Lecture 12: Cohomology: an introduction

Contents

35 Cohomology

Suppose that $C \in Ch_+$ is an ordinary chain complex, and that A is an abelian group.

There is a *cochain complex* $\hom(C,A)$ with

$$\hom(C,A)^n = \hom(C_n,A)$$

and *coboundary*

$$\delta : \hom(C_n,A) \to \hom(C_{n+1},A)$$

defined by precomposition with $\partial : C_{n+1} \to C_n$.

Generally, a **cochain complex** is an unbounded complex which is concentrated in negative degrees. See Section 1.

We use classical notation for $\hom(C,A)$: the corresponding complex in negative degrees is specified by

$$\hom(C,A)_{-n} = \hom(C_n,A).$$

The **cohomology group** $H^n \hom(C,A)$ is specified by

$$H^n \hom(C,A) := \frac{\ker(\delta : \hom(C_n,A) \to \hom(C_{n+1},A))}{\operatorname{im}(\delta : \hom(C_{n-1},A) \to \hom(C_n,A))}.$$

This group coincides with the group $H_{-n} \hom(C,A)$ for the complex in negative degrees.

Exercise: Show that there is a natural isomorphism

$$H^n \hom(C,A) \cong \pi(C,A(n))$$

where $A(n)$ is the chain complex consisting of the group A concentrated in degree n, and $\pi(C,A(n))$ is chain homotopy classes of maps.

Example: If X is a space, then the cohomology group $H^n(X,A)$ is defined by

$$H^n(X,A) = H^n \hom(\mathbb{Z}(X),A) \cong \pi(\mathbb{Z}(X),A(n)),$$

where $\mathbb{Z}(X)$ is the Moore complex for the free simplicial abelian group $\mathbb{Z}(X)$ on X.

Here is why the classical definition of $H^n(X,A)$ is not silly: all ordinary chain complexes are fibrant, and the Moore complex $\mathbb{Z}(X)$ is free in each degree, hence cofibrant, and so there is an isomorphism

$$\pi(\mathbb{Z}(X),A(n)) \cong [\mathbb{Z}(X),A(n)],$$

where the square brackets determine morphisms in the homotopy category for the standard model structure on Ch_+ (Theorem 3.1).

The normalized chain complex $N\mathbb{Z}(X)$ is naturally weakly equivalent to the Moore complex $\mathbb{Z}(X)$, and there are natural isomorphisms

$$\begin{aligned}
[\mathbb{Z}(X),A(n)] &\cong [N\mathbb{Z}(X),A(n)] \\
&\cong [\mathbb{Z}(X),K(A,n)] \quad \text{(Dold-Kan correspondence)} \\
&\cong [X,K(A,n)] \quad \text{(Quillen adjunction)}
\end{aligned}$$

Here, $[X,K(A,n)]$ is morphisms in the homotopy category for simplicial sets. We have proved the following:

Theorem 35.1. *There is a natural isomorphism*

$$H^n(X,A) \cong [X,K(A,n)]$$

for all simplicial sets X and abelian groups A.

In other words, $H^n(X,A)$ is representable by the Eilenberg-Mac Lane space $K(A,n)$ in the homotopy category.

Suppose that C is a chain complex and A is an abelian group. Define the **cohomology groups** (or hypercohomology groups) $H^n(C,A)$ of C with coefficients in A by

$$H^n(C,A) = [C,A(n)].$$

This is the derived functor definition of cohomology.

Example: Suppose that A and B are abelian groups. We compute the groups $H^n(A(0),B) = [A(0),B(n)]$. This is done by replacing $A(0)$ by a cofibrant model. There is a short exact sequence

$$0 \to F_1 \to F_0 \to A \to 0$$

with F_i free abelian. The chain complex F_* given by

$$\cdots \to 0 \to 0 \to F_1 \to F_0$$

is cofibrant, and the chain map $F_* \to A(0)$ is a weak equivalence, hence a cofibrant replacement for the complex $A(0)$.

