Lecture 10: Serre spectral sequence

Contents

28 The fundamental groupoid, revisited

The **path category** PX for a simplicial set X is the category generated by the graph $X_1 \rightrightarrows X_0$ of 1-simplices $x : d_1(x) \to d_0(x)$, subject to the relations

$$d_1(\sigma) = d_0(\sigma) \cdot d_2(\sigma)$$

given by the 2-simplices σ of X.

There is a natural bijection

$$\hom(PX, C) \cong \hom(X, BC),$$

so the functor $P : s\mathbf{Set} \to \mathbf{cat}$ is left adjoint to the nerve functor.

Write GPX for the groupoid freely associated to the path category. The functor $X \mapsto GP(X)$ is left adjoint to the nerve functor

$$B : \mathbf{Gpd} \to s\mathbf{Set}.$$

Say that a functor $f : G \to H$ between groupoids is a **weak equivalence** if the induced map $f : BG \to BH$ is a weak equivalence of simplicial sets.

Observe that $\mathrm{sk}_2(X) \subset X$ induces an isomorphism $P(\mathrm{sk}_2(X)) \cong P(X)$, and hence an isomorphism

$$GP(\mathrm{sk}_2(X)) \cong GP(X).$$

Nerves of groupoids are Kan complexes, so $f : G \to H$ is a weak equivalence if and only if

1) f induces bijections

$$f : \hom(a,b) \to \hom(f(a), f(b))$$

for all objects a, b of G, (ie. f is full and faithful) and

2) for every object c of H there is a morphism $c \to f(a)$ in H for some object a of G (f is surjective on π_0).

Thus, f is a weak equivalence of groupoids if and only if it is a categorical equivalence (exercise).

Lemma 28.1. *The functor $X \mapsto GP(X)$ takes weak equivalences of simplicial sets to weak equivalences of groupoids.*

Proof. 1) Claim: The inclusion $\Lambda^n_k \subset \Delta^n$ induces an isomorphism $GP(\Lambda^n_k) \cong GP(\Delta^n)$ if $n \geq 2$.

This is obvious if $n \geq 3$, for then $\mathrm{sk}_2(\Lambda^n_k) = \mathrm{sk}_2(\Delta^n)$.

If $n = 2$, then $GP(\Lambda^2_k)$ has a contracting homotopy onto the vertex k (exercise). It follows that $GP(\Lambda^2_k) \to GP(\Delta^2)$ is an isomorphism.

If $n = 1$, then Λ^1_k is a point, and $GP\Lambda^1_k$ is a strong deformation retraction of $GP(\Delta^1)$.

2) In all cases, $GP(\Lambda^n_k)$ is a strong deformation retraction of $GP(\Delta^n)$.

Strong deformation retractions are closed under pushout in the groupoid category (exercise).

Thus, every trivial cofibration $i : A \to B$ induces a weak equivalence $GP(A) \to GP(B)$, so every weak equivalence $X \to Y$ induces a weak equivalence $GP(X) \to GP(Y)$. $\square$

Suppose Y is a Kan complex, and recall that the fundamental groupoid $\pi(Y)$ for Y has objects given by the vertices of Y, morphisms given by homotopy classes of paths (1-simplices) $x \to y$ rel end points, and composition law defined by extending maps

$$(\beta, \ , \alpha) : \Lambda^2_1 \to Y$$

to maps $\sigma : \Delta^2 \to Y$: $[d_1(\sigma)] = [\beta] \cdot [\alpha]$.

There is a natural functor

$$GP(Y) \to \pi(Y)$$

which is the identity on vertices and takes a simplex $\Delta^1 \to Y$ to the corresponding homotopy class. This functor is an isomorphism of groupoids (exercise).

If X is a topological space then the combinatorial fundamental groupoid $\pi(S(X))$ coincides up to isomorphism with the usual fundamental groupoid $\pi(X)$ of X.

Corollary 28.2. *Suppose $i : X \to Z$ is a weak equivalence, such that Z is a Kan complex.*

Then i induces a weak equivalence of groupoids

$$GP(X) \xrightarrow{i_*} GP(Z) \xrightarrow{\cong} \pi(Z).$$

There is a functor

$$u_X : GP(X) \to G(\Delta/X)$$

that takes a 1-simplex $\omega : d_1(\omega) \to d_0(\omega)$ to the morphism $(d^0)^{-1}(d^1)$ in $G(\Delta/X)$ defined by the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "d^1"] \arrow[dr, "d_1(\omega)"'] & \Delta^1 \arrow[d, "\omega"] & \Delta^0 \arrow[l, "d^0"'] \arrow[dl, "d_0(\omega)"] \\ & X & \end{tikzcd}$$

This assignment takes 2-simplices to composition laws of $G(\Delta/X)$ [1, p.141].

