Lecture 09: Bisimplicial abelian groups

Contents

24 Derived functors

Homology

Suppose $A : I \to \mathbf{Ab}$ is a diagram of abelian groups, defined on a small category I.

There is a simplicial abelian group $E_I A$, with

$$E_I A_n = \bigoplus_{\sigma: i_0 \to \dots \to i_n} A(i_0)$$

and with simplicial structure maps θ^* defined for $\theta : \mathbf{m} \to \mathbf{n}$ by the commutative diagrams

$$\begin{tikzcd} A(i_0) \arrow[r, "\alpha_*"] \arrow[d, "in_\sigma"'] & A(i_{\theta(0)}) \arrow[d, "in_{\theta^*(\sigma)}"] \\ \bigoplus_{\sigma: i_0 \to \dots \to i_n} A(i_0) \arrow[r, "\theta^*"'] & \bigoplus_{\gamma: j_0 \to \dots \to j_m} A(j_0) \end{tikzcd}$$

where $\alpha : i_0 \to i_{\theta(0)}$ is the morphism of I defined by θ.

The simplicial abelian group $E_I A$ defines the homotopy colimit within simplicial abelian groups.

Specifically, every diagram $B : I \to s\mathbf{Ab}$ of simplicial abelian groups determines a bisimplicial abelian group $E_I B$ with horizontal objects

$$E_I B_n = \bigoplus_{\sigma : i_0 \to \cdots \to i_n} B(i_0).$$

There is a **projective model structure** on $s\mathbf{Ab}^I$, for which $f : A \to B$ is a weak equivalence (respectively fibration) if and only if each map $f : A_i \to B_i$ is a weak equivalence (respectively fibration) of simplicial abelian groups (exercise).

Lemma 24.1. *The canonical map*

$$E_I B \to \varinjlim_I B,$$

induces a weak equivalence of simplicial abelian groups

$$\pi : d(E_I B) \to \varinjlim_I B$$

if B is projective cofibrant.

Proof. The generating projective cofibrations are induced from the generating projective cofibrations

$$j \times 1 : K \times \hom(i, \ ) \to L \times \hom(i, \ )$$

of I-diagrams of simplicial sets by applying the free abelian group functor.

There is an isomorphism

$$E_I\mathbb{Z}(X) \cong \mathbb{Z}(\underrightarrow{\mathrm{holim}}_I X)$$

for all I-diagrams of simplicial sets X.

The map

$$E_I\mathbb{Z}(\hom(i,\ ) \times K) \to \varinjlim_I \mathbb{Z}(\hom(i,\ ) \times K)$$

is the result of applying the free abelian group functor to a diagonal weak equivalence of bisimplicial sets.

Every pointwise weak equivalence of I-diagrams $A \to B$ induces a diagonal weak equivalence

$$d(E_I A) \to d(E_I B).$$

This is a consequence of Lemma 24.2. $\square$

Lemma 24.2. *Every level weak equivalence $A \to B$ of bisimplicial abelian groups induces a weak equivalence $d(A) \to d(B)$.*

Lemma 24.2 follows from Lemma 21.1 (the bisimplicial sets result).

Corollary 24.3. *Suppose the level weak equivalence $p : A \to B$ is a projective cofibrant replacement of an I-diagram B.*

Then there are weak equivalences

$$d(E_I B) \xleftarrow{\simeq} d(E_I A) \xrightarrow{\simeq} \varinjlim_I A.$$

Example: Suppose $A : I \to \mathbf{Ab}$ is a diagram of abelian groups. Write $\mathbf{Ab}^I$ for the category of such I-diagrams and natural transformations.

$\mathbf{Ab}^I$ has a set of projective generators, ie. all functors $\mathbb{Z}(\hom(i,\ ))$ obtained by applying the free abelian group functor to the functors $\hom(i,\ )$, $i \in \mathrm{Ob}(I)$.

It follows that every I-diagram $A : I \to \mathbf{Ab}$ of abelian groups has a projective resolution

$$\cdots \to P_1 \to P_0 \to A \to 0.$$

The I-diagram $\Gamma(P_*)$ of simplicial abelian groups is projective cofibrant (exercise), so there are weak equivalences of simplicial abelian groups

$$E_I A = d(E_I A) \xleftarrow{\simeq} d(E_I \Gamma(P_*)) \xrightarrow{\simeq} \varinjlim_I \Gamma(P_*) \cong \Gamma(\varinjlim_I P_*).$$

Thus, there are isomorphisms

$$\pi_k(E_I A) \cong \pi_k(\Gamma(\varinjlim_I P_*)) \cong H_k(\varinjlim_I P_*).$$

We have proved the following:

Lemma 24.4. *There are natural isomorphisms*

$$\pi_k(E_I A) \cong L(\varinjlim_I)_k(A)$$

for all I-diagrams of abelian groups A.

In other words, the homotopy (or homology) groups of $E_I A$ coincide with the left derived functors of the colimit functor in abelian groups.

Remark: Exactly the same script works for diagrams of simplicial modules over an arbitrary commutative unitary ring R.

Example: Suppose G is a group, and let $R(G)$ be the corresponding group-algebra over R. An $R(G)$-module, or simply a G-module in $R-\mathbf{Mod}$, is a diagram

$$M : G \to R - \mathbf{Mod},$$

and the higher derived functors of $\varinjlim_G$ for M are the group homology groups $H_n(G,M)$, as defined classically.

In effect, one can show that there is an isomorphism of simplicial R-modules

$$\begin{aligned} L(\varinjlim_{G})_k(M) &= H_k(E_G M) \\ &\cong H_k(R(EG) \otimes_G M) = H_k(G,M). \end{aligned}$$

Here, $EG = B(*/G)$ is the standard contractible cover of BG so $R(EG) \to R$ is a free G-resolution of the trivial G-module R.

$R(EG) \otimes_G M$ is the **Borel construction** for the G-module M.

The R-module $\varinjlim_G M$ is the module of **coinvariants** of the G-module M, and it is common to write

$$M/G = \varinjlim_G M.$$

Example: Suppose that $A : \Delta^{op} \to s\mathbf{Ab}$ is a bisimplicial abelian group.