It follows that there are isomorphisms

$$[A(0),B(n)] \cong [F_*,B(n)] \cong \pi(F_*,B(n)) = H^n \hom(F_*,A),$$

and there is an exact sequence

$$\begin{aligned} 0 \to H^0 \hom(F_*,B) \to \hom(F_0,B) \to \hom(F_1,B) \\ \to H^1 \hom(F_*,B) \to 0. \end{aligned}$$

It follows that

$$[A(0),B(n)] = H^n \hom(F_*,B) = \begin{cases} \hom(A,B) & \text{if } n = 0, \\ \operatorname{Ext}^1(A,B) & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Similarly, there are isomorphisms

$$[A(p), B(n)] = \begin{cases} \hom(A,B) & \text{if } n = p, \\ \operatorname{Ext}^1(A,B) & \text{if } n = p+1, \\ 0 & \text{if } n > p+1 \text{ or } n < p. \end{cases}$$

Most generally, for ordinary chain complexes, we have the following:

Theorem 35.2. *Suppose that C is a chain complex, and B is an abelian group.*

There is a short exact sequence

$$0 \to \operatorname{Ext}^1(H_{n-1}(C), B) \to H^n(C,B) \xrightarrow{p} \hom(H_n(C), B) \to 0. \tag{1}$$

The map p is natural in C and B. This sequence is split, with a non-natural splitting.

Theorem 35.2 is the **universal coefficients theorem** for cohomology.

Proof. Let $Z_p = \ker(\partial : C_p \to C_{p-1})$. Pick a surjective homomorphism, $F_0^p \to Z_p$ with F_0^p free, and F_1^p be the kernel of the (surjective) composite

$$F_0^p \to Z_p \to H_p(C).$$

Then F_1^p is free, and there is a map $F_1^p \to C_{p+1}$

such that the diagram

$$\begin{tikzcd} F_1^p \arrow[rr] \arrow[d] & & C_{p+1} \arrow[d, "\partial"] \\ F_0^p \arrow[r] & Z_p \arrow[r] & C_p \end{tikzcd}$$

commutes. Write ϕ_p for the resulting chain map $F_*^p[-p] \to C$. Then the sum

$$\phi : \bigoplus_{p \geq 0} F_*^p[-p] \to C$$

(ϕ_n on the n^{th} summand) is a cofibrant replacement for the complex C.

At the same time, we have cofibrant resolutions $F_*^p[-p] \to H_p(C)(p)$, for $p \geq 0$.

It follows that there are isomorphisms

$$\begin{aligned} [C, B(n)] &\cong [\bigoplus_{p \geq 0} H_p(C)(p), B(n)] \\ &\cong \prod_{p \geq 0} [H_p(C)(p), B(n)] \\ &\cong \hom(H_n(C), B) \oplus \mathrm{Ext}^1(H_{n-1}(C), B). \end{aligned}$$

The induced map $p : [C, B(n)] \to \hom(H_p(C), B)$ is defined by restricting a chain map $F \to B(n)$ to the group homomorphism $Z_n(F) \subset F_n \to B$, where $F \to C$ is a cofibrant model of C. $\square$

Recall that there are various models for the space $K(A,n)$ in simplicial abelian groups. These include the object $\Gamma A(n)$ arising from the Dold-Kan correspondence, and the space

$$A \otimes S^n \cong A \otimes (S^1)^{\otimes n}$$

where

$$S^n = (S^1)^{\wedge n} = S^1 \wedge \cdots \wedge S^1 \ \ (n \text{ smash factors}).$$

In general, if K is a pointed simplicial set and A is a simplicial abelian group, we write

$$A \otimes K = A \otimes \tilde{\mathbb{Z}}(K),$$

where $\tilde{\mathbb{Z}}(K)$ is the reduced Moore complex for K.

Suppose given a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \tag{2}$$

of simplicial abelian groups.

The diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \otimes \Delta^1_* \arrow[d, "0"] \\ B \arrow[r, "p"'] & C \end{tikzcd}$$

is homotopy cocartesian, so there is a natural map $\delta : C \to A \otimes S^1$ in the homotopy category. Proceeding inductively gives the **Puppe sequence**

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \xrightarrow{\delta} A \otimes S^1 \xrightarrow{i \otimes 1} B \otimes S^1 \xrightarrow{p \otimes 1} \dots \quad (3)$$

and a long exact sequence

$$[E,A] \to [E,B] \to [E,C] \xrightarrow{\delta} [E, A \otimes S^1] \to [E, B \otimes S^1] \to \dots$$

or equivalently

$$H^0(E,A) \to H^0(E,B) \to H^0(E,C) \xrightarrow{\delta} H^1(E,A) \to H^1(E,B) \to \dots \quad (4)$$

in cohomology, for arbitrary simplicial abelian groups (or chain complexes) E.

The morphisms δ is the long exact sequence (4) are called **boundary** maps.

Specializing to $E = \mathbb{Z}(X)$ for a space X and a short exact sequence of groups (2) gives the standard long exact sequence

$$H^0(X,A) \to H^0(X,B) \to H^0(X,C) \xrightarrow{\delta} H^1(X,A) \to H^1(X,B) \to \dots \quad (5)$$

in cohomology for the space X.

There are other ways of constructing the long exact sequence (5) — exercise.

Lemma 36.1. *The twist automorphism*

$$\tau : S^1 \wedge S^1 \xrightarrow{\cong} S^1 \wedge S^1, \;\; x \wedge y \mapsto y \wedge x.$$

induces

$$\tau_* = \times(-1) : H_2(S^1 \wedge S^1, \mathbb{Z}) \to H_2(S^1 \wedge S^1, \mathbb{Z}).$$

Proof. There are two non-degenerate 2-simplices σ_1, σ_2 in $S^1 \wedge S^1$ and a single non-degenerate 1-simplex $\gamma = d_1\sigma_1 = d_1\sigma_2$.

It follows that the normalized chain complex $N\mathbb{Z}(S^1 \wedge S^1)$ has the form

$$\cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\nabla} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

where $\nabla(m,n) = m+n$. Thus, $H_2(S^1 \wedge S^1, \mathbb{Z}) \cong \mathbb{Z}$, generated by $\sigma_1 - \sigma_2$.

The twist τ satisfies $\tau(\sigma_1) = \sigma_2$ and fixes their common face γ.

Thus, $\tau_*(\sigma_1 - \sigma_2) = \sigma_2 - \sigma_1$. □

Corollary 36.2. *Suppose that $\sigma \in \Sigma_n$ acts on $(S^1)^{\wedge n}$ by shuffling smash factors.*

Then the induced automorphism

$$\sigma_* : H_n((S^1)^{\wedge n}, \mathbb{Z}) \to H_n((S^1)^{\wedge n}, \mathbb{Z}) \cong \mathbb{Z}$$

is multiplication by the sign of σ.

Explicitly, the action of σ on $(S^1)^{\wedge n}$ is specified by

$$\sigma(x_1 \wedge \cdots \wedge x_n) = x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}.$$

Suppose that A and B are abelian groups. There are natural isomorphisms of simplicial abelian groups

$$K(A,n) \otimes K(B,m) \xrightarrow{\cong} A \otimes B \otimes (S^1)^{\otimes(n+m)} = K(A \otimes B, n+m)$$

where the displayed isomorphism

$$(S^1)^{\otimes n} \otimes A \otimes (S^1)^{\otimes m} \otimes B \xrightarrow{\cong} (S^1)^{\otimes n} \otimes (S^1)^{\otimes m} \otimes A \otimes B$$

is defined by permuting the middle tensor factors.

Suppose that X and Y are simplicial sets, and suppose that $f : X \to K(A,n)$ and $g : Y \to K(B,m)$ are simplicial set maps.

There is a natural map

$$X \times Y \xrightarrow{\eta} \mathbb{Z}(X) \otimes \mathbb{Z}(Y),$$

which is defined by $(x,y) \mapsto x \otimes y$.

The composite

$$X \times Y \xrightarrow{\eta} \mathbb{Z}(X) \otimes \mathbb{Z}(Y) \xrightarrow{f_* \otimes g_*} K(A,n) \otimes K(B,m) \cong K(A \otimes B, n+m)$$

represents an element of $H^{n+m}(X \times Y, A \otimes B)$.

Warning: The isomorphism above has the form

$$a \otimes (x_1 \wedge \cdots \wedge x_n) \otimes b \otimes (y_1 \wedge \cdots \wedge y_m)$$
$$\mapsto a \otimes b \otimes (x_1 \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \wedge y_m).$$

Do **not** shuffle smash factors.

We have defined a pairing

$$\cup : H^n(X,A) \otimes H^m(Y,B) \to H^{n+m}(X \times Y, A \otimes B),$$

called the **external cup product**.

If R is a unitary ring, then the ring multiplication $m : R \otimes R \to R$ and the diagonal $\Delta : X \to X \times X$ together induce a composite

$$H^n(X,R) \otimes H^m(X,R) \xrightarrow{\cup} H^{n+m}(X \times X, R \otimes R) \xrightarrow{\Delta^* \cdot m_*} H^{n+m}(X,R)$$

which is the cup product

$$\cup : H^n(X,R) \otimes H^m(X,R) \to H^{n+m}(X,R)$$

for $H^*(X,R)$.

Exercise: Show that the cup product gives the cohomology $H^*(X,R)$ the structure of a graded commutative ring with identity. This ring structure is natural in spaces X and rings R.

The graded commutativity follows from Corollary 36.2.

Suppose that we have a short exact sequence of simplicial abelian groups

$$0 \to A \to B \to C \to 0$$

and that D is a flat simplicial abelian group in the sense that the functor $? \otimes D$ is exact. The sequence

$$0 \to A \otimes D \xrightarrow{i\otimes 1} B \otimes D \xrightarrow{p\otimes 1} C \otimes D \xrightarrow{\delta\otimes 1} A \otimes S^1 \otimes D \\ \xrightarrow{i\otimes 1} B \otimes S^1 \otimes D \xrightarrow{p\otimes 1} \dots$$

is equivalent to the Puppe sequence for the short exact sequence

$$0 \to A \otimes D \to B \otimes D \to C \otimes D \to 0$$

It follows that there is a commutative diagram

$$\begin{tikzcd}
[E,C] \otimes [F,D] \arrow[r, "\cup"] \arrow[d, "\delta\otimes 1"'] & {[E \otimes F, C \otimes D]} \arrow[d, "\delta"] \\
{[E, A \otimes S^1] \otimes [F,D]} \arrow[r, "\cup"'] & {[E \otimes F, A \otimes D \otimes S^1]}
\end{tikzcd}$$

In particular, if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules and X is a space, then there is a commutative diagram

$$\begin{tikzcd}
H^p(X,C) \otimes H^q(X,R) \arrow[r, "\cup"] \arrow[d, "\delta\otimes 1"'] & H^{p+q}(X,C) \arrow[d, "\delta"] \\
H^{p+1}(X,A) \otimes H^q(X,R) \arrow[r, "\cup"'] & H^{p+q+1}(X,A)
\end{tikzcd} \tag{6}$$

It an exercise to show that the diagram

$$\begin{tikzcd} H^q(X,R)\otimes H^p(X,C) \arrow[r, "\cup"] \arrow[d, "1\otimes\delta"'] & H^{q+p}(X,C) \arrow[d, "(-1)^q\delta"] \\ H^q(X,R)\otimes H^{p+1}(X,A) \arrow[r, "\cup"'] & H^{q+p+1}(X,A) \end{tikzcd} \tag{7}$$

commutes.

The diagrams (6) and (7) are cup product formulas for the boundary homomorphism.

37 Cohomology of cyclic groups

Suppose that ℓ is a prime $\neq 2$. What follows is directly applicable to cyclic groups of ℓ-primary roots of unity in fields.

We shall sketch the proof of the following:

Theorem 37.1. *There is a ring isomorphism*

$$H^*(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[x]\otimes\Lambda(y)$$

where $|x| = 2$ *and* $|y| = 1$.

We write $|z| = n$ for $z \in H^n(X,A)$. $|z|$ is the **degree** of z.

In the statement of Theorem 37.1, $x \in H^2(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell)$ and $y \in H^1(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell)$.

$\mathbb{Z}/\ell[x]$ is a graded polynomial ring with generator x in degree 2, and $\Lambda(y)$ is an exterior algebra with generator y in degree 1.

Fact: If $z \in H^{2k+1}(X, \mathbb{Z}/\ell)$ and $\ell \neq 2$, then

$$z \cdot z = (-1)^{(2k+1)(2k+1)} z \cdot z = (-1) z \cdot z,$$

so that $2(z \cdot z) = 0$, and $z \cdot z = 0$.

We know, from the Example at the end of Section 25, that there are isomorphisms

$$H_p(B\mathbb{Z}/\ell^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ \mathbb{Z}/\ell^n & \text{if } p = 2k+1,\ k \geq 0, \\ 0 & \text{if } p = 2k,\ k > 0. \end{cases}$$

It follows (exercise) that there are isomorphisms

$$H_p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell, \text{ for } p \geq 0.$$

There is an isomorphism

$$H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) \cong \hom(H_p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell), \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$$

for $p \geq 0$ (Theorem 35.2).

1) $x \in H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$ is dual to the generator of the ℓ-torsion subgroup of

$$\mathbb{Z}/\ell^n = H_1(B\mathbb{Z}/\ell^n, \mathbb{Z}).$$

2) $y \in H^1(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$ is dual to the generator of

$$\mathbb{Z}/\ell \cong \mathbb{Z}/\ell^n \otimes \mathbb{Z}/\ell = H_1(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell).$$

Here's an integral coefficients calculation:

Theorem 37.2. *There is a ring isomorphism*

$$H^*(B\mathbb{Z}/\ell^n,\mathbb{Z}) \cong \mathbb{Z}[x]/(\ell^n \cdot x)$$

where $|x| = 2$.

This result appears in a book of Snaith, [1]. The argument uses explicit cocycles, with the Alexander-Whitney map ((7) of Section 26).

We can verify the underlying additive statement, namely that

$$H^p(B\mathbb{Z}/\ell^n,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ \mathbb{Z}/\ell^n & \text{if } p = 2k,\ k > 0, \\ 0 & \text{if } p \text{ odd} \end{cases}$$

Apply $\hom(\ ,\mathbb{Z})$ to the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\ell^n} \mathbb{Z} \to \mathbb{Z}/\ell^n \to 0$$

to get the exact sequence

$$0 \to \hom(\mathbb{Z}/\ell^n,\mathbb{Z}) \to \mathbb{Z} \xrightarrow{\ell^n} \mathbb{Z} \to \operatorname{Ext}^1(\mathbb{Z}/\ell^n,\mathbb{Z}) \to 0$$

to show that $\hom(\mathbb{Z}/\ell^n,\mathbb{Z}) = 0$ (we knew this) and $\operatorname{Ext}^1(\mathbb{Z}/\ell^n,\mathbb{Z}) \cong \mathbb{Z}/\ell^n$.

Then

$$H^{2k}(B\mathbb{Z}/\ell^n,\mathbb{Z}) \cong \operatorname{Ext}^1(H_{2k-1}(B\mathbb{Z}/\ell^n,\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}/\ell^n$$

for $k > 0$ and

$H^{2k+1}(B\mathbb{Z}/\ell^n,\mathbb{Z}) \cong \hom(H_{2k+1}(B\mathbb{Z}/\ell^n,\mathbb{Z}),\mathbb{Z}) = 0$

for $k \geq 0$.

Proof of Theorem 37.1. The exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times\ell} \mathbb{Z} \to \mathbb{Z}/\ell \to 0$$

is an exact sequence of $\mathbb{Z}$-modules, so that the Puppe sequence

$$0 \to K(\mathbb{Z},0) \xrightarrow{\times\ell} K(\mathbb{Z},0) \to K(\mathbb{Z}/\ell,0) \xrightarrow{\delta} K(\mathbb{Z},1) \xrightarrow{\times\ell} K(\mathbb{Z},1) \to \dots$$

has an action by $K(\mathbb{Z},2)$.

It follows that there are commutative diagrams

$$\begin{tikzcd}
H^p(B\mathbb{Z}/\ell^n,\mathbb{Z}) \arrow[r, "\times\ell"] \arrow[d, "\cdot x"', "\cong"] & H^p(B\mathbb{Z}/\ell^n,\mathbb{Z}) \arrow[r] \arrow[d, "\cong"', "\cdot x"] & H^p(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell) \arrow[d, "\cdot x"] \\
H^{p+2}(B\mathbb{Z}/\ell^n,\mathbb{Z}) \arrow[r, "\times\ell"'] & H^{p+2}(B\mathbb{Z}/\ell^n,\mathbb{Z}) \arrow[r] & H^{p+2}(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^p(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell) \arrow[r, "\delta"] \arrow[d, "\cdot x"'] & H^{p+1}(B\mathbb{Z}/\ell^n,\mathbb{Z}) \arrow[r, "\times\ell"] \arrow[d, "\cdot x"', "\cong"] & H^{p+1}(B\mathbb{Z}/\ell^n,\mathbb{Z}) \arrow[d, "\cong"', "\cdot x"] \\
H^{p+2}(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell) \arrow[r, "\delta"'] & H^{p+3}(B\mathbb{Z}/\ell^n,\mathbb{Z}) \arrow[r, "\times\ell"'] & H^{p+3}(B\mathbb{Z}/\ell^n,\mathbb{Z})
\end{tikzcd}$$

for $p > 0$.

Thus, the cup product map

$$\cdot x : H^p(B\mathbb{Z}/\ell^n,\mathbb{Z}) \to H^{p+2}(B\mathbb{Z}/\ell^n,\mathbb{Z}/\ell)$$

is an isomorphism for all p.

Finally, the map

$$H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}) \to H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$$

is surjective, so the generator $x \in H^2(B\mathbb{Z}/\ell^n, \mathbb{Z})$ maps to a generator x of $H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$.

The ring homomorphism

$$\mathbb{Z}/\ell[x] \otimes \Lambda(y) \to H^*(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$$

defined by $x \in H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$ and a generator $y \in H^1(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$ is then an isomorphism of $\mathbb{Z}/\ell$-vector spaces in all degrees. $\square$

References

[1] Victor P. Snaith. *Topological methods in Galois representation theory*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1989. A Wiley-Interscience Publication.