There is a functor

$$v_X : G(\Delta/X) \to GP(X)$$

which associates to each object $\sigma : \Delta^n \to X$ its last vertex

$$\Delta^0 \xrightarrow{n} \Delta^n \xrightarrow{\sigma} X.$$

Then any map between simplices of Δ/X is mapped to a canonically defined path between last vertices, and compositions of Δ/X determine 2-simplices relating last vertices.

Then $v_X u_X$ is the identity on $GP(X)$ and the maps

$$\begin{tikzcd} \Delta^0 \arrow[d, "n"'] \arrow[dr, "\sigma \cdot n"] & \\ \Delta^n \arrow[r, "\sigma"'] & X \end{tikzcd}$$

determine a natural isomorphism (aka. homotopy)

$$u_X v_X \cong 1_{G(\Delta/X)}.$$

We have proved

Lemma 28.3. *There is an equivalence of groupoids*

$$u_X : GP(X) \leftrightarrows G(\Delta/X) : v_X,$$

which is natural in simplicial sets X.

Here's a summary. Suppose X is a simplicial set with fibrant model $i : X \to Z$. Then there is a picture of natural equivalences

$$\begin{tikzcd} GP(X) \arrow[r, "i_*", "\simeq"'] \arrow[d, "u_X"', "\simeq"] & GP(Z) \arrow[r, "\cong"] & \pi(Z) \\ G(\Delta/X) & & \pi(S|Z|) \arrow[u, "\simeq", "\varepsilon_*"'] \arrow[r, "\cong"'] & \pi(|Z|) \end{tikzcd}$$

You need the Milnor theorem (Theorem 13.2) to show that ε_* is an equivalence.

I refer to any of the three equivalent models $\pi(Z)$, $GP(X)$ or $G(\Delta/X)$ as the **fundamental groupoid** of X, and write $\pi(X)$ to denote any of these objects.

The adjunction map $X \to BGP(X)$ is often written

$$\eta : X \to B\pi(X).$$

Lemma 28.4. *Suppose C is a small category.*

Then there is an isomorphism

$$GP(BC) \cong G(C),$$

which is natural in C.

Proof. The adjunction functor $\varepsilon : P(BC) \to C$ is an isomorphism (exercise). $\square$

Remark: This result leads to a fast existence proof for the isomorphism

$$\pi_1(BQ\mathbf{M},0) \cong K_0(\mathbf{M})$$

(due to Quillen [3]) for an exact category $\mathbf{M}$, in algebraic K-theory.

It also follows that the adjunction functor

$$\varepsilon : GP(BG) \to G$$

is an isomorphism for all groupoids G.

Lemma 28.5. *Suppose X is a Kan complex.*

Then the adjunction map $\eta : X \to BGP(X)$ induces a bijection $\pi_0(X) \cong \pi_0(BGP(X))$ and isomorphisms

$$\pi_1(X,x) \xrightarrow{\cong} \pi_1(BGP(X),x)$$

for each vertex x of X.

Proof. This result is another corollary of Lemma 28.4.

There is a commutative diagram

$$\begin{tikzcd} \pi(X) \arrow[r, "\pi(\eta)"] & \pi(BGP(X)) \\ GP(X) \arrow[u, "\cong"] \arrow[r, "GP(\eta)"] \arrow[dr, "1"'] & GPBGP(X) \arrow[u, "\cong"'] \arrow[d, "\varepsilon", "\cong"'] \\ & GP(X) \end{tikzcd}$$

It follows that η induces an isomorphism

$$\pi(\eta) : \pi(X) \xrightarrow{\cong} \pi(BGPX).$$

Finish by comparing path components and automorphism groups, respectively. $\square$

Say that a morphism $p : G \to H$ of groupoids is a **fibration** if the induced map $BG \to BH$ is a fibration of simplicial sets.

Exercise: Show that a functor p is a fibration if and only if it has the **path lifting property** in the sense that all lifting problems

$$\begin{tikzcd} \mathbf{0} \arrow[r] \arrow[d] & G \arrow[d, "p"] \\ \mathbf{1} \arrow[r] \arrow[ur, dotted] & H \end{tikzcd}$$

(involving functors) can be solved.

Cofibrations of groupoids are defined by a left lifting property in the usual way.

There is a **function complex** construction $\mathbf{hom}(G,H)$ for groupoids, with

$$\mathbf{hom}(G,H) := \mathbf{hom}(BG,BH).$$

Lemma 28.6. *1) With these definitions, the category* **Gpd** *satisfies the axioms for a closed simplicial model category. This model structure is cofibrantly generated and right proper.*

2) The functors

$$GP : s\mathbf{Set} \leftrightarrows \mathbf{Gpd} : B$$

form a Quillen adjunction.

Proof. Use Lemma 28.1 and its proof. $\square$

29 The Serre spectral sequence

Suppose $f : X \to Y$ is a map of simplicial sets, and consider all pullback diagrams

$$\begin{tikzcd} f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

defined by the simplices of Y.