The colimit $\varinjlim_{\mathbf{n}} A_n$ is the coequalizer

$$A_1 \rightrightarrows A_0 \to \pi_0 A = \varinjlim_n A_n$$

of the face maps $d_0, d_1 : A_1 \to A_0$.

The bisimplicial set $\Delta^n \tilde{\times} K$ has (horizontal) colimit

$$\pi_0 \Delta^n \times K \cong K.$$

It follows that the map

$$\mathbf{Z}(\Delta^n \tilde{\times} K) \to \varinjlim_p \mathbb{Z}(\Delta^n_p \times K)$$

is a levelwise equivalence (in vertical degrees) of bisimplicial abelian groups. This implies that the bisimplicial abelian group map

$$A \to \pi_0 A = \varinjlim_n A_n$$

is a weak equivalence in all vertical degrees for all projective cofibrant objects A, and therefore induces a diagonal weak equivalence

$$d(A) \xrightarrow{\simeq} \varinjlim_n A_n$$

for all such objects A.

It follows that if $A \to B$ is a projective cofibrant resolution of a bisimplicial abelian group B, then there are weak equivalences

$$d(B) \xleftarrow{\simeq} d(A) \xrightarrow{\simeq} \varinjlim_n A_n,$$

and so the diagonal $d(B)$ is naturally equivalent to the homotopy colimit of the simplicial object A.

Cohomology

There is a cohomological version of the theory presented so far in this section. A little more technology is involved.

1) The category $\mathbf{Ab}^I$ of I-diagrams of abelian groups has enough injectives.

2) If A is an I-diagram of abelian groups, then there is an isomorphism of cochain complexes

$$\hom(B(I/?), A) \cong \prod{}^* A.$$

3) The functor $\hom(\ , J)$ is exact if J is injective (exercise), and thus takes weak equivalences $X \to$

Y of I-diagrams of simplicial sets to cohomology isomorphisms $\hom(Y,J) \to \hom(X,J)$.

The canonical map $B(I/?) \to *$ is a weak equivalence of I-diagrams, so the morphism

$$\hom(*,J) \to \hom(B(I/?),J)$$

is a cohomology isomorphism if J is injective. Thus, there are isomorphisms

$$H^k \prod{}^* J \cong \begin{cases} \varprojlim_I J & \text{if } k = 0, \text{ and} \\ 0 & \text{if } k > 0. \end{cases}$$

4) More generally, there are isomorphisms

$$H^k \prod{}^* A \cong R(\varprojlim{}_I)^k A =: \varprojlim{}_I^k A$$

for $k \geq 0$ and for all I-diagrams A.

In effect, A has an injective resolution $A \to J^*$ and both (cohomological) spectral sequences for the bicomplex $\prod{}^* J^*$ collapse.

5) If A is an I-diagram of abelian groups, then there is an isomorphism

$$[*, K(A,n)] \cong \varprojlim{}_I^n (A), \tag{1}$$

where $[\ ,\ ]$ denotes morphisms in the homotopy category of I-diagrams of simplicial sets.

The best argument that I know of for the isomorphism (1) appears in [4] (also [5]).

The theory of higher right derived functors of inverse limit is a type of sheaf cohomology theory.

6) There are isomorphisms

$$\begin{aligned}\pi_0 \underleftarrow{\mathrm{holim}}_I K(A,n) &\cong \pi_0 \underleftarrow{\mathrm{holim}}_I Z\\ &\cong \pi_0 \varprojlim_I Z\\ &\cong [*, Z]\\ &\cong [*, K(A,n)]\\ &\cong \varprojlim{}^n_I A,\end{aligned}$$

where $K(A,n) \to Z$ is an injective fibrant model of $K(A,n)$.

The object $K(A,n)$ is a de-looping of $K(A,n-1)$, so there are isomorphisms

$$\pi_k \underleftarrow{\mathrm{holim}}_I K(A,n) \cong \begin{cases} \varprojlim{}^{n-k}_I A & \text{if } 0 \leq k \leq n, \text{ and} \\ 0 & \text{if } k > n \end{cases}$$

25 Spectral sequences for a bicomplex

This section contains a very basic introduction to spectral sequences.

We shall only explicitly discuss the spectral sequences in homology which are associated to a bicomplex.

These spectral sequences, their cohomological analogs (used at the end of Section 24), and the Bousfield-Kan spectral sequence for a tower of fibrations [1], are the most common prototypes for spectral sequences that one meets in nature.

Most of the material of this section in Mac Lane's "Homology" [6]. There are many other sources.

A **bicomplex** C consists of an array of abelian groups $C_{p,q}$, $p,q \geq 0$ and morphisms

$$\partial_v : C_{p,q} \to C_{p,q-1} \text{ and } \partial_h : C_{p,q} \to C_{p-1,q},$$

such that

$$\partial_v^2 = \partial_h^2 = 0 \text{ and}$$
$$\partial_v \partial_h + \partial_h \partial_v = 0.$$

A **morphism** $f : C \to D$ of bicomplexes consists of morphisms $f : C_{p,q} \to D_{p,q}$ that respect the differentials.

Write Ch_+^2 for the corresponding category.

There is a functor

$$\mathrm{Tot} : Ch_+^2 \to Ch_+$$

taking values in ordinary chain complexes with

$$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

and with differential $\partial : \mathrm{Tot}(C)_n \to \mathrm{Tot}(C)_{n-1}$ defined on the summand $C_{p,q}$ by

$$\partial(x) = \partial_v(x) + \partial_h(x).$$

Every bicomplex C has two filtrations, horizontal and vertical.

The p^{th} stage F_pC of the **horizontal filtration** has

$$F_pC_{r,s} = \begin{cases} C_{r,s} & \text{if } r \leq p, \text{ and} \\ 0 & \text{if } r > p. \end{cases}$$

Then

$$0 = F_{-1}C \subset F_0C \subset F_1C \subset \dots$$

and

$$\bigcup_p F_p(C) = C$$

The functor $C \mapsto \mathrm{Tot}(C)$ is exact, so this filtration on C induces a filtration on $\mathrm{Tot}(C)$.

