Lecture 04: Simplicial sets

Contents

9 Simplicial sets

A **simplicial set** is a functor

$$X : \Delta^{op} \to \mathbf{Set},$$

ie. a contravariant set-valued functor defined on the ordinal number category Δ.

One usually writes $\mathbf{n} \mapsto X_n$.

X_n is the set of n-**simplices** of X.

A **simplicial map** $f : X \to Y$ is a natural transformation of such functors.

The simplicial sets and simplicial maps form the category of simplicial sets, denoted by $s\mathbf{Set}$ — one also sees the notation $\mathbf{S}$ for this category.

If $\mathscr{A}$ is some category, then a **simplicial object** in $\mathscr{A}$ is a functor

$$A : \Delta^{op} \to \mathscr{A}.$$

Maps between simplicial objects are natural transformations.

The simplicial objects in $\mathscr{A}$ and their morphisms form a category $s\mathscr{A}$.

Examples: 1) $s\mathbf{Gr}$ = simplicial groups.

2) $s\mathbf{Ab}$ = simplicial abelian groups.

3) $s(R-\mathbf{Mod})$ = simplicial R-modules.

4) $s(s\mathbf{Set}) = s^2\mathbf{Set}$ is the category of **bisimplicial sets**.

Simplicial objects are everywhere.

Examples of simplicial sets:

1) We've already met the *singular set* $S(X)$ for a topological space X, in Section 4.

$S(X)$ is defined by the *cosimplicial space* (covariant functor) $\mathbf{n} \mapsto |\Delta^n|$, by

$$S(X)_n = \hom(|\Delta^n|, X).$$

$\theta : \mathbf{m} \to \mathbf{n}$ defines a function

$$S(X)_n = \hom(|\Delta^n|, X) \xrightarrow{\theta^*} \hom(|\Delta^m|, X) = S(X)_m$$

by precomposition with the map $\theta : |\Delta^m| \to |\Delta^m|$.

The assignment $X \mapsto S(X)$ defines the **singular functor**

$$S : \mathbf{CGWH} \to s\mathbf{Set}.$$

2) The ordinal number $\mathbf{n}$ represents a contravariant functor

$$\Delta^n = \hom_\Delta(\ ,\mathbf{n}) : \Delta^{op} \to \mathbf{Set},$$

called the **standard n-simplex**.

$$\iota_n := 1_{\mathbf{n}} \in \hom_\Delta(\mathbf{n},\mathbf{n}).$$

The n-simplex ι_n is the **classifying n-simplex**.

The Yoneda Lemma implies that there is a natural bijection

$$\hom_{s\mathbf{Set}}(\Delta^n, Y) \cong Y_n$$

defined by sending the map $\sigma : \Delta^n \to Y$ to the element $\sigma(\iota_n) \in Y_n$.

A map $\Delta^n \to Y$ *is* an *n-simplex of Y*.

Every ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ induces a simplicial set map

$$\theta : \Delta^m \to \Delta^n,$$

defined by composition.

We have a covariant functor

$$\Delta : \Delta \to s\mathbf{Set}$$

with $\mathbf{n} \mapsto \Delta^n$. This is a *cosimplicial object* in $s\mathbf{Set}$.

If $\sigma : \Delta^n \to X$ is a simplex of X, the i^{th} **face** $d_i(\sigma)$ is the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X,$$

The j^{th} **degeneracy** $s_j(\sigma)$ is the composite

$$\Delta^{n+1} \xrightarrow{s^j} \Delta^n \xrightarrow{\sigma} X.$$

3) $\partial\Delta^n$ is the subobject of Δ^n which is generated by the $(n-1)$-simplices d^i, $0 \leq i \leq n$.

Λ^n_k isthe subobject of $\partial\Delta^n$ which is generated by the simplices d^i, $i \neq k$.

$\partial\Delta^n$ is the **boundary** of Δ^n, and Λ^n_k is the k^{th} **horn**.