We know (Lemma 23.1) that the bisimplicial set map

$$\bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} f^{-1}(\sigma_0) \to X$$

defines a (diagonal) weak equivalence

$$\underrightarrow{\mathrm{holim}}_{\,\sigma:\Delta^n \to Y} f^{-1}(\sigma) \to X$$

where the homotopy colimit defined on the simplex category Δ/Y.

The induced bisimplicial abelian group map

$$\bigoplus_{\sigma_0 \to \cdots \to \sigma_n} \mathbb{Z}(f^{-1}(\sigma_0)) \to \mathbb{Z}(X)$$

is also a diagonal weak equivalence.

It follows (see Lemma 24.4) that there is a spectral sequence with

$$E_2^{p,q} = L(\varinjlim_{\sigma:\Delta^n \to Y})_p H_q(f^{-1}(\sigma)) \Rightarrow H_{p+q}(X,\mathbb{Z}), \tag{1}$$

often called the *Grothendieck spectral sequence*.

Making sense of the spectral sequence (1) usually requires more assumptions on the map f.

A) Suppose $f : X \to Y$ is a fibration and that Y is connected.

By properness, the maps

$$\theta_* : f^{-1}(\sigma) \to f^{-1}(\tau)$$

induced by simplex morphisms $\theta : \sigma \to \tau$ are weak equivalences, and the maps

$$\theta_* : H_k(f^{-1}(\sigma),\mathbb{Z}) \to H_k(f^{-1}(\tau),\mathbb{Z})$$

are isomorphisms.

It follows that the functors $H_k : \Delta/Y \to \mathbf{Ab}$ which are defined by

$$\sigma \mapsto H_k(f^{-1}(\sigma),\mathbb{Z})$$

factor through an action of the fundamental groupoid of Y, in the sense that these functors extend uniquely to functors

$$H_k : G(\Delta/Y) \to \mathbf{Ab}.$$

Suppose x is a vertex of Y, and write $F = p^{-1}(x)$ for the fibre of f over x.

Since Y is connected there is a morphism $\omega_\sigma : x \to \sigma$ in $G(\Delta/Y)$ for each object σ of the simplex category. The maps ω_σ, induce isomorphisms

$$\omega_{\sigma *} : H_k(F,\mathbb{Z}) \to H_k(f^{-1}(\sigma),\mathbb{Z}),$$

and hence define a functor

$$H_k(F,\mathbb{Z}) : G(\Delta/Y) \to \mathbf{Ab}$$

which is naturally isomorphic to the functor H_k.

It follows that the spectral sequence (1) is isomorphic to

$$E_2^{p,q} = L(\varinjlim_{\Delta/Y})_p H_q(F,\mathbb{Z})) \Rightarrow H_{p+q}(X,\mathbb{Z}) \qquad (2)$$

under the assumption that $f : X \to Y$ is a fibration and Y is connected.

This is the general form of the **Serre spectral sequence**.

This form of the Serre spectral sequence is used, but calculations often involve more assumptions.

B) The fundamental groupoid $G(\Delta/Y)$ **acts trivially** on the homology fibres $H_k(f^{-1}(\sigma),\mathbb{Z})$ of f if any two morphisms $\alpha,\beta:\sigma\to\tau$ in $G(\Delta/Y)$ induce the same map

$$\alpha_* = \beta_* : H_k(f^{-1}(\sigma),\mathbb{Z}) \to H_k(f^{-1}(\tau),\mathbb{Z})$$

for all $k \geq 0$.

This happens, for example, if the fundamental group (or groupoid) of Y is trivial.

In that case, all maps $x \to x$ in $G(\Delta/Y)$ induce the identity

$$H_k(F,\mathbb{Z}) \to H_k(F,\mathbb{Z})$$

for all $k \geq 0$, and there are isomorphisms (exercise)

$$\begin{aligned} L(\varinjlim)_p H_q(F,\mathbb{Z}) &\cong H_p(B(\Delta/Y), H_q(F,\mathbb{Z})) \\ &\cong H_p(Y, H_q(F,\mathbb{Z})). \end{aligned}$$

Thus, we have the following:

Theorem 29.1. *Suppose $f : X \to Y$ is a fibration with Y connected, and let F be the fibre of f over a vertex x of Y. Suppose the fundamental groupoid $G(\Delta/Y)$ of Y acts trivially on the homology fibres of f.*

Then there is a spectral sequence with

$$E_2^{p,q} = H_p(Y, H_q(F,\mathbb{Z})) \Rightarrow H_{p+q}(X,\mathbb{Z}). \qquad (3)$$

This spectral sequence is natural in all such fibre sequences.

The spectral sequence given by Theorem 29.1 is the standard form of the homology Serre spectral sequence for a fibration.

Integral coefficients were used in the statement of Theorem 29.1 for display purposes — $\mathbb{Z}$ can be replaced by an arbitrary abelian group of coefficients.

Examples: Eilenberg-Mac Lanes spaces

Say that X is n**-connected** ($n \geq 0$) if $\pi_0 X = *$, and $\pi_k(X,x) = 0$ for all $k \leq n$ and all vertices x.

