & Y^J \arrow[d, "q_0"] \\ Y^I \arrow[r, "p_1"'] & Y \end{tikzcd}$$

The diagram

$$\begin{tikzcd} Y^I \times_Y Y^J \arrow[r, "p_*"] \arrow[d, "{(q_*,q_1 p_*)}"'] & Y^J \arrow[d, "{(q_0,q_1)}"] \\ Y^I \times Y \arrow[r, "p_1 \times 1"'] & Y \times Y \end{tikzcd}$$

is a pullback and $p_0 \times 1 : Y^I \times Y \to Y \times Y$ is a fibration, so the composite

$$Y^I \times_Y Y^J \xrightarrow{(p_0 q_*, q_1 p_*)} Y \times Y$$

is a fibration. The weak equivalences s, s' from the respective path objects determine a commutative diagram

$$\begin{tikzcd} & Y^I \times_Y Y^J \arrow[d, "(p_0q_*, q_1p_*)"] \\ Y \arrow[ur, "(s,s')"] \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

and the map (s,s') is a weak equivalence since p_0q_* is a trivial fibration.

The homotopies h,h' therefore determine a right homotopy

$$\begin{tikzcd} & Y^I \times_Y Y^J \arrow[d, "(p_0q_*, q_1p_*)"] \\ X \arrow[ur, "(h,h')"] \arrow[r, "(f_1,f_3)"'] & Y \times Y \end{tikzcd}$$

It follows that the right homotopy relation is transitive.

Right homotopy is symmetric, since the twist isomorphism $Y \times Y \xrightarrow{\cong} Y \times Y$ is a fibration.

Right homotopy is reflexive, since the morphism s in a path object is a right homotopy from the identity to itself. $\square$

Here's the result that ties the homotopical room together:

Lemma 7.6. *1) Suppose Y is fibrant and $X \otimes I$ is a fixed choice of cylinder object for an object X. Suppose $f, g : X \to Y$ are right homotopic. Then there is a left homotopy*

$$\begin{tikzcd} X \sqcup X \arrow[r, "(f,g)"] \arrow[d, "i"'] & Y \\ X \otimes I \arrow[ur, "h"'] & \end{tikzcd}$$

2) Suppose X is cofibrant and Y^I is a fixed choice of path object for an object Y. Suppose $f, g : X \to Y$ are left homotopic. Then there is a right homotopy

$$\begin{tikzcd} & Y^I \arrow[d, "p"] \\ X \arrow[ur, "h"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

Proof. Statement 2) is the dual of statement 1). We'll prove statement 1).

Suppose

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "i"'] & X \\ X \otimes I \arrow[ur, "\sigma"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y^I \arrow[d, "(p_0, p_1)"] \\ Y \arrow[ur, "s"] \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

are the fixed choice of cylinder and the path object involved in the right homotopy $f \sim_r g$, respectively, and let $h : X \to Y^I$ be the right homotopy. Form the diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "{(sf,h)}"] \arrow[d, "i"'] & Y^I \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \\ X \otimes I \arrow[r, "f\sigma"'] \arrow[ur, dotted, "\theta"] & Y & \end{tikzcd}$$

The lift θ exists because p_0 is a trivial fibration since Y is fibrant (exercise). The composite $p_1\theta$ is the desired left homotopy. □

Corollary 7.7. *Suppose $f, g : X \to Y$ are morphisms of $\mathscr{M}$, where X is cofibrant and Y is fibrant. Suppose*

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "i"'] & X \\ X \otimes I \arrow[ur, "\sigma"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & Y^I \arrow[d, "p"] \\ Y \arrow[ur, "s"] \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

are fixed choices of cylinder and path objects for X and Y respectively. Then the following are equivalent:

- *f is left homotopic to g.*
- *There is a right homotopy $h : X \to Y^I$ from f to g.*

- *f is right homotopic to g.*
- *There is a left homotopy $H : X \otimes I \to Y$ from f to g.*

Thus, if X is cofibrant and Y is fibrant, all notions of homotopy of maps $X \to Y$ collapse to the same thing.

Write $f \sim g$ to say that f is homotopic to g (by whatever means) in this case.

Here's the first big application:

Theorem 7.8 (Whitehead Theorem)**.** *Suppose $f : X \to Y$ is a weak equivalence, and the objects X and Y are both fibrant and cofibrant. Then f is a homotopy equivalence.*

Proof. We can assume that f is a trivial fibration: every weak equivalence is a composite of a trivial fibration with a trivial cofibration, and the trivial cofibration case is dual.

