Lecture 05: Fibrations, geometric realization

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12 Kan fibrations

A map $p : X \to Y$ is a **Kan fibration** if it has the RLP wrt all inclusions $\Lambda^n_k \subset \Delta^n$.

Example: A fibration of simplicial sets, (Section 11), is a Kan fibration, since $|\Lambda^n_k| \to |\Delta^n|$ is a weak equivalence.

The converse statement is also true: every Kan fibration is a fibration. This is Theorem 13.5 below.

Say that X is a **Kan complex** if the map $X \to *$ is a Kan fibration.

Exercise: Suppose C is a small category. Show that the nerve BC is a Kan complex if and only if C is a groupoid.

Example: The ordinal number posets $\mathbf{n}$ are not groupoids if $n \geq 1$, so the simplices $\Delta^n = B\mathbf{n}$ *are not* Kan complexes.

The saturation of the set of cofibrations $\Lambda^n_k \subset \Delta^n$ is normally called the class of **anodyne extensions**.

This is the class of cofibrations which has the LLP wrt all Kan fibrations.

Lemma 12.1. *The following sets of cofibrations have the same saturations:*

- $\mathbf{A}_1 =$ *all maps* $\Lambda^n_k \subset \Delta^n$,
- $\mathbf{A}_2 =$ *all inclusions*

$$(\Delta^1 \times \partial\Delta^n) \cup (\{\varepsilon\} \times \Delta^n) \subset \Delta^1 \times \Delta^n,\ \varepsilon = 0, 1.$$

Proof. 1) The saturation of $\mathbf{A}_2$ includes all maps

$$(\Delta^1 \times K) \cup (\{\varepsilon\} \times L) \subset \Delta^1 \times L,\ \varepsilon = 0, 1.$$

induced by inclusions $K \subset L$, since L is built from K by attaching cells.

The functor $r_k : \mathbf{n} \times \mathbf{1} \to \mathbf{n}$ specified by the picture

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \dots \arrow[r] & k \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & \dots \arrow[r] & k \arrow[d] \\ 0 \arrow[r] & 1 \arrow[r] & \dots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \dots \arrow[r] & n \end{tikzcd}$$

and the functor $i : \mathbf{n} \to \mathbf{n} \times \mathbf{1}$ defined by $i(j) = (j, 1)$ together determine a retraction diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & (\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d] \\ \Delta^n \arrow[r] & \Delta^n \times \Delta^1 \arrow[r] & \Delta^n \end{tikzcd}$$

(NB: $\Delta^n \times \{0\}$ is mapped into Λ^n_k) so $\Lambda^n_k \subset \Delta^n$ is in the saturation of the family $\mathbf{A}_2$ if $k < n$.

The map $\Lambda^n_k \subset \Delta^n$ is a retraction of

$$(\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{1\}) \subset \Delta^n \times \Delta^1$$

if $k > 0$. Thus, the saturation of $\mathbf{A}_1$ is contained in the saturation of $\mathbf{A}_2$.

2) The non-degenerate $(n+1)$-simplices of $h_i : \Delta^n \times \Delta^1$ are functors $\mathbf{n}+\mathbf{1} \to \mathbf{n} \times \mathbf{1}$ defined by the pictures

$$\begin{tikzcd}
(0,0) \arrow[r] & (1,0) \arrow[r] & \dots \arrow[r] & (i,0) \arrow[d] & & \\
& & & (i,1) \arrow[r] & \dots \arrow[r] & (i,n)
\end{tikzcd}$$

Let $(\Delta^n \times \Delta^1)^{(i)}$ be the subcomplex of $\Delta^n \times \Delta^1)$ generated by $\partial\Delta^n \times \Delta^1$ and the simplices $h_0, \dots, h_i$. Let

$$(\Delta^n \times \Delta^1)^{(-1)} = (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}).$$

Then $(\Delta^n \times \Delta^1)^{(n)} = \Delta^n \times \Delta^1$, and there are pushouts

$$\begin{tikzcd}
\Lambda^{n+1}_{i+2} \arrow[r] \arrow[d] & (\Delta^n \times \Delta^1)^{(i)} \arrow[d] \\
\Delta^{n+1} \arrow[r] & (\Delta^n \times \Delta^1)^{(i+1)}
\end{tikzcd}$$

It follows that the members of $\mathbf{A}_2$ are in the saturation of the set $\mathbf{A}_1$. $\square$

Lemma 12.2. *Suppose $i : K \to L$ is an anodyne extension and $j : A \to B$ is a cofibration.*

Then the inclusion

$$(K \times B) \cup (L \times A) \subset L \times B$$

is anodyne.