One often says that X is **simply connected** if it is 1-connected.

X is simply connected if and only if it has a trivial fundamental groupoid $\pi(X)$ (exercise).

Here's a general fact:

Lemma 29.2. *Suppose X is a Kan complex, $n \geq 0$, and that X is n-connected. Pick a vertex $x \in X$.*

Then X has a subcomplex Y such that $Y_k = \{x\}$ for $k \leq n$, and Y is a strong deformation retract of X.

The proof is an exercise.

Corollary 29.3. *Suppose X is n-connected. Then there are isomorphisms*

$$H_k(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } 0 < k \leq n. \end{cases}$$

Example: There is a fibre sequence

$$K(\mathbb{Z},1) \to WK(\mathbb{Z},1) \to K(\mathbb{Z},2) \qquad (4)$$

such that $WK(\mathbb{Z},1) \simeq *$.

$K(\mathbb{Z},2)$ is simply connected, so the Serre spectral sequence for (4) has the form

$$H_p(K(\mathbb{Z},2), H_q(K(\mathbb{Z},1),\mathbb{Z})) \Rightarrow H_{p+q}(*,\mathbb{Z}).$$

1) $H_1(K(\mathbb{Z},2),A) = 0$ by Corollary 29.3, so $E_2^{1,q} = 0$ for all q.

2) $K(\mathbb{Z},1) \simeq S^1$, so $E_2^{p,q} = 0$ for $q > 1$.

The quotient of the differential

$$d_2 : E_2^{2,0} \to E_2^{0,1} \cong \mathbb{Z}$$

survives to $E_\infty^{0,1} \subset H_1(*) = 0$, so d_2 is surjective. The kernel of d_2 survives to $E_\infty^{2,0} = 0$, so d_2 is an isomorphism and

$$H_2(K(\mathbb{Z},2),\mathbb{Z}) \cong \mathbb{Z}.$$

Inductively, we find isomorphisms

$$H_n(K(\mathbb{Z},2),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 2k,\ k \geq 0, \text{ and} \\ 0 & \text{if } n = 2k+1,\ k \geq 0. \end{cases}$$

Example: There is a fibre sequence

$$K(\mathbb{Z}/n,1) \to WK(\mathbb{Z}/n,1) \to K(\mathbb{Z}/n,2) \qquad (5)$$

such that $WK(\mathbb{Z}/n,1) \simeq *$.

$K(\mathbb{Z}/n,2)$ is simply connected, so the Serre spectral sequence for (5) has the form

$$H_p(K(\mathbb{Z}/n,2), H_q(K(\mathbb{Z}/n,1),\mathbb{Z})) \Rightarrow H_{p+q}(*,\mathbb{Z}).$$

We showed (see (6) of Section 25) that there are isomorphisms

$$H_p(B\mathbb{Z}/n,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 0, \\ 0 & \text{if } p = 2n,\ n > 0, \text{ and} \\ \mathbb{Z}/n & \text{if } p = 2n+1,\ n \geq 0. \end{cases}$$

There are isomorphisms

$$E_2^{1,q} \cong 0$$

for $q \geq 0$ and

$$H_2(K(\mathbb{Z}/n,2),\mathbb{Z}) \xrightarrow[\cong]{d_2} H_1(K(\mathbb{Z}/n,1),\mathbb{Z}) \cong \mathbb{Z}/n.$$

$E_2^{0,2} = H_2(K(\mathbb{Z}/n,1),\mathbb{Z}) = 0$, so all differentials on $E_2^{3,0}$ are trivial. Thus, $E_2^{3,0} = E_\infty^{3,0} = 0$ because $H_3(*) = 0$, and

$$H_3(K(\mathbb{Z}/n,2),\mathbb{Z}) = E_2^{3,0} = 0.$$

We shall need the following later:

Lemma 29.4. *Suppose A is an abelian group. Then there is an isomorphism*

$$H_3(K(A,2),\mathbb{Z}) \cong 0.$$

Proof. Suppose X and Y are connected spaces such that

$$H_i(X,\mathbb{Z}) \cong 0 \cong H_i(Y,\mathbb{Z})$$

for $i = 1,3$. Then a Künneth formula argument (exercise — use Theorem 27.2) shows that $X \times Y$ has the same property.

The spaces $K(\mathbb{Z},2)$ and $K(\mathbb{Z}/n,2)$ are connected and have vanishing integral H_1 and H_3, so the same holds for all $K(A,2)$ if A is finitely generated.

Every abelian group is a filtered colimit of its finitely generated subgroups, and the functors $H_*(\ ,\mathbb{Z})$ preserve filtered colimits. $\square$

Lemma 29.5. *Suppose A is an abelian group and that $n \geq 2$. Then there is an isomorphism*

$$H_{n+1}(K(A,n),\mathbb{Z}) \cong 0.$$

Proof. The proof is by induction on n. The case $n = 2$ follows from Lemma 29.4.