One filters $\mathrm{Tot}(C)_n$ in finitely many stages:

$$0 = F_1 \mathrm{Tot}(C)_n \subset F_0 \mathrm{Tot}(C)_n \subset \cdots \subset F_n \mathrm{Tot}(C)_n = \mathrm{Tot}(C)_n.$$

Generally, the long exact seqences in homology associated to the exact sequences

$$0 \to F_{p-1}C \xrightarrow{i} F_pC \xrightarrow{p} F_pC/F_{p-1}C \to 0$$

arising from a filtration $\{F_pC\}$ on a chain complex C fit together to define a **spectral sequence** for the filtered complex.

This spectral sequence arises from the "ladder diagram"

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & \\
H_{p+q}(F_{p-2}) \arrow[d, "i_*"'] & & H_{p+q-1}(F_{p-3}) \arrow[d, "i_*"] & \\
H_{p+q}(F_{p-1}) \arrow[d, "i_*"'] & & H_{p+q-1}(F_{p-2}) \arrow[d, "i_*"] \arrow[r] & H_{p+q-1}(F_{p-2}/F_{p-3}) \\
H_{p+q}(F_p) \arrow[d, "i_*"'] \arrow[r, "p_*"] & H_{p+q}(F_p/F_{p-1}) \arrow[r, "\partial"] & H_{p+q-1}(F_{p-1}) \arrow[d, "i_*"] & \\
H_{p+q}(F_{p+1}) \arrow[d, "i_*"'] & & H_{p+q-1}(F_p) \arrow[d, "i_*"] \arrow[r, "p_*"] & H_{p+q-1}(F_p/F_{p-1}) \\
\vdots \arrow[d] & & \vdots \arrow[d] & \\
H_{p+q}(C) & & H_{p+q-1}(C) &
\end{tikzcd}$$

Set

$$Z_r^{p,q} = \{x \in H_{p+q}(F_p/F_{p-1}) \mid \partial(x) \in \operatorname{im}(i_*^{r-1})\}$$

and

$$B_r^{p,q} = p_*(\ker(i_*^{r-1})),$$

and then define

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}}.$$

for $r \geq 1$. Here, we adopt the convention that $i_*^0 = 1$, so that

$$E_1^{p,q} = H_{p+q}(F_p/F_{p-1})$$

This is cheating slightly (this only works for bi-complexes), but set

$$E_\infty^{p,q} = \frac{\ker(\partial)}{p_*(\ker(H_{p+q}(F_p) \to H_{p+q}(C)))}.$$

Finally, define

$$F_p H_{p+q}(C) = \operatorname{im}(H_{p+q}(F_p) \to H_{p+q}(C)).$$

Given $[x] \in E_r^{p,q}$ represented by $x \in Z_r^{p,q}$ choose $y \in H_{p+q}(F_{p-r})$ such that $i_*^{r-1}(y) = \partial(x)$. Then the assignment $[x] \mapsto [p_*(y)]$ defines a homomorphism

$$d_r : E_r^{p,q} \to E_r^{p-r,q+r-1},$$

and this homomorphism is natural in filtered complexes.
Then we have the following:

Lemma 25.1. *1) We have the relation $d_r^2 = 0$, and there is an isomorphism*

$$E_{r+1}^{p,q} \cong \frac{\ker(d_r : E_r^{p,q} \to E_r^{p-r,q+r-1})}{\mathrm{im}(d_r : E_r^{p+r,q-r+1} \to E_r^{p,q})}.$$

2) There are isomorphisms

$$E_r^{p,q} \cong E_\infty^{p,q}$$

for $r > p, q+2$.

3) There are short exact sequences

$$0 \to F_{p-1}H_{p+q}(C) \to F_pH_{p+q}(C) \to E_\infty^{p,q} \to 0.$$

The proof is an exercise — chase some elements.

In general (ie. for general filtered complexes),

$$E_1^{p,q} = H_{p+q}(F_p/F_{p-1}),$$

and $E_2^{p,q}$ is the homology of the complex with differentials $E_1^{p,q} \to E_1^{p-1,q}$ given by the composites

$$H_{p+q}(F_p/F_{p-1}) \xrightarrow{\partial} H_{p+q-1}(F_{p-1}) \xrightarrow{p_*} H_{p+q-1}(F_{p-1}/F_{p-2})$$

In the case of the horizontal filtration $F_p \operatorname{Tot}(C)$ for a bicomplex C, there is a natural isomorphism

$$F_p \operatorname{Tot}(C)/F_{p-1} \operatorname{Tot}(C) \cong C_{p,*}[p],$$

so there is an isomorphism

$$E_1^{p,q} \cong H_q(C_{p,*})$$

The differential d_1 is the homomorphism

$$H_q(C_{p,*}) \xrightarrow{\partial_{h*}} H_q(C_{p-1,*})$$

which is induced by the horizontal differential.

It follows, that for the horizontal filtration on the total complex $\mathrm{Tot}(C)$ of a bicomplex C, there is a spectral sequence with

$$E_2^{p,q} = H_p^h(H_q^v C) \Rightarrow H_{p+q}(\mathrm{Tot}(C)).$$

In particular, the spectral sequence converges to $H_*(\mathrm{Tot}(C))$ in the sense that the filtration quotients $E_\infty^{p,q}$ determine $H_*(\mathrm{Tot}(C))$.

Here's an example of how it all works:

Lemma 25.2. *Suppose $f : C \to D$ is a morphism of bicomplexes such that for some $r \geq 1$ the induced morphisms $E_r^{p,q}(C) \to E_r^{p,q}(D)$ are isomorphisms for all $p, q \geq 0$.*

Then the induced map $\mathrm{Tot}(C) \to \mathrm{Tot}(D)$ is a homology isomorphism.

Lemma 25.2 is sometimes called the *Zeeman comparison theorem*.