The faces $d^i : \Delta^{n-1} \to \Delta^n$ determine a covering

$$\bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n,$$

and for each $i < j$ there are pullback diagrams

$$\begin{tikzcd}
\Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \Delta^{n-1} \arrow[d, "d^i"] \\
\Delta^{n-1} \arrow[r, "d^j"'] & \Delta^n
\end{tikzcd}$$

(Excercise!). It follows that there is a coequalizer

$$\bigsqcup_{i<j, 0\leq i,j\leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{0\leq i\leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

in $s\mathbf{Set}$.

Similarly, there is a coequalizer

$$\bigsqcup_{i<j,i,j\neq k}\Delta^{n-2} \rightrightarrows \bigsqcup_{0\leq i\leq n,i\neq k}\Delta^{n-1} \longrightarrow \Lambda^n_k.$$

4) Suppose the category C is **small**, ie. the morphisms $\mathrm{Mor}(C)$ (and objects $\mathrm{Ob}(C)$) form a set.

Examples include all finite ordinal numbers $\mathbf{n}$ (because they are posets), all monoids (small categories having one object), and all groups.

There is a simplicial set BC with n-simplices

$$BC_n = \hom(\mathbf{n}, C),$$

ie. the functors $\mathbf{n} \to C$.

The simplicial structure on BC is defined by precomposition with ordinal number maps: if $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map (aka. functor) and $\sigma : \mathbf{n} \to C$ is an n-simplex, then $\theta^*(\sigma)$ is the composite functor

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\sigma} C.$$

The object BC is called the **classifying space** or **nerve** of C (the notation NC is also common).

If G is a (discrete) group, BG "is" the standard classifying space for G in **CGWH**, which classifies principal G-bundles.

NB: $B\mathbf{n} = \Delta^n$.

5) Suppose I is a small category, and $X : I \to \mathbf{Set}$ is a set-valued functor (aka. a diagram in sets).

The **translation category** ("category of elements") $E_I(X)$ has objects given by all pairs (i,x) with $x \in X(i)$.

A morphism $\alpha : (i,x) \to (j,y)$ is a morphism $\alpha : i \to j$ of I such that $\alpha_*(x) = y$.

The simplicial set $B(E_I X)$ is the **homotopy colimit** for the functor X. One often writes

$$\underrightarrow{\mathrm{holim}}_I\, X = B(E_I X).$$

Here's a different description of the nerve BI:

$$BI = \underrightarrow{\mathrm{holim}}_I\, *.$$

BI is the homotopy colimit of the (constant) functor $I \to \mathbf{Set}$ which associates the one-point set $*$ to every object of I.

There is a functor

$$E_I X \to I,$$

defined by the assignment $(i,x) \mapsto i$.

This functor induces a simplicial set map

$$\pi : B(E_I X) = \underrightarrow{\mathrm{holim}}_I\, X \to BI.$$

A functor $\mathbf{n} \to C$ is specified by a string of arrows

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

in C, for then all composites of these arrows are uniquely determined.

The functors $\mathbf{n} \to E_I X$ can be identified with strings

$$(i_0, x_0) \xrightarrow{\alpha_1} (i_1, x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n, x_n).$$

Such a string is specified by the underlying string $i_0 \to \dots \to i_n$ in the index category Y and $x_0 \in X(i_0)$.

It follows that there is an identification

$$(\underrightarrow{\mathrm{holim}}_I X)_n = B(E_I X)_n = \bigsqcup_{i_0 \to \dots \to i_n} X(i_0).$$

The construction is functorial with respect to natural transformations in diagrams X.

A diagram $X : I \to s\mathbf{Set}$ in simplicial sets (a simplicial object in set-valued functors) determines a simplicial category $m \mapsto E_I(X_m)$ and a corresponding bisimplicial set with (n,m) simplices

$$B(E_I X)_m = \bigsqcup_{i_0 \to \dots \to i_n} X(i_0)_m.$$

The **diagonal** $d(Y)$ of a bisimplicial set Y is the simplicial set with n-simplices $Y_{n,n}$. Equivalently,

$d(Y)$ is the composite functor

$$\Delta^{op} \xrightarrow{\Delta} \Delta^{op} \times \Delta^{op} \xrightarrow{Y} \mathbf{Set}$$

where Δ is the diagonal functor.