Consider the fibre sequence

$$K(A,n) \to WK(A,n) \to K(A,n+1),$$

with contractible total space $WK(A,n)$.

$E_2^{p,n+1-p} = 0$ for $p < n+1$ (the case $p = 0$ is the inductive assumption). All differentials defined on $E_2^{n+2,0}$ are therefore 0 maps, so

$$H_{n+2}(K(A,n+1),\mathbb{Z}) \cong E_2^{n+2,0} \cong E_\infty^{n+2,0} = 0,$$

since $E_\infty^{n+2,0}$ is a quotient of $H_{n+2}(*) = 0$. $\square$

30 The transgression

Suppose $p : X \to Y$ is a fibration with connected base space Y, and let $F = p^{-1}(*)$ be the fibre of p over some vertex $*$ of Y. Suppose that F is connected.

Consider the bicomplex

$$\bigoplus_{\sigma_0 \to \cdots \to \sigma_n} \mathbb{Z}(p^{-1}(\sigma_0))$$

defining the Serre spectral sequence for $H_*(X, \mathbb{Z})$, and write F_p for its horizontal filtration stages.

$\mathbb{Z}(F)$ is a subobject of F_0.

The differential $d_n : E_n^{n,0} \to E_n^{0,n-1}$ is called the **transgresssion**, and is represented by the picture

$$\begin{tikzcd} & H_{n-1}F_0 \arrow[r, "\cong"] \arrow[d, "i_*"] & H_{n-1}(F_0/F_{-1}) \arrow[r] & E_n^{0,n-1} \\ H_n(F_n/F_{n-1}) \arrow[r, "\partial"'] & H_{n-1}F_{n-1} & & \end{tikzcd}$$

Here,

$$E_n^{n,0} = \partial^{-1}(\mathrm{im}(i_*))/\mathrm{im}(\ker(i_*)),$$
$$E_n^{0,n-1} = H_{n-1}(F_0)/ker(i_*),$$

and $d_n([x]) = [y]$ where $i_*(y) = \partial(x)$.

One says (in old language) that $[x]$ **transgresses** to $[y]$ if $d_n([x]) = [y]$.

Note that

$$E_n^{0,n-1} \cong H_{n-1}(F_0)/\ker(i_*).$$

Given $[x] \in E_n^{n,0}$ and $z \in E_n^{0,n-1}$, then $d_n([x]) = z$ if and only if there is an element $y \in H_{n-1}(F_0)$ such that $i_*(y) = \partial(x)$ and $y \mapsto z$ under the composite

$$H_{n-1}(F_0) \xrightarrow{\cong} H_{n-1}(F_0/F_{-1}) \to E_n^{0,n-1}.$$

The inclusion $j : \mathbb{Z}(F) \subset F_0$ induces a composite map

$$j' : H_{n-1}(F) \to \varinjlim_{\sigma} H_{n-1}(F_\sigma) = E_2^{0,n-1} \twoheadrightarrow E_n^{0,n-1},$$

and j' is surjective since Y is connected (exercise).

Suppose $x \in H_n(F_n/F_{n-1})$ represents an element of $E_n^{n,0}$. Then $\partial(x) = i_*(y)$ for some $y \in H_{n-1}(F_0)$. Write z for the image of y in $E_n^{0,n-1}$.

Choose $v \in H_{n-1}(F)$ such that $j'(v) = z$. Then $j_*(v)$ and y have the same image in $E_n^{0,n-1}$ so $i_* j_*(z) = i_*(y)$ in $H_{n-1}(F_{n-1})$. This means that $\partial(x)$ is in the image of the map $H_{n-1}(F) \to H_{n-1}(F_{n-1})$.

It follows from the comparison of exact sequences

$$\begin{tikzcd}
H_n(F_n) \arrow[r] \arrow[d, "="'] & H_n(F_n/F) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(F) \arrow[r] \arrow[d] & H_{n-1}(F_n) \arrow[d, "="] \\
H_n(F_n) \arrow[r] & H_n(F_n/F_{n-1}) \arrow[r, "\partial"'] & H_{n-1}(F_{n-1}) \arrow[r] & H_{n-1}(F_n)
\end{tikzcd}$$

that x is in the image of the map

$$H_n(F_n/F) \to H_n(F_n/F_{n-1}).$$

In particular, the induced map

$$H_n(F_n/F) \to E_n^{n,0}$$

is surjective.

Thus, $d_n(x) = y$ if and only if there is an element w of $H_n(F_n/F)$ such that w maps to x and y, respectively, under the maps

$$E_n^{n,0} \leftarrow H_n(F_n/F) \xrightarrow{\partial} H_{n-1}(F) \xrightarrow{j'} E_n^{0,n-1}.$$

$H_n(F_n) \to H_n(X)$ is surjective, and $H_{n-1}(F_n) \to H_{n-1}(X)$ is an isomorphism, so a comparison of long exact sequences also shows that the map

$$H_n(F_n/F) \to H_n(X/F)$$

is surjective.

In summary, there is a commutative diagram

$$\begin{tikzcd}
E_2^{n,0} & E_n^{n,0} \arrow[l, hook'] & & H_n(F_n/F) \arrow[ll, two heads] \arrow[r, "\partial"] \arrow[dd, two heads] & H_{n-1}(F) \arrow[r, "j'"] & E_n^{0,n-1} \\
& & H_n(F_n) \arrow[ur] \arrow[dl] \arrow[d, two heads] & & & \\
& E_\infty^{n,0} \arrow[uu, hook] & H_n(X) \arrow[l, two heads] \arrow[r] & H_n(X/F) \arrow[uur, "\partial"'] & &
\end{tikzcd} \tag{6}$$

This diagram is natural in fibrations p.

There is a comparison of Serre spectral sequences arising from the diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "p"'] & Y \arrow[d, "1"] \\ Y \arrow[r, "1"'] & Y \end{tikzcd} \tag{7}$$

All fibres of p are connected, so it follows that the map

$$p_* : E_2^{n,0} \to H_n(Y)$$

is an isomorphism.

Write $F_n(Y)$ and $E_r^{p,q}(Y)$ for the filtration and spectral sequences, respectively, for the total complex associated to the map $1 : Y \to Y$.

There is a commutative diagram

$$\begin{tikzcd} E_2^{n,0} \arrow[d, "\cong"'] & E_n^{n,0} \arrow[l] \arrow[d, "p_*"] \\ E_2^{n,0}(Y) & E_n^{n,0}(Y) \arrow[l, "\cong"] \end{tikzcd}$$

that is induced by the comparison (7).

It follows that $p_* : E_n^{n,0} \to E_n^{n,0}(Y)$ injective, and that $E_n^{n,0}$ is identified with a subobject of $H_n(Y/*)$ via the composite

$$E_n^{n,0} \overset{p_*}{\subset} E_n^{n,0}(Y) \xleftarrow{\cong} E_\infty^{n,0}(Y) \xleftarrow{\cong} H_n(Y) \xrightarrow{\cong} H_n(Y/*).$$

Lemma 30.1. *Suppose $p : X \to Y$ is a fibration with connected base Y and connected fibre F over $* \in Y_0$. Suppose $x \in E_n^{n,0} \subset H_n(Y/*)$, $n \geq 1$, and that $y \in E_n^{0,n-1}$.*

Then $d_n(x) = y$ if and only if there is an element $z \in H_n(X/F)$ such that $p_*(z) = x \in H_n(Y/*)$ and $z \mapsto y$ under the composite

$$H_n(X/F) \xrightarrow{\partial} H_{n-1}(F) \xrightarrow{j'} E_n^{0,n-1}.$$

Proof. Use the fact that the map

$$H_n(F_n/F) \to H_n(X/F)$$

is surjective, and chase elements through the comparison induced by (7) of the diagram (6) with the diagram

$$\begin{tikzcd}
E_n^{n,0}(Y) & & H_n(F_n(Y)/*) \arrow[ll, two heads] \arrow[dd, two heads] \\
& H_n(F_n(Y)) \arrow[ur, "\cong"] \arrow[dl] \arrow[d, two heads] & \\
E_\infty^{n,0}(Y) \arrow[uu, "\cong"] & H_n(Y) \arrow[l, "\cong"'] \arrow[r, "\cong"'] & H_n(Y/*)
\end{tikzcd}$$

to prove the result. $\square$

31 The path-loop fibre sequence

We will use the model structure for the category $s_*\mathbf{Set}$ of pointed simplicial sets (aka. pointed spaces).

This model structure is easily constructed, since $s_*\mathbf{Set} = */s\mathbf{Set}$ is a slice category: a pointed simplicial set is a simplicial set map $* \to X$, and a pointed map is a diagram

$$\begin{tikzcd} & X \arrow[dd, "g"] \\ * \arrow[ur, "x"] \arrow[dr, "y"'] & \\ & Y \end{tikzcd} \tag{8}$$

In general, if $\mathscr{M}$ is a closed model category, with object A, then the slice category $A/\mathscr{M}$ has a closed model structure, for which a morphism

$$\begin{tikzcd} & X \arrow[dd, "f"] \\ A \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd}$$

is a weak equivalence (resp. fibration, cofibration) if the map $f : X \to Y$ is a weak equivalence (resp. fibration, cofibration).

Exercise: 1) Verify the existence of the model structure for the slice category $A/\mathscr{M}$.

2) The dual assertion is the existence of a model structure for the category $\mathscr{M}/B$ for all objects $B \in \mathscr{M}$. Formulate the result.

Warning: A map $g : X \to Y$ of pointed simplicial sets is a weak equivalence if and only if it induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and isomorphisms

$$\pi_n(X,z) \cong \pi_n(Y,g(z))$$

for **all** base points $z \in X_0$.

The model structure for $s_*\mathbf{Set}$ is a closed simplicial model structure, with function complex $\mathbf{hom}_*(X,Y)$ defined by

$$\mathbf{hom}_*(X,Y)_n = \hom(X \wedge \Delta^n_+, Y),$$

where

$$\Delta^n_+ = \Delta^n \sqcup \{*\}$$

is the simplex Δ^n with a disjoint base point.

The **smash product** of pointed spaces X, Y is defined by

$$X \wedge Y = \frac{X \times Y}{X \vee Y},$$

where the **wedge** $X \vee Y$ or **one-point union** of X and Y is the coproduct of X and Y in the pointed category.

The **loop space** ΩX of a pointed Kan complex X is the pointed function complex

$$\Omega X = \mathbf{hom}_*(S^1, X),$$

where $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle with the obvious choice of base point.

Write Δ^1_* for the simplex Δ^1, pointed by the vertex 1, and let

$$S^0 = \partial\Delta^1 = \{0,1\},$$

pointed by 1. Then the cofibre sequence

$$S^0 \subset \Delta^1_* \xrightarrow{\pi} S^1 \tag{9}$$

of pointed spaces induces a fibre sequence

$$\Omega X = \mathbf{hom}_*(S^1, X) \to \mathbf{hom}_*(\Delta^1_*, X) \xrightarrow{p} \mathbf{hom}_*(S^0, X) \cong X \tag{10}$$

provided X is fibrant.

The pointed inclusion $\{1\} \subset \Delta^1_*$ is a weak equivalence, so the space

$$PX = \mathbf{hom}_*(\Delta^1_*, X)$$

is contractible if X is fibrant.

The simplicial set PX is the **pointed path space** for X, and the fibre sequence (10) is the **path-loop fibre sequence** for X.

It follows that, if X is fibrant and $*$ denotes the base point for all spaces in the fibre sequence (10), then there are isomorphisms

$$\pi_n(X,*) \cong \pi_{n-1}(\Omega X,*)$$

for $n \geq 2$ and a bijection

$$\pi_1(X,*) \cong \pi_0(\Omega X).$$

Dually, one can take a pointed space Y and smash with the cofibre sequence (9) to form a natural cofibre sequence

$$Y \cong S^0 \wedge Y \to \Delta^1_* \wedge Y \to S^1 \wedge Y.$$

The space $\Delta^1_* \wedge Y$ is contractible (exercise) — it is the **pointed cone** for Y, and one writes

$$CX = X \wedge \Delta^1_*.$$

One often writes

$$\Sigma X = X \wedge S^1.$$

This object is called the **suspension** of X, although saying this is a bit dangerous because there's more than one suspension construction for simplicial sets — see [1, III.5], [2, 4.4].

The suspension functor is left adjoint to the loop functor. More generally, there is a natural isomorphism

$$\mathbf{hom}_*(X \wedge K, Y) \cong \mathbf{hom}_*(K, \mathbf{hom}_*(X, Y))$$

of pointed simplicial sets (exercise).

Lemma 31.1. *Suppose $f : X \to \Omega Y$ is a pointed map, and let $f' : \Sigma X \to Y$ denote its adjoint. Then there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & CX \arrow[r] \arrow[d, "h(f)"] & \Sigma X \arrow[d, "f'"] \\ \Omega Y \arrow[r] & PY \arrow[r, "p"'] & Y \end{tikzcd}$$

Proof. We'll say how $h(f)$ is defined. Checking that the diagram commutes is an exercise.

The pointed map (contracting homotopy)

$$h : \Delta^1_* \wedge \Delta^1_* \to \Delta^1_*$$

is defined by the relations

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

Then the map

$$h(f) : X \wedge \Delta^1_* \to \mathbf{hom}_*(\Delta^1_*, Y)$$

is adjoint to the composite

$$X \wedge \Delta^1_* \wedge \Delta^1_* \xrightarrow{1 \wedge h} X \wedge \Delta^1_* \xrightarrow{f \wedge 1} \mathbf{hom}_*(S^1, Y) \wedge \Delta^1_*$$
$$\xrightarrow{1 \wedge \pi} \mathbf{hom}_*(S^1, Y) \wedge S^1 \xrightarrow{ev} Y.$$

$\square$

Lemma 31.2. *Suppose Y is a pointed Kan complex which is n-connected for $n \geq 1$.*

Then the transgression d_i induces isomorphisms

$$H_i(Y) \cong H_{i-1}(\Omega Y)$$

for $2 \leq i \leq 2n$.

Proof. Y is at least simply connected, and the homotopy groups $\pi_i(Y,*)$ vanish for $i \leq n$.

The Serre spectral sequence for the path-loop fibration for Y has the form

$$E_2^{p,q} = H_p(Y, H_q(\Omega Y)) \Rightarrow H_{p+q}(PY).$$

The space ΩY is $(n-1)$-connected, so $E_2^{p,q} = 0$ for $0 < q \leq n-1$ or $0 < p \leq n$.