Y is cofibrant, so the lifting exists in the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "1"'] \arrow[ur, dotted, "j"] & Y \end{tikzcd}$$

Suppose

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "i"'] & X \\ X \otimes I \arrow[ur, "\sigma"'] & \end{tikzcd}$$

is a cylinder object for X, and then form the diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "{(jf,1)}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ X \otimes I \arrow[r, "f\sigma"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

The indicated lift (and required homotopy) exists because f is a trivial fibration. $\square$

Examples: 1) (traditional Whitehead Theorem) Every weak equivalence $f : X \to Y$ between CW-complexes is a homotopy equivalence.

2) Every weak equivalence $f : C \to D$ in $Ch_+(R)$ between complexes of projective R-modules is a chain homotopy equivalence.

3) Any two projective resolutions $p : P \to M(0)$, $q : Q \to M(0)$ of a module M are chain homotopy equivalent.

The maps p and q are trivial fibrations, and both P and Q are cofibrant chain complexes, so the lift θ

exists in the diagram

$$
\begin{tikzcd}
 & Q \arrow[d, "q"] \\
P \arrow[r, "p"'] \arrow[ur, dotted, "\theta"] & M(0)
\end{tikzcd}
$$

The map θ is a weak equivalence of cofibrant complexes, hence a chain homotopy equivalence.

3 bis) $f : M \to N$ a homomorphism of modules. $p : P \to M(0)$, $q : Q \to N(0)$ projective resolutions.

The lift exists in the diagram

$$
\begin{tikzcd}
0 \arrow[rr] \arrow[d] & & Q \arrow[d, "q"] \\
P \arrow[r, "p"'] \arrow[urr, dotted, "f_1"] & M(0) \arrow[r, "f"'] & N(0)
\end{tikzcd}
$$

since P is cofibrant and q is a trivial fibration, so f lifts to a chain complex map f_1.

If f also lifts to some other chain complex map $f_2 : P \to Q$, there is a commutative diagram

$$
\begin{tikzcd}
P \oplus P \arrow[rrr, "(f_1,f_2)"] \arrow[d, "i"'] & & & Q \arrow[d, "q"] \\
P \otimes I \arrow[r, "\sigma"'] \arrow[urrr, dotted, "h"] & P \arrow[r, "p"'] & M(0) \arrow[r, "f"'] & N(0)
\end{tikzcd}
$$

for some (any) choice of cylinder $P \otimes I$.

Then $f_1 \simeq_l f_2$, so f_1 and f_2 are *chain homotopic* since P is cofibrant and Q is fibrant.

4) $X =$ a space. There is a trivial fibration $p : U \to X$ such that U is a CW complex (exercise).

Suppose Y is a cofibrant space. Then Y is a retract of a CW-complex (exercise).

Suppose $f : X \to Y$ and choose trivial fibrations $p : U \to X$ and $q : V \to Y$ such that U and V are CW-complexes. Then there is a map $f' : U \to V$ which lifts f in the sense that the diagram

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes, and any two such maps are "naively" homotopic (exercise).

8 The homotopy category

For all $X \in \mathscr{M}$ find maps

$$X \xleftarrow{p_X} QX \xrightarrow{j_X} RQX$$

such that

- p_X is a trivial fibration and QX is cofibrant, and j_X is a trivial cofibration and RQX is fibrant (and cofibrant),
- $QX = X$ and $p_X = 1_X$ if X is cofibrant, and $RQX = QX$ and $j_X = 1_{QX}$ if QX is fibrant.

Every map $f : X \to Y$ determines a diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & QX \arrow[l, "p_X"'] \arrow[r, "j_X"] \arrow[d, "f_1"] & RQX \arrow[d, "f_2"] \\ Y & QY \arrow[l, "p_Y"] \arrow[r, "j_Y"'] & RQY \end{tikzcd}$$

since QX is cofibrant and RQY is fibrant.

Lemma 8.1. *The map f_2 is uniquely determined up to homotopy.*

Proof. Suppose f_1' and f_2' are different choices for f_1 and f_2 respectively.

There is a diagram

$$\begin{tikzcd} QX \sqcup QX \arrow[r, "(f_1,f_1')"] \arrow[d, "i"'] & & QY \arrow[d, "p_Y"] \\ QX \otimes I \arrow[urr, dotted] \arrow[r, "\sigma"'] & QX \arrow[r, "f p_X"'] & Y \end{tikzcd}$$

for any choice of cylinder $QX \otimes I$ for QX, so f_1 and f_1' are left homotopic.