Proof. The class of cofibrations $K' \to L'$ such that

$$(K' \times B) \cup (L' \times A) \subset L' \times B$$

is anodyne is saturated, and includes all cofibrations

$$(\Delta^1 \times \partial\Delta^n) \cup (\{\varepsilon\} \times \Delta^n) \subset \Delta^1 \times \Delta^n, \ \varepsilon = 0, 1,$$

by rebracketing (see [2, I.4.6]). □

Corollary 12.3. *The cofibrations*

$$(\Lambda^n_k \times \Delta^m) \cup (\Delta^n \times \partial\Delta^m) \subset \Delta^n \times \Delta^m$$

are anodyne.

Here's something else that Lemma 12.2 buys you:

Corollary 12.4. *Suppose $p : X \to Y$ is a Kan fibration and $j : A \to B$ is a cofibration.*

Then the map

$$\mathbf{hom}(B,X) \xrightarrow{(j^*,p_*)} \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,Y)$$

is a Kan fibration.

If either p is a trivial fibration or j is anodyne, then the map (j^*,p_*) is a trivial fibration.

Proof. Solutions of the lifting problem

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & \mathbf{hom}(B,X) \arrow[d, "(j^*,p_*)"] \\ L \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,Y) \end{tikzcd}$$

are equivalent to solutions of the lifting problem

$$\begin{tikzcd} (L \times A) \cup (K \times B) \arrow[r] \arrow[d, "(i,j)_*"'] & X \arrow[d, "p"] \\ L \times B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

by the exponential law. The map $(i,j)_*$ is anodyne if either i or j is anodyne, by Lemma 12.2. $\square$

Corollary 12.5. *The function complex $\mathbf{hom}(X,Y)$ is a Kan complex if Y is a Kan complex.*

The proof of Corollary 12.5 is an exercise.

Lemma 12.6. *Simplicial homotopy of maps*

$$X \to Y$$

is an equivalence relation if Y is a Kan complex.

Proof. It's enough to show that simplicial homotopy classes of vertices $\Delta^0 \to Z$ is an equivalence relation if Z is a Kan complex, since $\mathbf{hom}(X,Y)$ is a Kan complex.

The paths

$$x \xrightarrow{\omega_2} y \xrightarrow{\omega_0} z$$

define a map $(\omega_0, \ , \omega_2) : \Lambda^2_1 \to Z$ which extends to a 2-simplex $\sigma : \Delta^2 \to Z$. The 1-simplex $d_1\sigma$ is a path $x \to z$. Thus, the path relation is transitive.

Suppose $\omega_2 : x \to y$ is a path in a Kan complex Z. Let $x : x \to x$ denote the constant path (degenerate 1-simplex) at x. Then there is a diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "{(\ ,x,\omega_2)}"] \arrow[d] & Z \\ \Delta^2 \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

so there is a path $d_0\theta : y \to x$. The path relation is therefore symmetric.

The constant path $\Delta^1 \xrightarrow{s^0} \Delta^0 \xrightarrow{x} X$ is a path from x to x, so the relation is reflexive. $\square$

Path components

Write $\pi_0(Z)$ for the path components, aka. simplicial homotopy classes of vertices $\Delta^0 \to Z$ for a Kan complex of Z.

The argument for Lemma 12.6 implies that there is a coequalizer

$$Z_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} Z_0 \longrightarrow \pi_0(Z)$$

in **Set**.