Thus, the first possible non-trivial group off the edges in the E_2-term is in bidegree $(n+1, n)$.

All differentials reduce total degree by 1 so

- the differentials $d_r : E_r^{i,0} \to E_r^{i-r,r-1}$ vanish for $i \leq 2n$ and $r < i$,
- the differentials $d_r : E_r^{r,i-r} \to E_r^{0,i-1}$ vanish for $r < i$ and $i \leq 2n$.

It follows that there is an exact sequence

$$0 \to E_\infty^{i,0} \to E_i^{i,0} \xrightarrow{d_i} E_i^{0,i-1} \to E_\infty^{0,i-1} \to 0$$

for $0 < i \leq 2n$, and

$$\begin{aligned} E_i^{i,0} &\cong E_2^{i,0} \cong H_i(Y), \text{ and} \\ E_i^{0,i-1} &\cong E_2^{0,i-1} \cong H_{i-1}(\Omega Y) \end{aligned}$$

for $0 < i \leq 2n$.

All groups $E_\infty^{p,q}$ vanish for $(p,q) \neq (0,0)$. $\square$

Lemma 31.3. *Suppose $f : X \to \Omega Y$ is a map of pointed simplicial sets, where Y is fibrant. Suppose Y is n-connected, where $n \geq 1$.*

Then for $2 \leq i \leq 2n$ there is a commutative diagram

$$\begin{tikzcd} H_i(\Sigma X) \arrow[r, "\partial", "\cong"'] \arrow[d, "f'_*"'] & H_{i-1}(X) \arrow[d, "f_*"] \\ H_i(Y) \arrow[r, "\cong", "d_i"'] & H_{i-1}(\Omega Y) \end{tikzcd} \tag{11}$$

where $f' : \Sigma X \to Y$ is the adjoint of f.

Proof. From the diagram of Lemma 31.1, there is a commutative diagram

$$\begin{tikzcd} H_i(\Sigma X/*) \arrow[d, "f'_*"'] & H_i(CX/X) \arrow[l, "\cong"'] \arrow[r, "\partial"] \arrow[d, "h(f')_*"] & H_{i-1}(X) \arrow[d, "f_*"] \\ H_i(Y/*) & H_i(PY/\Omega Y) \arrow[l, "p_*"] \arrow[r, "\partial"'] & H_{i-1}(\Omega Y) \end{tikzcd} \tag{12}$$

After the standard identifications

$$E_i^{i,0} \cong H_i(Y/*), \text{ and}$$
$$E_i^{0,i-1} \cong H_{i-1}(\Omega Y).$$

and given $x \in H_i(Y/*)$ and $y \in H_{i-1}(\Omega Y)$, Lemma 30.1 implies that $d_i(x) = y$ if there is a $z \in H_i(PY/\Omega Y)$ such that $p_*(z) = x$ and $\partial(z) = y$.

This is true for $f'_*(v)$ and $f_*(\partial(v))$ for $v \in H_i(\Sigma X)$, given the isomorphism in the diagram (12).

The map d_i is an isomorphism for $2 \leq i \leq 2n$ by Lemma 31.2. ∂ is always an isomorphism. $\square$

Corollary 31.4. *Suppose Y is an n-connected pointed Kan complex with $n \geq 1$.*

Then there is a commutative diagram

$$\begin{tikzcd} H_i(\Sigma\Omega Y) \arrow[r, "\partial", "\cong"'] \arrow[d, "\varepsilon_*"'] & H_{i-1}(\Omega Y) \\ H_i(Y) \arrow[ur, "d_i"', "\cong"] & \end{tikzcd}$$

for $2 \leq i \leq 2n$.

The adjunction map $\varepsilon : \Sigma\Omega Y \to Y$ induces an isomorphism $H_i(\Sigma\Omega Y) \cong H_i(Y)$ for $2 \leq i \leq 2n$.

Proof. This is the case $f = 1_{\Omega Y}$ of Lemma 31.3. $\square$

If Y is a 1-connected pointed Kan complex, then ΩY is connected.

We can say more about the map ε_*. The following result implies that $\Sigma\Omega Y$ is simply connected, so the adjunction map ε in the statement of Corollary 31.4 induces isomorphisms

$$\varepsilon_* : H_i(\Sigma\Omega Y) \xrightarrow{\cong} H_i(Y)$$

for $0 \leq i \leq 2n$.

Lemma 31.5. *Suppose X is a connected pointed simplicial set.*

Then the fundamental groupoid $\pi(\Sigma X)$ is a trivial groupoid.

Proof. The proof is an exercise.

Use the assumption that X is connected to show that the functor $\pi(CX) \to \pi(\Sigma X)$ is full. $\square$

References

[1] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[2] J. F. Jardine. *Generalized Étale Cohomology Theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[3] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.