The diagonal $dB(E_I X)$ of the bisimplicial set $B(E_I X)$ is the **homotopy colimit** $\underrightarrow{\mathrm{holim}}_I\, X$ of the functor $X : I \to s\mathbf{Set}$.

There is a natural simplicial set map

$$\pi : \underrightarrow{\mathrm{holim}}_I\, X \to BI.$$

6) Suppose X and Y are simplicial sets. The **function complex**

$$\mathbf{hom}(X,Y)$$

has n-simplices

$$\mathbf{hom}(X,Y)_n = \hom(X \times \Delta^n, Y).$$

If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map and $f : X \times \Delta^n \to Y$ is an n-simplex of $\mathbf{hom}(X,Y)$, then $\theta^*(f)$ is the composite

$$X \times \Delta^m \xrightarrow{1 \times \theta} X \times \Delta^m \xrightarrow{f} Y.$$

There is a natural simplicial set map

$$ev : X \times \mathbf{hom}(X,Y) \to Y$$

defined by

$$(x, f : X \times \Delta^n \to Y) \mapsto f(x, \iota_n).$$

Suppose K is a simplicial set.

The function

$$ev_* : \hom(K, \mathbf{hom}(X,Y)) \to \hom(X \times K, Y),$$

is defined by sending $g : K \to \mathbf{hom}(X,Y)$ to the composite

$$X \times K \xrightarrow{1 \times g} X \times \mathbf{hom}(X,Y) \xrightarrow{ev} Y.$$

The function ev_* is a *bijection*, with inverse that takes $f : X \times K \to Y$ to the morphism $f_* : K \to \mathbf{hom}(X,Y)$, where $f_*(y)$ is the composite

$$X \times \Delta^n \xrightarrow{1 \times y} X \times K \xrightarrow{f} Y.$$

The natural bijection

$$\hom(X \times K, Y) \cong \hom(K, \mathbf{hom}(X,Y))$$

is called the **exponential law**.

*s***Set** is a *cartesian closed category*.

The function complexes also give *s***Set** the structure of a *category enriched in simplicial sets*.

10 The simplex category and realization

Suppose X is a simplicial set.

The **simplex category** Δ/X has for objects all simplices $\Delta^n \to X$.

Its morphisms are the *incidence relations* between the simplices, meaning all commutative diagrams

$$\begin{tikzcd} \Delta^m \arrow[d, "\theta"'] \arrow[dr, "\tau"] & \\ \Delta^n \arrow[r, "\sigma"'] & X \end{tikzcd} \tag{1}$$

Δ/X is a type of *slice category*. It is denoted by $\Delta \downarrow X$ in [2]. See also [6].

In the broader context of homotopy theories associated to a test category (long story — see [4]) one says that the simplex category is a *cell category*.

Exercise: Show that a simplicial set X is a colimit of its simplices, ie. the simplices $\Delta^n \to X$ define a simplicial set map

$$\varinjlim_{\Delta^n \to X} \Delta^n \to X,$$

which is an isomorphism.

There is a space $|X|$, called the **realization** of the simplicial set X, which is defined by

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

Here $|\Delta^n|$ is the topological standard n-simplex, as described in Section 4.

$|X|$ is the colimit of the functor $\Delta/X \to \mathbf{CGWH}$ which takes the morphism (1) to the map

$$|\Delta^m| \xrightarrow{\theta} |\Delta^n|.$$

The assignment $X \mapsto |X|$ defines a functor

$$|\ | : s\mathbf{Set} \to \mathbf{CGWH},$$

called the **realization functor**.