More generally, the set $\pi_0 X$ of **path components** is defined for an arbitrary simplicial set X by the coequalizer

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0(X)$$

Exercise: Show that there is a natural bijection

$$\pi_0(X) \cong \pi_0(|X|)$$

for simplicial sets X.

Combinatorial homotopy groups

Suppose Y is a Kan complex, and $x \in Y_0$ is a vertex.

The map i^* in the pullback diagram

$$\begin{tikzcd} F_x \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^n, Y) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "x"'] & \mathbf{hom}(\partial\Delta^n, Y) \end{tikzcd}$$

is a Kan fibration by Corollary 12.4.
The vertices of the Kan complex F_x are diagrams

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "x"] \arrow[d, "i"'] & Y \\ \Delta^n \arrow[ur] & \end{tikzcd}$$

or simplices $\alpha : \Delta^n \to Y$ which restrict to the trivial map $\partial\Delta^n \to \Delta^0 \xrightarrow{x} Y$ on the boundary.

The path components $\pi_0(F_x)$ are the simplicial homotopy classes of maps

$$(\Delta^n, \partial\Delta^n) \to (Y, x)$$

rel $\partial\Delta^n$.

$\pi_n^s(Y,x)$ denotes this set of simplicial homotopy classes.

The set $\pi_n^s(Y,x)$ has the structure of a group for $n \geq 1$, and this group is abelian if $n \geq 2$. These are the *simplicial homotopy groups* of a Kan complex.

The multiplication is specified for $[\alpha],[\beta]\in\pi_n^s(Y,x)$ by

$$[\alpha]*[\beta]=[d_n\sigma],$$

where $\sigma:\Delta^{n+1}\to Y$ is a lifting

$$\begin{tikzcd}
\Lambda_n^{n+1} \arrow[r, "{(x,\dots,x,\alpha,?,\beta)}"] \arrow[d] & Y \\
\Delta^{n+1} \arrow[ur, dotted, "\sigma"'] &
\end{tikzcd}$$

Equivalently, $\pi_n^s(Y,x)$ can be identified with homotopy classes of maps

$$((\Delta^1)^{\times n},\partial((\Delta^1)^{\times n}))\to(Y,x)$$

by the same (prismatic) argument as the corresponding result for topological spaces (Section 5).

The group $\pi_2^s(Y,x)$ is the group of automorphisms of the constant loop $x\to x$ in the combinatorial fundamental groupoid $\pi^s(\Omega(Y))$ for the loop object $\Omega(Y)$ at x. The loop "space" $\Omega(Y)$ is defined by the pullback diagram

$$\begin{tikzcd}
\Omega(Y) \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^1,Y) \arrow[d] \\
\Delta^0 \arrow[r, "x"'] & \mathbf{hom}(\partial\Delta^1,Y)
\end{tikzcd}$$

The **combinatorial fundamental groupoid** $\pi^s(Z)$ is defined for a Kan complex Z by analogy with the definition of the fundamental groupoid of a space. Exercise: Construct $\pi^s(Z)$.

This group multiplication is defined by one of the two directions implicit in the maps

$$((\Delta^1)^{\times 2}, \partial(\Delta^1)^{\times 2}) \to (Y,x).$$

The second multiplication coincides with this one (and has the same identity), since the inclusions

$$\Lambda^2_1 \times \Lambda^2_1 \to \Delta^2 \times \Delta^2$$

are anodyne (exercise). It follows that $\pi_2^s(Y,x)$ is an abelian group.

The group laws for all $\pi_n^s(Y,x), n \geq 2$, are constructed similarly, and are abelian. $\pi_n^s(Y,x)$ is an automorphism group of the combinatorial fundamental groupoid $\pi^s\Omega^{n-1}(Y)$ of the iterated loop space $\Omega^{n-1}(Y)$ (at x).

Long exact sequence

Suppose $p : X \to Y$ is a Kan fibration such that Y (hence X) is a Kan complex.

Define the **fibre** F over a vertex $y \in Y$ by the pull-back diagram

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

Suppose x is a vertex of F. There is a boundary homomorphism

$$\partial : \pi_{n+1}^s(Y,y) \to \pi_n^s(F,x)$$

which is defined for $[\alpha] \in \pi_{n+1}^s(Y,y)$ by setting $\partial([\alpha]) = [d_0\theta]$, where θ is a choice of lifting making the diagram

$$\begin{tikzcd} \Lambda_0^{n+1} \arrow[r, "x"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[r, "\alpha"'] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

commute.

The same arguments as for Lemma 5.2 apply, giving

Lemma 12.7. *$p : X \to Y$ is a Kan fibration such that Y is a Kan complex, and F is the fibre over a vertex $y \in Y$.*

1) For each vertex $x \in F$ there is a sequence of pointed sets

$$\dots \pi_n^s(F,x) \xrightarrow{i_*} \pi_n^s(X,x) \xrightarrow{p_*} \pi_n^s(Y,p(x)) \xrightarrow{\partial} \pi_{n-1}^s(F,x) \to \dots$$
$$\dots \pi_1^s(Y,f(x)) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y)$$

which is exact in the sense that $\ker = \mathrm{im}$ *everywhere.*

2) There is a group action

$$* : \pi_1^s(Y,p(x)) \times \pi_0(F) \to \pi_0(F)$$

such that $\partial([\alpha]) = [\alpha] * [x]$, and $i_*[z] = i_*[w]$ iff there is $[\beta] \in \pi_1(Y,p(x))$ st $[\beta] * [z] = [w]$.

Here's a combinatorial analogue of Lemma 5.1:

Lemma 12.8. *$p : X \to Y$ is a Kan fibration and Y is a Kan complex. Suppose p induces a bijection $\pi_0(X) \cong \pi_0(Y)$, and isomorphisms $\pi_n^s(X,x) \cong \pi_n^s(Y,p(x))$ for all $n \geq 1$ and all vertices x of X. Then p is a trivial fibration of s***Set**.

Proof. Show that p has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. The argument is the same as for Lemma 5.1. $\square$

A **combinatorial weak equivalence** is a map $f : X \to Y$ of *Kan complexes* that induces an isomorphism in all possible simplicial homotopy groups, ie. f induces a bijection and isomrphisms

$$\pi_n^s(X,x) \cong \pi_n^s(Y,f(x)),\ x \in X_0, n \geq 1.$$

Equivalently, f induces a bijection

$$\pi_0(X) \cong \pi_0(Y)$$

and all diagrams

$$\begin{tikzcd} \pi_n^s(X) \arrow[r] \arrow[d] & \pi_n^s(Y) \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

are pullbacks of sets. Here,

$$\pi_n^s(X) := \bigsqcup_{x \in X_0} \pi_n^s(X,x).$$

By Lemma 12.8, a map p that is a Kan fibration and a combinatorial weak equivalence between Kan complexes must also be a trivial fibration.

Here's a major theorem, due to Quillen:

Theorem 13.1. *The realization of a Kan fibration is a Serre fibration.*

Proof. This will only be a brief sketch — the details can be found, for example, [2, I.10].

The idea is to use the theory of minimal fibrations to show that every Kan fibration $p : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where g is a trivial fibration (ie. has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$) and q is a minimal Kan fibration.

Garbriel and Zisman show [1], [2] that the realization of a minimial fibration $q : Z \to Y$ is a Serre fibration: the idea is that every pullback $q^{-1}(\sigma)$ of a simplex $\sigma : \Delta^n \to Y$ is isomorphic over Δ^n to a simplicial set $F \times \Delta^n$, where F is a fibre over some vertex Δ^n, and it follows that the realization of q is locally a projection, hence a Serre fibration.

The trivial fibration g sits in a diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "(1_X, g)"'] & X \arrow[d, "g"] \\ X \times Z \arrow[r, "pr"'] \arrow[ur, dotted] & Z \end{tikzcd}$$

and is therefore a retract of a projection. $\square$

A Kan fibration $p : X \to Y$ is said to be **minimal** if, given simplices $\alpha, \beta : \Delta^n \to Y$ (with $\partial(\alpha) = \partial(\beta)$ and $p(\alpha) = p(\beta)$), then the existence of a diagram

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r, "pr"] \arrow[d, "i \times 1"'] & \partial\Delta^n \arrow[d] \\ \Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

(fibrewise homotopy rel boundary) forces $\alpha = \beta$.

Every Kan fibration has a minimal Kan fibration as a strong fibrewise deformation retract, and every fibrewise weak equivalence of minimal fibrations is an isomorphism. See [2, I.10].

The *Milnor Theorem* is a consequence of Quillen's theorem:

Theorem 13.2 (Milnor)**.** *Suppose that Y is a Kan complex and $\eta : Y \to S(|Y|)$ is the adjunction homomorphism.*

Then η is a combinatorial weak equivalence.

We need the path-loop fibre sequence for the proof of Theorem 13.2.

If Y is a Kan complex, then the map $\partial\Delta^1 \subset \Delta^1$ induces a Kan fibration

$$\mathbf{hom}(\Delta^1, Y) \xrightarrow{(p_0,p_1)} Y \times Y \cong \mathbf{hom}(\partial\Delta^1, Y),$$

and the induced maps p_0, p_1 are trivial fibrations, by Corollary 12.4.

Take a vertex $x \in Y$, and form the pullback

$$\begin{tikzcd} P_xY \arrow[r, "i"] \arrow[d, "p_{0*}"'] & \mathbf{hom}(\Delta^1, Y) \arrow[d, "p_0"] \\ \Delta^0 \arrow[r, "x"'] & Y \end{tikzcd}$$

The map p_{0*} is a trivial fibration, so P_xY is contractible.

There is a pullback

$$\begin{tikzcd} P_xY \arrow[r, "i"] \arrow[d, "(p_{0*}, p_1 i)"'] & \mathbf{hom}(\Delta^1, Y) \arrow[d, "(p_0, p_1)"] \\ \Delta^0 \times Y \arrow[r, "(x, 1_Y)"'] & Y \times Y \end{tikzcd}$$

so $\pi = p_1 i : P_xY \to Y$ is a Kan fibration. The **loop space** ΩY is the fibre of π over $x \in Y$.

We have the Kan fibre sequence

$$\Omega Y \to P_xY \xrightarrow{\pi} Y$$

This is the **path-loop fibre sequence** for the Kan complex Y.

P_xY is the **path space** at x.

Proof of Theorem 13.2. The map $\eta : Y \to S(|Y|)$ induces a bijection $\pi_0(Y) \cong \pi_0(S(|Y|))$.

The maps

$$S(|\Omega Y|) \to S(|P_xY|) \to S(|Y|)$$

form a Kan fibre sequence by Theorem 13.1 and the exactness of the realization functor (Lemma 10.1).

The Kan complex $S(|P_xY|)$ is contractible.

There is a commutative diagram of functions

$$\begin{tikzcd}
\pi_1^s(Y,x) \arrow[r, "\eta_*"] \arrow[d, "\partial"', "\cong"] & \pi_1^s(S(|Y|),x) \arrow[d, "\cong"', "\partial"] \\
\pi_0(\Omega Y) \arrow[r, "\cong", "\eta_*"'] & \pi_0(S(|\Omega Y|))
\end{tikzcd}$$

so $\eta_* : \pi_1^s(Y,x) \to \pi_1^s(S(|Y|),x)$ is an isomorphism.

Inductively, all maps $\pi_n^s(Y,x) \to \pi_n^s(S(|Y|),x)$ are isomorphisms, for all vertices x of Y. $\square$

Corollary 13.3. *There are natural isomorphisms*

$$\pi_n^s(Y,x) \cong \pi_n(|Y|,x)$$

at all vertices x for all Kan complexes Y.

Proof. The adjunction isomorphism

$$[(\Delta^n, \partial\Delta^n), (S(X),x)] \cong [(|\Delta^n|, |\partial\Delta^n|), (X,x)]$$

gives an isomorphism

$$\pi_n^s(S(X),x) \cong \pi_n(X,x)$$

for each space X. □

Lemma 13.4. *Suppose $p : X \to Y$ is a Kan fibration and a weak equivalence. Then p is a trivial fibration.*

Proof. The class of maps which are both Kan fibrations and weak equivalences is stable under pullback.

In effect, given a pullback diagram

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "p_*"'] & X \arrow[d, "p"] \\ Z \arrow[r] & Y \end{tikzcd}$$

the realization $|p|$ is a trivial Serre fibration by Theorem 13.1, so $|p_*|$ is also a trivial Serre fibration, since realization preserves pullbacks.

It is enough to show (by a lifting argument) that, if $p : X \to \Delta^n$ is a Kan fibration and a weak equivalence, then p is a trivial fibration.

As in the proof of Theorem 13.1, p has a factorization

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "p"'] & F \times \Delta^n \arrow[d, "pr"] \\ & \Delta^n \end{tikzcd}$$

where g is a trivial fibration and the projection pr is minimal.

pr is a weak equivalence, so all homotopy groups of the space $|F|$ vanish, and Theorem 13.2 (Milnor Theorem) implies that all simplicial homotopy groups of F vanish.

By Lemma 12.8, all lifting problems

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & F \times \Delta^n \arrow[d, "pr"] \\ \Delta^m \arrow[r] \arrow[ur, dotted] & \Delta^n \end{tikzcd}$$

have solutions. $\square$

Theorem 13.5. *Every Kan fibration is a fibration.*

Proof. Suppose $i : A \to B$ is a trivial cofibration. Then i has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & Z \arrow[d, "p"] \\ & B \end{tikzcd}$$

such that j is an anodyne extension and p is a Kan fibration.

Then j is a weak equivalence, so p is a weak equivalence, and is a trivial fibration by Lemma 13.4.

The lifting exists in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ B \arrow[r, "1_B"'] \arrow[ur, dotted] & B \end{tikzcd}$$

so i is a retract of an anodyne extension and is therefore an anodyne extension.

Thus, every Kan fibration has the right lifting property with respect to all trivial cofibrations. $\square$

Remark: The approach to constructing the model structure for simplicial sets that is given here is non-standard.

Normally, as in [2], one decrees at the outset that the Kan fibrations are the fibrations, and the weak equivalences and cofibrations are as defined here.

The model structure is produced much more quickly in these notes (as in [3]), at the expense of knowing that the Kan fibrations are the fibrations until the very end.

Replacing maps by fibrations

Suppose $f : X \to Y$ is a map of Kan complexes.

Form the pullback diagram

$$\begin{tikzcd} X \times_Y \mathbf{hom}(\Delta^1, Y) \arrow[r, "f_*"] \arrow[d, "p_{0*}"'] & \mathbf{hom}(\Delta^1, Y) \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \\ X \arrow[r, "f"'] & Y & \end{tikzcd}$$

where p_0 and p_1 are the trivial fibrations arising from the standard path object

$$\begin{tikzcd} & \mathbf{hom}(\Delta^1, Y) \arrow[d, "(p_0,p_1)"] \\ Y \arrow[ur, "s"] \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

for the Kan complex Y.

Remark: The right homotopy relation associated to this path object is classical simplicial homotopy.

There is a pullback diagram

$$\begin{tikzcd} X \times_Y \mathbf{hom}(\Delta^1, Y) \arrow[r, "f_*"] \arrow[d, "(p_{0*}, p_1 f_*)"'] & \mathbf{hom}(\Delta^1, Y) \arrow[d, "(p_0, p_1)"] \\ X \times Y \arrow[r, "f \times 1_Y"'] & Y \times Y \end{tikzcd}$$

and X is fibrant, so $\pi := p_1 f_*$ is a fibration.

p_{0*} is a trivial fibration. The map sf defines a section s_* of p_{0*}, so s_* is a weak equivalence.

Finally, $\pi s_* = p_1 s f = f$.

Thus, every map $f : X \to Y$ between Kan complexes has a functorial factorization

$$\begin{tikzcd} X \arrow[r, "s_*"] \arrow[dr, "f"'] & X \times_Y \mathbf{hom}(\Delta^1, Y) \arrow[d, "\pi"] \\ & Y \end{tikzcd} \tag{1}$$

such that π is a fibration and s_* is a section of a trivial fibration.

Remark: This construction is an abstraction of the classical replacement of a map by a fibration, and works for the subcategory of fibrant objects in an arbitrary simplicial model category.

The dual of this construction is the mapping cylinder, which replaces a map by a cofibration up to weak equivalence (exercise).

Simplicial sets and spaces

Theorem 13.6. *The adjunction maps $\eta : X \to S(|X|)$ and $\varepsilon : |S(Y)| \to Y$ are weak equivalences, for all simplicial sets X and spaces Y, respectively.*

Proof. Every combinatorial weak equivalence $f : X \to Y$ between Kan complexes is a weak equivalence.

In effect, every map which is a fibration and a combinatorial weak equivalence is a weak equivalence by Lemma 12.8, and then one finishes by replacing the map f with a fibration as above.

The adjunction map $\eta : X \to S(|X|)$ is a weak equivalence if X is fibrant (Theorem 13.2).

Choose a fibrant model for an arbitrary simplicial set X, ie. a weak equivalence $j : X \to Z$ such that Z is fibrant.

Then in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\simeq"'] & S(|X|) \arrow[d, "\simeq"] \\ Z \arrow[r, "\simeq", "\eta"'] & S(|Z|) \end{tikzcd}$$

the indicated maps are weak equivalences, so $\eta : X \to S(|X|)$ is a weak equivalence too.

Suppose Y is a space. In the triangle identity

$$\begin{tikzcd} S(Y) \arrow[r, "\eta"] \arrow[dr, "1"'] & S(|S(Y)|) \arrow[d, "s(\varepsilon)"] \\ & S(Y) \end{tikzcd}$$

$S(\varepsilon)$ is a weak equiv. of Kan complexes, so $\varepsilon : |S(Y)| \to Y$ is a weak equiv. of spaces. $\square$

The realization and singular functor adjunction

$$| \ | : s\mathbf{Set} \leftrightarrows \mathbf{CGWH} : S$$

is a classic example of a Quillen equivalence. In particular we have the following:

Corollary 13.7. *The realization and singular functors induce an adjoint equivalence*

$$| \ | : \mathrm{Ho}(s\mathbf{Set}) \leftrightarrows \mathrm{Ho}(\mathbf{CGWH}) : S.$$

The final result of this section gives the closed "simplicial' model structure for the $s\mathbf{Set}$.

Lemma 13.8. *Suppose $p : X \to Y$ is a fibration and $i : A \to B$ is a cofibration.*

Then the induced map

$$\mathbf{hom}(B,X) \xrightarrow{(i^*,p_*)} \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,X) \tag{2}$$

is a fibration. This map is a trivial fibration if either i or p is a weak equivalence.

Proof. If $j : K \to L$ is a cofibration, then the induced map

$$(B \times K) \cup_{(A \times K)} (A \times L) \to B \times L \tag{3}$$

is a cofibration, which is a weak equivalence if either i or j is a weak equivalence (exercise).

Use an adjunction argument to show that the map (2) has the RLP wrt $j : K \to L$ if and only if the map $p : X \to Y$ has the RLP wrt the map (3). $\square$

Roughly speaking (see [2] for a full definition), a **closed simplicial model category** is a closed model category $\mathscr{M}$ together with an internal function space construction with exponential law such that the following holds:

SM7: Suppose $p : X \to Y$ is a fibration and $i : A \to B$ is a cofibration. Then the map

$$\mathbf{hom}(B,X) \xrightarrow{(i^*,p_*)} \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,X)$$

is a fibration, which is trivial if either i or p is a weak equivalence.

Second example: The category **CGWH** has a closed simplicial model category structure, with the usual mapping space construction. The statement **SM7** follows from the observation that two cofibrations $i : A \to B$ and $j : C \to D$ induce a cofibration

$$(B \times C) \cup_{(A \times C)} (A \times D) \to B \times D,$$

which is trivial if either i or j is trivial (exercise).

References

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