Proof. The map f induces isomorphisms

$$E_s^{p,q}(C) \xrightarrow{\cong} E_s^{p,q}(D)$$

for all $s \geq r$ (because all such E_s-terms are computed by taking homology groups, inductively in $s \geq r+1$. It follows that all induced maps

$$E_\infty^{p,q}(C) \to E_\infty^{p,q}(D)$$

are isomorphisms. But then, starting with the morphism

$$\begin{tikzcd}
E_\infty^{0,p+q}(C) \arrow[r, "\cong"] \arrow[d, "\cong"'] & E_\infty^{0,p+q}(D) \arrow[d, "\cong"] \\
F_0H_{p+q}(\mathrm{Tot}(C)) \arrow[r] & F_0H_{p+q}(\mathrm{Tot}(D))
\end{tikzcd}$$

and using the fact that the induced maps on successive filtration quotients are the isomorphisms

$$E_\infty^{r,p+q-r}(C) \xrightarrow{\cong} E_\infty^{r,p+q-r}(D),$$

one shows inductively that all maps

$$F_rH_{p+q}(\mathrm{Tot}(C)) \to F_rH_{p+q}(\mathrm{Tot}(D))$$

are isomorphisms, including the case $r = p+q$ which is the map

$$H_{p+q}(\mathrm{Tot}(C)) \to H_{p+q}(\mathrm{Tot}(D)).$$

This is true for all total degrees $p+q$, so the map $\mathrm{Tot}(C) \to \mathrm{Tot}(D)$ is a quasi-isomorphism. □

Example: Suppose A is a bisimplicial abelian group. Then the Generalized Eilenberg-Zilber Theorem (Theorem 26.1) asserts that there is a natural chain homotopy equivalence of chain complexes

$$d(A) \simeq \mathrm{Tot}(A)$$

where $\mathrm{Tot}(A)$ is the total complex of the associated (Moore) bicomplex. Filtering A in the horizontal direction therefore gives a spectral sequence with

$$E_2^{p,q} = \pi_p^h(\pi_q^v(A)) \Rightarrow \pi_{p+q} d(A). \qquad (2)$$

This spectral sequence is natural in bisimplicial abelian groups A.

This spectral sequence can be used to give an alternate proof of Lemma 24.4. If $A \to B$ is a level equivalence of bisimplicial abelian groups, then there is an E_1-level isomophism

$$\pi_q(A_{p,*}) \xrightarrow{\cong} \pi_q(B_{p,*})$$

for all $p, q \geq 0$.

Use Lemma 25.2 for the spectral sequence (2) to show that the map $d(A) \to d(B)$ is a weak equivalence.

Application: The Lyndon-Hochschild-Serre spectral sequence

Suppose $f : C \to D$ is a functor between small categories, and recall the bisimplicial set map

$$\bigsqcup_{d_0 \to \cdots \to d_n} B(f/d_0) \to BC$$

of Section 23. Lemma 23.3 says that this map is a diagonal weak equivalence.

The free abelian group functor preserves diagonal weak equivalences, so there is a spectral sequence

$$E_2^{p,q} = L(\varinjlim)_p H_q(B(f/?), \mathbb{Z}) \Rightarrow H_{p+q}(BC, \mathbb{Z}). \tag{3}$$

The derived colimit functors are computed over the base category D.

In the special case where f is a surjective group homomorphism $G \to H$ with kernel K, this is a form of the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H_p(H, H_q(BK, \mathbb{Z})) \Rightarrow H_{p+q}(BG, \mathbb{Z}). \tag{4}$$

We can put in other coefficients if we want.

To see that the E_2-term of (4) has the indicated form, take a set-theoretic section $\sigma : H \to G$ of the group homomorphism f such that $\sigma(e) = e$.

Conjugation $x \mapsto \sigma(h)x\sigma(h)^{-1}$, defines a group isomorphism $c_{\sigma(h)} : K \to K$ and hence an isomorphism

$$h_* : H_*(BK, \mathbb{Z}) \to H_*(BK, \mathbb{Z})$$

in homology. The map h_* is independent of the choice of section σ because any two pre-images of h determine homotopic maps $K \to K$ (ie. the two isomorphisms differ by conjugation by an element of K).

This action of H on $H_*(BK, \mathbb{Z})$ is the one appearing in the description of the E_2-term of (4).

The objects of $f/*$ are the elements of H, and a morphism $g : h \to h'$ in $f/*$ is an element $g \in G$ such that $h' f(g) = h$.

There is a functor $K \to f/*$ defined by sending $k \in K$ to the morphism $k : e \to e$.

There is a functor $f/* \to K$ which is defined by sending the morphism $g : h \to h'$ to the element $\sigma(h')g\sigma(h)^{-1}$.

$\sigma(e) = e$, so the composite functor

$$K \to f/* \to K$$

is the identity, while the elements $\sigma(h)$, $h \in H$ define a homotopy from the composite

$$f/* \to K \to f/*$$

to the identity on $f/*$.

Finally, composition with $\alpha \in H$ defines the functor $\alpha_* : f/* \to f/*$ in the decription of the bisimplicial set for $f : G \to H$, and there is a homotopy commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & f/* \\ {\scriptstyle c_{\sigma(\alpha)}}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ K & \longrightarrow & f/* \end{array}$$

Thus, the action of α on $H_*(B(f/*),\mathbb{Z})$ coincides with the morphism $\alpha_* : H_*(BK,\mathbb{Z}) \to H_*(BK,\mathbb{Z})$ displayed above, up to isomorphism.

Example: Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

All three groups are abelian, so all conjugation actions are trivial and there is a spectral sequence with

$$E_2^{p,q} = H_p(B(\mathbb{Q}/\mathbb{Z}), H_q(B\mathbb{Z},\mathbb{Q})) \Rightarrow H_{p+q}(B\mathbb{Q},\mathbb{Q}).$$

$S^1 \simeq B\mathbb{Z}$, so there are isomorphisms

$$H_q(B\mathbb{Z},\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } q = 0,1, \text{ and} \\ 0 & \text{if } q > 1. \end{cases}$$

The group $\mathbb{Q}/\mathbb{Z}$ is all torsion, so that

$$H_q(B(\mathbb{Q}/\mathbb{Z}),\mathbb{Q}) = 0$$

for $q \geq 1$.

The E_2-term for the spectral sequence therefore collapses, so the "edge homomorphism"

$$H_*(B\mathbb{Z},\mathbb{Q}) \to H_*(B\mathbb{Q},\mathbb{Q})$$

is an isomorphism.

To see the claim about torsion groups, observe that torsion abelian groups are filtered colimits of finitely generated torsion abelian groups, and a finitely generated torsion abelian group is a finite direct sum of cyclic groups.

It therefore suffices, by a Künneth formula argument (see (11) below) to show that

$$H_p(\mathbb{Z}/n,\mathbb{Q}) = H_p(B(\mathbb{Z}/n),\mathbb{Q}) = 0$$

for $p > 0$.

The abelian group $\mathbb{Z}$, as a trivial $\mathbb{Z}/n$-module, has a free resolution by $\mathbb{Z}/n$-modules

$$\dots \xrightarrow{N} \mathbb{Z}(\mathbb{Z}/n) \xrightarrow{1-t} \mathbb{Z}(\mathbb{Z}/n) \to \mathbb{Z} \to 0 \qquad (5)$$

where $1-t$ is multiplication by group-ring element $1-t$ and t is the generator of the group $\mathbb{Z}/n$.

The map N is multiplication by the "norm element"

$$N = 1 + t + t^2 + \cdots + t^{n-1}.$$

Tensoring the resolution with the trivial $\mathbb{Z}/n$-module $\mathbb{Z}$ gives the chain complex

$$\ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

and it follows that

$$H_p(B\mathbb{Z}/n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 0, \\ 0 & \text{if } p = 2n,\ n > 0, \text{ and} \\ \mathbb{Z}/n & \text{if } p = 2n+1,\ n \geq 0. \end{cases} \tag{6}$$

Tensoring with $\mathbb{Q}$ (which is exact) therefore shows that $H_p(B\mathbb{Z}/n, \mathbb{Q}) = 0$ for $p > 0$.

We could equally well tensor the resolution (5) with the trivial $\mathbb{Z}/n$-module $\mathbb{Q}$ and get the same answer, because $\mathbb{Q}$ is uniquely n-divisible.

Every bisimplicial abelian group A has a naturally associated bicomplex $M(A)$ with

$$M(A)_{m,n} = A_{m,n},$$

and with horizontal boundaries

$$\partial_h = \sum_{i=0}^{m} (-1)^i d_i : A_{m,n} \to A_{m-1,n}$$

and vertical boundaries

$$\partial_v = \sum_{i=0}^{n} (-1)^{m+i} d_i : A_{m,n} \to A_{m,n-1}.$$

One checks that

$$\partial_h \partial_v + \partial_v \partial_h = 0$$

in all bidegrees — the signs were put in to achieve this formula.

Here is the Generalized Eilenberg-Zilber Theorem of Dold-Puppe [2], [3, IV.2.2]:

Theorem 26.1 (Dold-Puppe)**.** *Suppose that A is a bisimplicial abelian group.*

Then the chain complexes $d(A)$ and $\mathrm{Tot}(A)$ *are naturally chain homotopy equivalent.*

Proof. The standard Eilenberg-Zilber Theorem says that there are natural chain maps

$$f : \mathbb{Z}(K \times L) \to \mathrm{Tot}(\mathbb{Z}(K) \otimes \mathbb{Z}(L))$$

(Moore complexes) and

$$g : \mathrm{Tot}(\mathbb{Z}(K) \otimes \mathbb{Z}(L)) \to \mathbb{Z}(K \times L),$$

and there are natural chain homotopies $fg \simeq 1$ and $gf \simeq 1$ for simplicial sets K and L.

The Eilenberg-Zilber Theorem specializes to (is equivalent to — exercise) the existence of chain maps

$$d(\mathbb{Z}(\Delta^{p,q})) = \mathbb{Z}(\Delta^p \times \Delta^q) \xrightarrow{f} \mathrm{Tot}(\mathbb{Z}(\Delta^p) \otimes \mathbb{Z}(\Delta^q)) = \mathrm{Tot}(\mathbb{Z}(\Delta^{p,q}))$$

and

$$g : \mathrm{Tot}(\mathbb{Z}(\Delta^{p,q})) \to d(\mathbb{Z}(\Delta^{p,q}))$$

and chain homotopies $fg \simeq 1$ and $gf \simeq 1$ which are natural in bisimplices $\Delta^{p,q}$.

Every bisimplicial abelian group A is a natural colimit of the diagrams

$$\begin{tikzcd} A_{p,q} \otimes \mathbb{Z}(\Delta^{r,s}) \arrow[r, "1\otimes(\gamma,\theta)"] \arrow[d, "(\gamma,\theta)^*\otimes 1"'] & A_{p,q} \otimes \mathbb{Z}(\Delta^{p,q}) \\ A_{r,s} \otimes \mathbb{Z}(\Delta^{r,s}) & \end{tikzcd}$$

where $(\gamma,\theta):(\mathbf{r},\mathbf{s})\to(\mathbf{p},\mathbf{q})$ varies over the morphisms of $\Delta\times\Delta$, and the maps

$$\gamma_{p,q}:A_{p,q}\otimes\mathbb{Z}(\Delta^{p,q})\to A$$

given by $(a,(\gamma,\theta))\mapsto(\gamma,\theta)^*(a)$ define the colimit.

There are isomorphisms

$$\begin{aligned} d(B\otimes A) &\cong B\otimes d(A),\\ \mathrm{Tot}(B\otimes A) &\cong B\otimes\mathrm{Tot}(A) \end{aligned}$$

for bisimplicial abelian groups A and abelian groups B, and these isomorphisms are natural in both A and B. The functors Tot and d are also right exact.

It follows that f and g induce natural chain maps

$$f_*:d(A)\to\mathrm{Tot}(A),\ g_*:\mathrm{Tot}(A)\to d(A)$$

for all simplicial abelian groups A.

The chain homotopies $fg\simeq 1$ and $gf\simeq 1$ induce natural chain homotopies

$$\begin{aligned} f_*g_* &\simeq 1:\mathrm{Tot}(A)\to\mathrm{Tot}(A),\\ g_*f_* &\simeq 1:d(A)\to d(A) \end{aligned}$$

for all bisimplicial abelian groups A. $\square$

Remarks: 1) The proof of Theorem 26.1 which appears in [3, p.205] contains an error: the sequence

$$\bigoplus_{\tau\to\sigma} \mathbb{Z}(\Delta^{r,s}) \to \bigoplus_{\Delta^{p,q}\to A} \mathbb{Z}(\Delta^{p,q}) \to A \to 0$$

is not exact, which means that A is not a colimit of its bisimplices in general. The problem is fixed by using the co-end description of A that you see above.

2) The maps f and g have classical explicit models, namely the Alexander-Whitney map and shuffle map, respectively. See [6, VIII.8] for a full discussion.

Recall that $M(A)$ denotes the Moore chain complex of a simplicial abelian group A.

The Alexander-Whitney map

$$f : M(A \otimes B) \to \mathrm{Tot}(M(A) \otimes M(B)) \qquad (7)$$

is defined, for simplicial abelian groups A and B by

$$f(a \otimes b) = \sum_{0 \leq p \leq n} a|_{[0,\dots,p]} \otimes b|_{[p,\dots,n]}.$$

Here, $a \in A_n$ and $b \in B_n$ are n-simplices. The "front p-face" $a|_{[0,\dots,p]}$ is defined by

$$\Delta^p \xrightarrow{[0,\dots,p]} \Delta^n \xrightarrow{a} A.$$

The "back $(n-p)$-face" $b|_{[p,\dots,n]}$ is defined by

$$\Delta^{n-p} \xrightarrow{[p,\dots,n]} \Delta^n \xrightarrow{b} B.$$

The Eilenberg-Zilber Theorem follows from

Lemma 26.2. *1) The object*

$$(\mathbf{p},\mathbf{q}) \mapsto \mathbb{Z}(\Delta^p \times \Delta^q)$$

is a projective cofibrant $(\Delta\times\Delta)$-diagram of simplicial abelian groups.

2) The object

$$(\mathbf{p},\mathbf{q}) \mapsto \mathrm{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q))$$

is a projective cofibrant $(\Delta\times\Delta)$-diagram of chain complexes.

To see that Lemma 26.2 implies the Eilenberg-Zilber Theorem, observe that there is a natural chain homotopy equivalence

$$\mathrm{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)) \simeq \mathrm{Tot}(M\mathbb{Z}(\Delta^p) \otimes M\mathbb{Z}(\Delta^q))$$

of bicosimplicial chain complexes which is induced by the natural chain homotopy equivalence of Theorem 15.4 (Lecture 06) between normalized and Moore chain complexes.

There is a similar natural chain homotopy equivalence

$$M\mathbb{Z}(\Delta^p \times \Delta^q) \simeq N\mathbb{Z}(\Delta^p \times \Delta^q).$$

Finally, there is a natural chain homotopy equivalence

$$N\mathbb{Z}(\Delta^p \times \Delta^q) \simeq \mathrm{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)),$$

since both objects are projective cofibrant resolutions of the constant diagram of chain complexes $\mathbb{Z}(0)$ on $\Delta \times \Delta$ by Lemma 26.2.

We use the following result to prove Lemma 26.2:

Lemma 26.3. *Suppose $p : A \to B$ is a trivial projective fibration of cosimplicial simplicial abelian groups. Then all induced maps*

$$(p,s) : A^{n+1} \to B^{n+1} \times_{M^nB} M^nA$$

are trivial fibrations of simplicial abelian groups.

Proof. The map $s : B^{n+1} \to M^nB$ is surjective for all cosimplicial abelian groups B. In effect, if $x =$

$(0,\ldots,0,x_i,\ldots,x_n) \in M^nB$ then

$$s(d^{i+1}x_i) = (0,\ldots,0,x_i,y_{i+1},\ldots,y_n),$$

and $x - s(d^{i+1}x_i)$ is of the form

$$x - s(d^{i+1}x_i) = (0,\ldots,0,z_{i+1},\ldots,z_n) =: z.$$

Thus, inductively, if $z = s(v)$ for some $v \in A^{n+1}$ then $x = s(d^{i+1}x_i + v)$.

Write

$M^n_{(0,i)}A = \{(x_0,\ldots,x_i) \mid x_i \in A^n, s^i x_j = s^{j-1}x_i \text{for } i < j\}$.

Then $M^nA = M^n_{(0,n)}A$, and there are pullback diagrams

$$\begin{tikzcd}
M^n_{(0,i+1)} \arrow[r] \arrow[d] & A^n \arrow[d, "s"] \\
M^n_{(0,i)} \arrow[r, "s^i"'] & M^{n-1}_{(0,i)}A
\end{tikzcd}$$

in which the two unnamed arrows are projections.

Suppose K is a cosimplicial object in $s\mathbf{Ab}$ such that all objects K^n are acyclic.

Then under the inductive assumption that $s : K^n \to M^{n-1}_{(0,i)}K$ is a trivial fibration we see that the projection $M^n_{(0,i+1)}K \to K^n$ is a trivial fibration, and so the map $s : K^{n+1} \to M^n_{(0,i+1)}K$ is a weak equivalence.

This is true for all $i < n$, and it follows that the map $s : K^{n+1} \to M^n K$ is a trivial fibration.

If K is the kernel of the projective trivial fibration $p : A \to B$, then there is an induced comparison of short exact sequences

$$\begin{tikzcd}
O \arrow[r] & K^{n+1} \arrow[r] \arrow[d, "s"'] & A^{n+1} \arrow[r] \arrow[d, "(p,s)"'] & B^{n+1} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & M^n K \arrow[r] & B^{n+1} \times_{M^n B} M^n A \arrow[r] & B^{n+1} \arrow[r] & 0
\end{tikzcd}$$

so the map (p, s) is a weak equivalence. $\square$

Corollary 26.4. *The cosimplicial simplicial abelian group* $\mathbf{n} \mapsto \mathbb{Z}(\Delta^n)$ *is projective cofibrant.*

Proof. Suppose $p : A \to B$ is a projective trivial fibration. Solving a lifting problem

$$\begin{tikzcd}
 & A \arrow[d, "p"] \\
\mathbb{Z}(\Delta) \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd}$$

amounts to inductively solving lifting problems

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "(p,s)"] \\
\Delta^{n+1} \arrow[r] \arrow[ur, dotted] & B^{n+1} \times_{M^n B} M^n A
\end{tikzcd}$$

and this can be done by the previous Lemma. $\square$

Proof of Lemma 26.2. Suppose $q : C \to D$ is a projective trivial fibration of $(\Delta \times \Delta)$-diagrams of simplicial abelian groups. Then all maps

$$(q,s) : C^{n+1} \to D^{n+1} \times_{M^n D} M^n C$$

are projective trivial fibrations of cosimplicial simplicial abelian groups, by Lemma 26.3.

Write $\Delta \times \Delta$ for the bicosimplicial diagram

$$(\mathbf{p},\mathbf{q}) \mapsto \Delta^p \times \Delta^q$$

of simplicial sets.

Then lifting problems

$$\begin{tikzcd} & C \arrow[d, "q"] \\ \Delta \times \Delta \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

can be solved by inductively solving the lifting problems

$$\begin{tikzcd} \mathbb{Z}(\Delta \times \partial\Delta^{n+1}) \arrow[r] \arrow[d] & C^{n+1} \arrow[d, "{(q,s)}"] \\ \mathbb{Z}(\Delta \times \Delta^{n+1}) \arrow[r] \arrow[ur, dotted] & D^{n+1} \times_{M^n D} M^n C \end{tikzcd}$$

in cosimplicial simplicial abelian groups.

For that, it suffices to show that the map

$$\mathbb{Z}(\Delta \times \partial\Delta^{n+1}) \to \mathbb{Z}(\Delta \times \Delta^{n+1})$$

is a projective cofibration, but this follows from the observation that the maps

$$\mathbb{Z}(\partial\Delta^m \times \Delta^{n+1}) \cup \mathbb{Z}(\Delta^m \times \partial\Delta^{m+1}) \to \mathbb{Z}(\Delta^m \times \Delta^{n+1})$$

are cofibrations of simplicial abelian groups xfor $m \geq 0$, with Lemma 26.3.

We have proved statement 1) of Lemma 26.2.

The second statement of Lemma 26.2 has a very similar proof. If $q : C \to D$ is a projective trivial fibration of bicosimplicial chain complexes, then all maps

$$(p,s) : C^{n+1} \to D^{n+1} \times_{M^nD} M^nC$$

are projective trivial fibrations of cosimplicial chain complexes, by Lemma 26.3. Write

$$\mathrm{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta))$$

for the bicosimplicial chain complex

$$(\mathbf{p},\mathbf{q}) \mapsto \mathrm{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)).$$

Then solving lifting problems

$$\begin{tikzcd} & C \arrow[d, "q"] \\ \mathrm{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta)) \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

amounts to inductively solving lifting problems

$$\begin{tikzcd}
\mathrm{Tot}(N\mathbb{Z}(\Delta)\otimes N\mathbb{Z}(\partial\Delta^{n+1})) \arrow[r] \arrow[d, "i"'] & C^{n+1} \arrow[d, "{(q,s)}"] \\
\mathrm{Tot}(N\mathbb{Z}(\Delta)\otimes N\mathbb{Z}(\Delta^{n+1})) \arrow[r] \arrow[ur, dotted] & D^{n+1}\times_{M^nD} M^nC
\end{tikzcd}$$

For this, we show that all maps i are projective cofibrations of cosimplicial chain complexes, but this reduces to showing that each of the maps

$$\begin{aligned}
&\mathrm{Tot}(N\mathbb{Z}(\Delta^m)\otimes N\mathbb{Z}(\partial\Delta^{n+1}))\cup\mathrm{Tot}(N\mathbb{Z}(\partial\Delta^m)\otimes N\mathbb{Z}(\Delta^{n+1})) \\
&\quad\to\mathrm{Tot}(N\mathbb{Z}(\Delta^m)\otimes N\mathbb{Z}(\Delta^{n+1}))
\end{aligned}$$

are cofibrations of chain complexes.

This last morphism is defined by freely adjoining the chain $\iota_m\otimes\iota_n$, so it is a cofibration. □

Remark: The proof of the Eilenberg-Zilber Theorem that one finds in old textbooks uses the method of acyclic models.

27 Universal coefficients, Künneth formula

Suppose X is a simplicial set, and that A is an abelian group.

Recall that the n^{th} homology group $H_n(X,A)$ of X with coefficients in A is defined by

$$H_n(X,A) = H_n(\mathbb{Z}(X) \otimes_{\mathbb{Z}} A),$$

where $\mathbb{Z}(X)$ denotes both a free simplicial abelian group and its associated Moore complex.

The ring $\mathbb{Z}$ is a principal ideal domain, so A (a $\mathbb{Z}$-module) has a free resolution

$$0 \to F_2 \xrightarrow{i} F_1 \xrightarrow{p} A \to 0.$$

All abelian groups $\mathbb{Z}(X_n)$ are free, and tensoring with a free abelian group is exact, so there is a short exact sequence of chain complexes

$$0 \to \mathbb{Z}(X) \otimes F_2 \xrightarrow{1\otimes i} \mathbb{Z}(X) \otimes F_1 \xrightarrow{1\otimes p} \mathbb{Z}(X) \otimes A \to 0.$$

The long exact sequence in H_* has the form

$$\dots \xrightarrow{\partial} H_n(X,F_2) \xrightarrow{(1\otimes i)_*} H_n(X,F_1) \xrightarrow{(1\otimes p)_*} H_n(X,A) \xrightarrow{\partial} \dots$$

There are commutative diagrams

$$\begin{tikzcd}
H_n(X,F_2) \arrow[r, "(1\otimes i)_*"] \arrow[d, "\cong"'] & H_n(X,F_1) \arrow[d, "\cong"] \\
H_n(X,\mathbb{Z}) \otimes F_2 \arrow[r, "1\otimes i"'] & H_n(X,\mathbb{Z}) \otimes F_1
\end{tikzcd}$$

It follows that there are short exact sequences

$$0 \to H_n(X,\mathbb{Z}) \otimes A \to H_n(X,A) \to \operatorname{Tor}(H_{n-1}(X,\mathbb{Z}),A) \to 0. \tag{8}$$

These are the **universal coefficients** exact sequences.

Both Z_n (n-cycles) and B_n (n-boundaries) are free abelian groups, and so there is a map $\phi_n : B_n \to \mathbb{Z}(X)_{n+1}$ such that the diagram of abelian group homomorphisms

$$\begin{tikzcd} B_n \arrow[r, "\phi_n"] \arrow[d, "j"'] & \mathbb{Z}(X)_{n+1} \arrow[d] \\ Z_n \arrow[r, "i"'] & \mathbb{Z}(X)_n \end{tikzcd} \tag{9}$$

commutes, where i and j are canonical inclusions.

Write $\tilde{Z}_n$ for the chain complex which is concentrated in degrees n and $n+1$ and with boundary morphism given by the inclusion j.

The diagram (9) defines a chain map

$$\phi_n : \tilde{Z}_n \to \mathbb{Z}(X),$$

which induces an isomorphism

$$H_n(\tilde{Z}_n) \cong H_n(\mathbb{Z}(X)),$$

while $H_k(\tilde{Z}_n) = 0$ for $k \neq n$.

Adding up the maps ϕ_n therefore determines a (non-natural) weak equivalence

$$\phi : \bigoplus_{n \geq 0} \tilde{Z}_n \to \mathbb{Z}(X).$$

The two complexes are cofibrant, so ϕ is a chain homotopy equivalence and in particular there is a chain homotopy inverse

$$\psi : \mathbb{Z}(X) \to \bigoplus_{n \geq 0} \tilde{Z}_n.$$

The map ψ and projection onto the complex $\tilde{Z}_n$ therefore determine a chain map

$$\mathbb{Z}(X) \otimes A \to \tilde{Z}_n \otimes A$$

Comparing universal coefficients sequences gives a commutative diagram

$$\begin{tikzcd} H_n(X) \otimes A \arrow[r] \arrow[d, "\cong"'] & H_n(X, A) \arrow[d] \\ H_n(X) \otimes A \arrow[r, "\cong"'] & H_n(\tilde{Z}_n \otimes A) \end{tikzcd}$$

It follows that the natural map

$$H_n(X) \otimes A \to H_n(X, A)$$

from the universal coefficients sequence (8) is non-naturally split.

We have proved

Theorem 27.1 (Universal Coefficients Theorem)**.** *Suppose X is a simplicial set and A is an abelian group.*

There is a short exact sequence

$$\begin{aligned} 0 \to & H_n(X,\mathbb{Z}) \otimes A \to H_n(X,A) \\ & \to \operatorname{Tor}(H_{n-1}(X,\mathbb{Z}),A) \to 0. \end{aligned}$$

for each $n \geq 1$. This sequence is natural in X, and has a non-natural splitting.

Here's a different take on universal coefficients:

The chain complex $\tilde{Z}_n \otimes A$ has homology

$$H_k(\tilde{Z}_n \otimes A) \cong \begin{cases} \operatorname{Tor}(H_n(X),A) & \text{if } k = n+1, \\ H_n(X) \otimes A & \text{if } k = n, \\ 0 & \text{if } k \neq n, n+1, \end{cases}$$

and the chain homotopy equivalence ϕ induces isomorphisms

$$\begin{aligned} H_n(X,A) &\cong H_n(\bigoplus_{n \geq 0} \tilde{Z}_n \otimes A) \\ &\cong (H_n(X) \otimes A) \oplus \operatorname{Tor}(H_{n-1}(X),A). \end{aligned}$$

Remark: The simplicial set underlying a simplicial abelian group has the homotopy type (non-naturally) of a product of Eilenberg-Mac Lane spaces — see [3, III.2.20].

Suppose C is a chain complex.

Then the chain homotopy equivalence ϕ induces a homology isomorphism

$$\mathrm{Tot}((\bigoplus_{n\geq 0}\tilde{Z}_n)\otimes C) \xrightarrow{\simeq} \mathrm{Tot}(\mathbb{Z}(X)\otimes C).$$

We can assume that C is cofibrant, even free in each degree.

Form cofibrant chain complexes F_kC and maps $F_kC \to C$ such that the maps $H_kF_kC \to H_kC$ are isomorphisms, and such that $H_p(F_kC) = 0$ for $p \neq k$.

It follows that there is a chain homotopy equivalence

$$\bigoplus_{k\geq 0} F_kC \xrightarrow{\simeq} C.$$

There are isomorphisms

$$H_p(\tilde{Z}_n \otimes F_kC) \cong \begin{cases} H_n(X)\otimes H_k(C) & \text{if } p = n+k, \\ \mathrm{Tor}(H_n(X), H_k(C)) & \text{if } p = n+k+1, \\ 0 & \text{otherwise} \end{cases}$$

Exercise: Do you need a spectral sequence?
Hint: Filter $F_k C$.

Adding up these isomorphisms gives split short exact sequences

$$\begin{aligned} 0 \to & H_n(X) \otimes H_k(C) \to H_{n+k}\operatorname{Tot}(\tilde{Z}_n \otimes C) \\ & \to \operatorname{Tor}(H_n(X), H_{k-1}(C)) \to 0 \end{aligned} \tag{10}$$

for $k \geq 0$, and $H_k \operatorname{Tot}(\tilde{Z}_n \otimes C) = 0$ for $k < 0$.

Taking a direct sum of the sequences (10) (and reindexing) gives short exact sequences

$$\begin{aligned} 0 \to & \bigoplus_{0 \leq p \leq n} H_{n-p}(X) \otimes H_p(C) \to H_n \operatorname{Tot}(\mathbb{Z}(X) \otimes C)) \\ & \to \bigoplus_{0 \leq q \leq n-1} \operatorname{Tor}(H_{n-1-q}(X), H_q(C)) \to 0 \end{aligned} \tag{11}$$

The sequence (11) and the Eilenberg-Zilber Theorem (Theorem 26.1) together imply the following:

Theorem 27.2 (Künneth Theorem)**.** *Suppose X and Y are simplicial sets. Then there is a natural short exact sequence*

$$\begin{aligned} 0 \to & \bigoplus_{0 \leq p \leq n} H_{n-p}(X) \otimes H_p(Y) \to H_n(X \times Y) \\ & \to \bigoplus_{0 \leq q \leq n-1} \operatorname{Tor}(H_{n-1-q}(X), H_q(Y)) \to 0. \end{aligned}$$

This sequence splits, but not naturally.

The coefficient ring $\mathbb{Z}$ in the statement of Theorem 27.2 can be replaced by a principal ideal domain R. The same theorem holds for $H_*(X \times Y, R)$, with the same proof.

If $R = F$ is a field, all F-modules are free and the Tor terms in the Theorem vanish, so

$$H_n(X \times Y, F) \cong \bigoplus_{0 \leq p \leq n} H_{n-p}(X, F) \otimes_F H_p(Y, F). \tag{12}$$

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