Lemma 10.1. *The realization functor is left adjoint to the singular functor $S : \mathbf{CGWH} \to s\mathbf{Set}$.*

Proof. A simplicial set X is a colimit of its simplices. Thus, for a simplicial set X and a space Y, there are natural isomorphisms

$$\begin{aligned}
\hom(X, S(Y)) &\cong \hom(\varinjlim_{\Delta^n \to X} \Delta^n, S(Y)) \\
&\cong \varprojlim_{\Delta^n \to X} \hom(\Delta^n, S(Y)) \\
&\cong \varprojlim_{\Delta^n \to X} \hom(|\Delta^n|, Y) \\
&\cong \hom(\varinjlim_{\Delta^n \to X} |\Delta^n|, Y) \\
&= \hom(|X|, Y).
\end{aligned}$$

$\square$

Remark: Kan introduced the concept of adjoint functors to describe the relation between the realization and singular functors.

Examples:

1) $|\Delta^n| = |\Delta^n|$, since the simplex category Δ/Δ^n has a terminal object, namely $1 : \Delta^n \to \Delta^n$.

2) $|\partial\Delta^n| = |\partial\Delta^n|$ and $|\Lambda^n_k| = |\Lambda^n_k|$, since the realization functor is a left adjoint and therefore preserves coequalizers and coproducts.

The n^{th} **skeleton** $\mathrm{sk}_n X$ of a simplicial set X is the subobject generated by the simplices X_i, $0 \leq i \leq n$. The ascending sequence of subcomplexes

$$\mathrm{sk}_0 X \subset \mathrm{sk}_1 X \subset \mathrm{sk}_2 X \subset \dots$$

defines a filtration of X, and there are pushout diagrams

$$\begin{tikzcd} \bigsqcup_{x \in NX_n} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\ \bigsqcup_{x \in NX_n} \Delta^n \arrow[r] & \mathrm{sk}_n X \end{tikzcd} \tag{2}$$

NX_n is the set of non-degenerate n-simplices of X.

$\sigma \in X_n$ is **non-degenerate** if it is not of the form $s_j(y)$ for some $(n-1)$-simplex y and some j.

Exercise: Show that the diagram (2) is indeed a pushout.

For this, it's helpful to know that the functor $X \mapsto \mathrm{sk}_n X$ is left adjoint to truncation up to level n.

For *that*, you should know that every simplex x of a simplicial set X has a unique representation $x = s^*(y)$ where $s : \mathbf{n} \twoheadrightarrow \mathbf{k}$ is an ordinal number epi and $y \in X_k$ is non-degenerate.

Corollary 10.2. *The realization $|X|$ of a simplicial set X is a CW-complex.*

Every monomorphism $A \to B$ of simplicial sets induces a cofibration $|A| \to |B|$ of spaces. ie. $|B|$ is constructed from $|A|$ by attaching cells.

Lemma 10.3. *The realization functor preserves finite limits.*

Proof. There are isomorphisms

$$\begin{aligned}
|X \times Y| &\cong |\varinjlim_{\Delta^n \to X, \Delta^m \to Y} \Delta^n \times \Delta^m| \\
&\cong \varinjlim_{\Delta^n \to X, \Delta^m \to Y} |\Delta^n \times \Delta^m| \\
&\cong \varinjlim_{\Delta^n \to X, \Delta^m \to Y} |\Delta^n| \times |\Delta^m| \\
&\cong |X| \times |Y|
\end{aligned}$$

One shows that the canonical maps

$$|\Delta^n \times \Delta^m| \to |\Delta^n| \times |\Delta^m|$$

are isomorphisms with an argument involving shuffles — see [1, p.52].

If $\sigma, \tau : \Delta^n \to Y$ are simplices such that

$$|\sigma| = |\tau| : |\Delta^n| \to |Y|,$$

then $\sigma = \tau$ (exercise).

Suppose $f, g : X \to Y$ are simplicial set maps, and $x \in |X|$ is an element such that $f_*(x) = g_*(x)$.

If σ is the "carrier" of x (ie. non-degenerate simplex of X such that x is interior to the cell defined by σ), then $f_*(y) = g_*(y)$ for all y in the interior of

$|\sigma|$ (by transforming by a suitable automorphism of the cosimplicial space $|\Delta|$ — see [1, p.51]).

But then

$$|f\sigma| = |g\sigma| : |\Delta^n| \to |Y|,$$

so $f\sigma = g\sigma$ and $x \in |E|$, where E is the equalizer of f and g in $s\mathbf{Set}$. $\square$

11 Model structure for simplicial sets

A map $f : X \to Y$ of simplicial sets is a **weak equivalence** if $f_* : |X| \to |Y|$ is a weak equivalence of **CGWH**.

A map $i : A \to B$ of simplicial sets is a **cofibration** if and only if it is a monomorphism, ie. all functions $i : A_n \to B_n$ are injective.

A simplicial set map $p : X \to Y$ is a **fibration** if it has the RLP wrt all trivial cofibrations.

Remark: There is a natural commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "\nabla"] \arrow[d, "{(i_0,i_1)}"'] & X \\ X \times \Delta^1 \arrow[ur, "pr"'] & \end{tikzcd} \quad (3)$$

for simplicial sets X. (i_0, i_1) is the cofibration

$$1_X \times i : X \times \partial\Delta^1 \to X \times \Delta^1$$

induced by the inclusion $i : \partial\Delta^1 \subset \Delta^1$. The two inclusions i_ε of the end points of the cylinder are weak equivalences, as is $pr : X \times \Delta^1 \to X$.

The diagram (3) is a natural cylinder object for the model structure on simplicial sets (see Theorem 11.6). Left homotopy with respect to this cylinder is classical **simplicial homotopy**.

Lemma 11.1. *A map $p : X \to Y$ is a trivial fibration if and only if it has the RLP wrt all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

Proof. 1) Suppose p has the lifting property.

Then p has the RLP wrt all cofibrations (exercise: induct through relative skeleta), so the lifting s exists in the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "1_Y"'] \arrow[ur, "s"] & Y \end{tikzcd}$$

since all simplicial sets are cofibrant.

The lifting h exists in the diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "{(sp,1)}"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ X \times \Delta^1 \arrow[r, "p \cdot pr"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

so the map $p_* : |X| \to |Y|$ is a homotopy equivalence, hence a weak equivalence.

2) Suppose p is a trivial fibration and choose a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "p"'] & U \arrow[d, "q"] \\ & Y \end{tikzcd}$$

such that j is a cofibration and q has the RLP wrt all maps $\partial\Delta^n \subset \Delta^n$ (such things exist by a small object argument).

q is a weak equivalence by part 1), so j is a trivial cofibration and the lift r exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ U \arrow[r, "q"'] \arrow[ur, "r"] & Y \end{tikzcd}$$

Then p is a retract of q, and has the RLP. $\square$

Say that a simplicial set A is **countable** if it has countably many non-degenerate simplices.

A simplicial set K is **finite** if it has only finitely many non-degenerate simplices, eg. Δ^n, $\partial\Delta^n$, Λ^n_k.

Fact: If X is countable (resp. finite), then all subcomplexes of X are countable (resp. finite).

The following result is proved with simplicial approximation techniques:

Lemma 11.2. *Suppose that X has countably many non-degenerate simplices.*

Then $\pi_0|X|$ and all homotopy groups $\pi_n(|X|,x)$ are countable.

Proof. Suppose x is a vertex of X, identified with $x \in |X|$.

A continuous map

$$(|\Delta^k|, |\partial\Delta^k|) \to (|X|, x)$$

is homotopic, rel boundary, to the realization of a simplicial set map

$$(\mathrm{sd}^N \Delta^k, \mathrm{sd}^N \partial\Delta^k) \to (X, x),$$

by simplicial approximation [3].

The (iterated) subdivisions $\mathrm{sd}^M \Delta^k$ are finite complexes, and there are only countably many maps $\mathrm{sd}^M \Delta^k \to X$ for $M \geq 0$. $\square$

Here's a consequence:

Lemma 11.3 (Bounded cofibration lemma)**.** *Suppose given cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow i \\ A & \longrightarrow & Y \end{array}$$

where i is trivial and A is countable.

Then there is a countable $B \subset Y$ with $A \subset B$, such that the map $B \cap X \to B$ is a trivial cofibration.

Proof. Write $B_0 = A$ and consider the map

$$B_0 \cap X \to B_0.$$

The homotopy groups of $|B_0|$ and $|B_0 \cap X|$ are countable, by Lemma 11.2.

Y is a union of its countable subcomplexes.

Suppose that

$$\alpha, \beta : (|\Delta^n|, |\partial \Delta^n|) \to (|B_0 \cap X|, x)$$

become homotopic in $|B_0|$ hence in $|X|$.

The map defining the homotopy in $|X|$ is compact (ie. defined on a CW-complex with finitely many cells), so there is a countable $B' \subset Y$ with $B_0 \subset B'$ such that the homotopy lives in $|B' \cap X|$.

The image in $|Y|$ of any morphism

$$\gamma : (|\Delta^n|, |\partial\Delta^n|) \to (|B_0|, x)$$

lifts to $|X|$ up to homotopy, and that homotopy lives in $|B''|$ for some countable subcomplex $B'' \subset Y$ with $B_0 \subset B''$.

It follows that there is a countable subcomplex $B_1 \subset Y$ with $B_0 \subset B_1$ such that any two elements

$$[\alpha], [\beta] \in \pi_n(|B_0 \cap X|, x)$$

which map to the same element in $\pi_n(|B_0|, x)$ must also map to the same element of $\pi_n(|B_1 \cap X|, x)$, and every element

$$[\gamma] \in \pi_n(|B_0|, x)$$

lifts to an element of $\pi_n(|B_1 \cap X|, x)$, and this for all $n \geq 0$ and all (countably many) vertices x.

Repeat the construction inductively, to form a countable collection

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of subcomplexes of Y.

Then $B = \bigcup B_i$ is a countable subcomplex of Y, and the map $B \cap X \to B$ is a weak equivalence. $\square$

Say that a cofibration $A \to B$ is **countable** if B is countable.

Lemma 11.4. *Every simplicial set map $f : X \to Y$ has a factorization*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

such that q has the RLP wrt all countable trivial cofibrations, and i is constructed from countable trivial cofibrations by pushout and composition.

The proof of Lemma 11.4 is an example of a *transfinite small object argument*.

Lang's *Algebra* [5] has a quick introduction to cardinal arithmetic.

Proof. Choose an uncountable cardinal number κ, interpreted as the (totally ordered) poset of ordinal numbers $s < \kappa$.

Construct a system of factorizations

$$\begin{tikzcd} X \arrow[r, "i_s"] \arrow[dr, "f"'] & Z_s \arrow[d, "q_s"] \\ & Y \end{tikzcd} \tag{4}$$

of f with j_s a trivial cofibration as follows:

- given factorization of the form (4) consider all diagrams

$$D: \quad \begin{array}{ccc} A_D & \longrightarrow & Z_s \\ {\scriptstyle i_D}\downarrow & & \downarrow{\scriptstyle q_s} \\ B_D & \longrightarrow & Y \end{array}$$

such that i_D is a countable trivial cofibration, and form the pushout

$$\begin{array}{ccc} \bigsqcup_D A_D & \longrightarrow & Z_s \\ \downarrow & & \downarrow{\scriptstyle j_s} \\ \bigsqcup_D B_D & \longrightarrow & Z_{s+1} \end{array}$$

Then the map j_s is a trivial cofibration, and the diagrams together induce a map $q_{s+1} : Z_{s+1} \to Y$. Let $i_{s+1} = j_s i_s$.

- if $\gamma < \kappa$ is a limit ordinal, let $Z_\gamma = \varinjlim_{t<\gamma} Z_t$.

Now let $Z = \varinjlim_{s<\kappa} Z_s$ with induced factorization

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

Suppose given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & Z \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

with $j : A \to B$ a countable trivial cofibration. Then $\alpha(A)$ is a countable subcomplex of X, so $\alpha(A) \subset$

Z_s for some $s < \kappa$, for otherwise $\alpha(A)$ has too many elements.

The lifting problem is solved in Z_{s+1}. $\square$

Remark: The map $j : X \to Z$ is in the saturation of the set of countable trivial cofibrations.

The **saturation** of a set of cofibrations I is the smallest class of cofibrations containing I which is closed under pushout, coproducts, (long) compositions and retraction.

If a map p has the RLP wrt all maps of I then it has the RLP wrt all maps in the saturation of I. (exercise)

Classes of cofibrations which are defined by a left lifting property with respect to some family of maps are saturated in this sense. (exercise)

Lemma 11.5. *A map $q : X \to Y$ is a fibration if and only if it has the RLP wrt (the set of) all countable trivial cofibrations.*

We use a recurring trick for the proof of this result. It amounts to verifying a "solution set condition".

Proof. 1) Suppose given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where j is a cofibration, B is countable and f is a weak equivalence.

Lemma 11.1 says that f has a factorization $f = q \cdot i$, where i is a trivial cofibration and q has the RLP wrt all cofibrations.

The lift exists in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "j"'] & X \arrow[d, "i"] \\ & Z \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

$\theta(B)$ is countable, so there is a countable subcomplex $D \subset Z$ with $\theta(B) \subset D$ such that the map $D \cap X \to D$ is a trivial cofibration.

We have a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & D \cap X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

of the original diagram through a countable trivial cofibration.

2) Suppose that $i : C \to D$ is a trivial cofibration.

Then i has a factorization

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[dr, "i"'] & E \arrow[d, "p"] \\ & D \end{tikzcd}$$

such that p has the RLP wrt all countable trivial cofibrations, and j is built from countable trivial cofibrations by pushout and composition. Then j is a weak equivalence, so p is a weak equivalence.

Part 1) implies that p has the RLP wrt all countable cofibrations, and hence wrt all cofibrations.

The lift therefore exists in the diagram

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ D \arrow[r, "1_D"'] \arrow[ur, dotted, "\theta"] & D \end{tikzcd}$$

so i is a retract of j.

Thus, if $q : Z \to W$ has the RLP wrt all countable trivial cofibrations, then it has the RLP wrt all trivial cofibrations. $\square$

Exercise: Find a different, simpler proof for Lemma 11.5. Hint: use Zorn's lemma.

Theorem 11.6. *With the definitions of weak equivalence, cofibration and fibration given above the category s***Set** *of simplicial sets satisfies the axioms for a closed model category.*

Proof. The axioms **CM1**, **CM2** and **CM3** are easy to verify.

Every map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

such that j is a cofibration and q is a trivial fibration — this follows from Lemma 11.1 and a standard small object argument. The other half of the factorization axiom **CM5** is a consequence of Lemma 11.4 and Lemma 11.5.

CM4 also follows from Lemma 11.1. $\square$

Remark: In the adjoint pair of functors

$$|\ | : s\mathbf{Set} \leftrightarrows \mathbf{CGWH} : S$$

the realization functor (the left adjoint part) preserves cofibrations and trivial cofibrations. It's an immediate consequence that the singular functor S preserves fibrations and trivial fibrations.

Adjunctions like this between closed model category are called **Quillen adjunctions** or **Quillen pairs**. We'll see later on, and this is a huge result, that these functors form a Quillen equivalence.

Remark: We defined the weak equivalences of simplicial sets to be those maps whose realizations are weak equivalences of spaces. In this way, the model structure for $s\mathbf{Set}$, as it is described here, is *induced* from the model structure for **CGWH** via the realization functor $|\ |$.

Alternatively, one says that the model structure on simplicial sets is obtained from that on spaces by *transfer*.

References

[1] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[2] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[3] J. F. Jardine. Simplicial approximation. *Theory Appl. Categ.*, 12:No. 2, 34–72 (electronic), 2004.

[4] J. F. Jardine. Categorical homotopy theory. *Homology, Homotopy Appl.*, 8(1):71–144 (electronic), 2006.

[5] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[6] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.