The maps $j_Y f_1$ and $j_Y f_1'$ are left homotopic, hence right homotopic because QX is cofibrant and RQY is fibrant. Thus, there is a right homotopy

$$\begin{tikzcd} & RQY^I \arrow[d, "p"] \\ QX \arrow[ur, "h"] \arrow[r, "(j_Y f_1, j_Y f_1')"'] & RQY \times RQY \end{tikzcd}$$

for some (actually any) path object RQY^I. Form the diagram

$$\begin{tikzcd} QX \arrow[r, "h"] \arrow[d, "j_X"'] & RQY^I \arrow[d, "p"] \\ RQX \arrow[ur, dotted, "H"] \arrow[r, "(f_2,f_2')"'] & RQY \times RQY \end{tikzcd}$$

Then f_2 and f_2' are homotopic. $\square$

$\pi(\mathscr{M})_{cf}$ is the category whose objects are the cofibrant-fibrant objects of $\mathscr{M}$, and whose morphisms are homotopy classes of maps.

Lemma 8.1 implies that there is a well-defined functor

$$\mathscr{M} \to \pi(\mathscr{M})_{cf}$$

defined by $X \mapsto RQX$ and $f \mapsto [RQ(f)]$, where

$$RQ(f) = f_2.$$

The *homotopy category* $\mathrm{Ho}(\mathscr{M})$ of $\mathscr{M}$ has the same objects as $\mathscr{M}$, and has

$$\hom_{\mathrm{Ho}(\mathscr{M})}(X,Y) = \hom_{\pi(\mathscr{M})_{cf}}(RQX, RQY).$$

There is a functor

$$\gamma : \mathscr{M} \to \mathrm{Ho}(\mathscr{M})$$

that is the identity on objects, and sends $f : X \to Y$ to the homotopy class $[RQ(f)]$.

γ takes weak equivalences to isomorphisms in $\mathrm{Ho}(\mathscr{M})$, by the Whitehead Theorem (Theorem 7.8).

Lemma 8.2. *Suppose $f : RQX \to RQY$ represents a morphism $[f] : X \to Y$ of* $\mathrm{Ho}(\mathscr{M})$*. Then there is a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "{[f]}"'] & QX \arrow[l, "\gamma(p_X)"'] \arrow[r, "\gamma(j_X)"] \arrow[d, "{[f]}"] & RQX \arrow[d, "\gamma(f)"] \\ Y & QY \arrow[l, "\gamma(p_Y)"] \arrow[r, "\gamma(j_Y)"'] & RQY \end{tikzcd}$$

in $\mathrm{Ho}(\mathscr{M})$.

Proof. The maps $\gamma(p_X)$ and $\gamma(j_X)$ are isomorphisms defined by the class $[1_{RQX}]$ in $\pi(\mathscr{M})_{cf}$. $\square$

Theorem 8.3. *Suppose $\mathscr{M}$ is a closed model category, and $F : \mathscr{M} \to D$ takes weak equivalences to isomorphisms.*

There is a unique functor $F_* : \mathrm{Ho}(\mathscr{M}) \to D$ such that the diagram of functors

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\gamma"] \arrow[dr, "F"'] & \mathrm{Ho}(\mathscr{M}) \arrow[d, "F_*"] \\ & D \end{tikzcd}$$

commutes.

Proof. This result is a corollary of Lemma 8.2. $\square$

Remarks: 1) $\mathrm{Ho}(\mathscr{M})$ is a model for the category $\mathscr{M}[WE]^{-1}$ obtained from $\mathscr{M}$ by formally inverting all weak equivalences.

2) $\gamma : \mathscr{M} \to \mathrm{Ho}(\mathscr{M})$ induces a fully faithful functor $\gamma_* : \pi(\mathscr{M}_{cf}) \to \mathrm{Ho}(\mathscr{M})$. Every object of $\mathrm{Ho}(\mathscr{M})$ is isomorphic to a (cofibrant fibrant) object in the image of γ_*.

It follows that the functor γ_* is an equivalence of categories.

This last observation specializes to well known phenomena:

- The homotopy category of **CGWH** is equivalent to the category of CW-complexes and ordinary homotopy classes of maps between them.
- The derived category of $Ch_+(R)$ is equivalent to the category of chain complexes of projectives and chain homotopy classes of maps between them.

One final thing: the functor $\gamma : \mathscr{M} \to \mathrm{Ho}(\mathscr{M})$ reflects weak equivalences:

Proposition 8.4. *Suppose that $\mathscr{M}$ is a closed model category, and that $f : X \to Y$ is a morphism such that $\gamma(f)$ is an isomorphism in* $\mathrm{Ho}(\mathscr{M})$. *Then f is a weak equivalence of $\mathscr{M}$.*

For the proof, it is enough to suppose that both X and Y are fibrant and cofibrant and that f is a fibration with a homotopy inverse $g : Y \to X$. Then the idea is to show that f is a weak equivalence.

This claim is a triviality in almost all cases of interest, but it is a bit tricky to prove in full generality. This result appears as Proposition II.1.14 in [1].

References

[1] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition.