Lecture 14: Basic properties

Contents

41 Suspensions and shift

The suspension $X \wedge S^1$ and the fake suspension ΣX of a spectrum X were defined in Section 38 — the constructions differ by a non-trivial twist of bonding maps.

The **loop** spectrum for X is the function complex object

$$\mathbf{hom}_*(S^1, X).$$

There is a natural bijection

$$\hom(X \wedge S^1, Y) \cong \hom(X, \mathbf{hom}_*(S^1, Y))$$

so that the suspension and loop functors are adjoint.

The **fake loop** spectrum ΩY for a spectrum Y consists of the pointed spaces ΩY^n, $n \geq 0$, with adjoint bonding maps

$$\Omega\sigma_* : \Omega Y^n \to \Omega^2 Y^{n+1}.$$

There is a natural bijection

$$\hom(\Sigma X, Y) \cong \hom(X, \Omega Y),$$

so the fake suspension functor is left adjoint to fake loops.

The adjoint bonding maps $\sigma_* : Y^n \to \Omega Y^{n+1}$ define a natural map

$$\gamma : Y \to \Omega Y[1].$$

for spectra Y.

The map $\omega : Y \to \Omega^\infty Y$ of Section 40 is the filtered colimit of the maps

$$Y \xrightarrow{\gamma} \Omega Y[1] \xrightarrow{\Omega\gamma[1]} \Omega^2 Y[2] \xrightarrow{\Omega^2\gamma[2]} \dots$$

Recall the statement of the Freudenthal suspension theorem (Theorem 34.2):

Theorem 41.1. *Suppose that a pointed space X is n-connected, where $n \geq 0$.*

Then the homotopy fibre F of the canonical map $\eta : X \to \Omega(X \wedge S^1)$ is $2n$-connected.

In particular, the suspension homomorphism

$$\pi_i X \to \pi_i(\Omega(X \wedge S^1)) \cong \pi_{i+1}(X \wedge S^1)$$

is an isomorphism for $i \leq 2n$ and is an epimorphism for $i = 2n+1$, provided that X is n-connected.

In general (ie. with no connectivity assumptions on Y), the space $S^n \wedge Y$ is $(n-1)$-connected, by Lemma 31.5 and Corollary 34.1.

Thus, the suspension homomorphism

$$\pi_{i+k}(S^{n+k} \wedge Y) \to \pi_{i+k+1}(S^{n+k+1} \wedge Y)$$

is an isomorphism if $i \leq 2n-2+k$, and it follows that the map

$$\pi_i(S^n \wedge Y) \to \pi^s_{i-n}(\Sigma^\infty Y)$$

is an isomorphism for $i \leq 2(n-1)$.

Here's an easy observation:

Lemma 41.2. *The natural map $\gamma : X \to \Omega X[1]$ is a stable equivalence if X is strictly fibrant.*

Proof. This is a cofinality argument, which uses the fact that $\Omega^\infty X$ is the filtered colimit of the system

$$X \to \Omega X[1] \to \Omega^2 X[2] \to \dots$$

□

Lemma 41.3. *Suppose that Y is a pointed space. Then the canonical map*

$$\eta : \Sigma^{\infty} Y \to \Omega\Sigma(\Sigma^{\infty} Y)$$

is a stable homotopy equivalence.

Proof. The map

$$\pi_k(S^n \wedge Y) \to \pi^s_{k-n}(\Sigma^{\infty} Y)$$

is an isomorphism for $k \leq 2(n-1)$.

Similarly (exercise), the map

$$\pi_k(\Omega(S^{n+1} \wedge X)) \to \pi^s_{k-n}(\Omega\Sigma(\Sigma^{\infty} X))$$

is an isomorphism for $k+1 \leq 2n$ or $k \leq 2n-1$.
There is a commutative diagram

$$\begin{tikzcd}
\pi_k(S^n \wedge Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi^s_{k-n}(\Sigma^{\infty} Y) \arrow[d] \\
\pi_k(\Omega(S^{n+1} \wedge Y)) \arrow[r, "\cong"'] & \pi^s_{k-n}(\Omega\Sigma(\Sigma^{\infty} Y))
\end{tikzcd}$$

in which the indicated maps are isomorphisms for $k \leq 2(n-1)$.

It follows that the map

$$\pi^s_p(\Sigma^{\infty} Y) \to \pi^s_p(\Omega\Sigma(\Sigma^{\infty} Y))$$

is an isomorphism for $p \leq n-2$.

Finish by letting n vary. $\square$

Remark: What we've really shown in Lemma 41.3 is that the composite

$$\Sigma^\infty X \xrightarrow{\eta} \Omega\Sigma(\Sigma^\infty X) \xrightarrow{\Omega j} \Omega F(\Sigma(\Sigma^\infty X))$$

is a natural stable equivalence.

Lemma 41.4. *Suppose that Y is a spectrum. Then the composite*

$$Y \xrightarrow{\eta} \Omega\Sigma Y \xrightarrow{\Omega j} \Omega F(\Sigma Y)$$

is a stable equivalence.

Proof. We show that the maps

$$L_nY \xrightarrow{\eta} \Omega\Sigma L_nY \xrightarrow{\Omega j} \Omega F(\Sigma L_nY)$$

arising from the layer filtration for Y are stable equivalences.

In the layer filtration

$$L_nY:\ Y^0,\dots,Y^n,S^1\wedge Y^n,S^2\wedge Y^n,\dots$$

the maps

$$(\Sigma^\infty Y^n[-n])^r \to L_nY^r$$

are isomorphisms for $r \geq n$.

Thus, the maps

$$(\Omega F(\Sigma(\Sigma^\infty Y^n[-n])))^r \to \Omega F(\Sigma(L_nY))^r$$

are weak equivalences for $r \geq n$, so that

$$\Omega F(\Sigma(\Sigma^\infty Y^n[-n])) \to \Omega F(\Sigma(L_n Y))$$

is a stable equivalence.

The map $\eta : X \to \Omega\Sigma X$ respects shifts, so Lemma 41.3 implies that the composite

$$\Sigma^\infty Y^n[-n] \to \Omega\Sigma(\Sigma^\infty Y^n[-n]) \to \Omega F(\Sigma(\Sigma^\infty Y^n[-n]))$$

is a stable equivalence. □

Theorem 41.5. *Suppose that X is a spectrum. Then the canonical map*

$$\sigma : \Sigma X \to X[1]$$

is a stable equivalence.

Proof. The map σ is adjoint to the map $\sigma_* : X \to \Omega X[1]$, so that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\sigma_*"'] & \Omega\Sigma X \arrow[r, "\Omega j"] \arrow[d, "\Omega\sigma"] & \Omega F(\Sigma X) \arrow[d, "\Omega F\sigma"] \\ & \Omega X[1] \arrow[r, "\Omega j"'] & \Omega F(X[1]) \end{tikzcd}$$

where $j : \Sigma X \to F(\Sigma X)$ is a strictly fibrant model.

The composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega(FX)[1]$$

is a stable equivalence by Lemma 41.2, and the shifted map $j[1] : X[1] \to (FX)[1]$ is a strictly fibrant model of $X[1]$.

It follows that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j} \Omega F(X[1])$$

is a stable equivalence.

The composite

$$X \xrightarrow{\eta} \Omega\Sigma X \xrightarrow{\Omega j} \Omega F(\Sigma X)$$

is a stable equivalence by Lemma 41.4.

The map $\Omega F\sigma$ is therefore a stable equivalence, so Lemma 41.2 implies that

$$F\sigma : F(\Sigma X) \to F(X[1])$$

is a stable equivalence. □

Here's another, still elementary but fussier result:

Theorem 41.6. *The functors $X \mapsto X \wedge S^1$ and $X \mapsto \Sigma X$ are naturally stably equivalent.*

Sketch Proof: ([1], Lemma 1.9, p.7) The isomorphisms $\tau : S^1 \wedge X^n \to X^n \wedge S^1$ and the bonding maps $\sigma \wedge S^1$ together define a spectrum with the space

$S^1 \wedge X^n$ in level n, and with bonding maps $\tilde{\sigma}$ defined by the diagrams

$$\begin{tikzcd}
S^1 \wedge S^1 \wedge X^n \arrow[r, "\tilde{\sigma}"] \arrow[d, "S^1 \wedge \tau"', "\cong"] & S^1 \wedge X^{n+1} \arrow[d, "\cong"', "\tau"] \\
S^1 \wedge X^n \wedge S^1 \arrow[r, "\sigma \wedge S^1"'] & X^{n+1} \wedge S^1
\end{tikzcd}$$

There are commutative diagrams

$$\begin{tikzcd}
S^1 \wedge S^1 \wedge X^n \arrow[dr, "S^1 \wedge \sigma"] \arrow[dd, "\tau \wedge X^n"'] & \\
& S^1 \wedge X^{n+1} \\
S^1 \wedge S^1 \wedge X^n \arrow[ur, "\tilde{\sigma}"'] &
\end{tikzcd}$$

Composing then gives a diagram

$$\begin{tikzcd}
S^1 \wedge S^1 \wedge S^1 \wedge X^n \arrow[dr, "(S^1 \wedge \sigma)(S^1 \wedge S^1 \wedge \sigma)"] \arrow[dd, "(3{,}2{,}1) \wedge X^n"'] & \\
& S^1 \wedge X^{n+2} \\
S^1 \wedge S^1 \wedge S^1 \wedge X^n \arrow[ur, "\tilde{\sigma} \cdot (S^1 \wedge \tilde{\sigma})"'] &
\end{tikzcd}$$

where $(3,2,1)$ is induced on the smash factors making up S^3 by the corresponding cyclic permutation of order 3.

The spaces $S^1 \wedge X^0, S^1 \wedge X^2, \dots$ and the respective composite bonding maps $(S^1 \wedge \sigma)(S^1 \wedge S^1 \wedge \sigma)$ and $\tilde{\sigma} \cdot (S^1 \wedge \tilde{\sigma})$ define "partial" spectrum structures from which the stable homotopy types of the original spectra can be recovered.

The self map $(3,2,1)$ of the 3-sphere S^3 has degree 1 and is therefore homotopic to the identity.

This homotopy can be used to describe a telescope construction (see [1], p.11-15, and the next section) which is stably equivalent to both of these partial spectra. $\square$

Remark: The proof of Theorem 41.6 that is sketched here is essentially classical. See Prop. 10.53 of [2] for an alternative.

Corollary 41.7. *1) The functors $X \mapsto X[1]$, $X \mapsto \Sigma X$ and $X \mapsto X \wedge S^1$ are naturally stably equivalent.*

2) The functors $X \mapsto X[-1]$, $X \mapsto \Omega X$ and $X \mapsto \mathbf{hom}_*(S^1, X)$ are naturally stably equivalent.

Proof. Lemma 41.2 implies that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega FX[1]$$

is a stable equivalence for all spectra X, where $j : X \to FX$ is a strictly fibrant model.

Shift preserves stable equivalences, so the induced composite

$$X[-1] \xrightarrow{\sigma_*[-1]} \Omega X \xrightarrow{\Omega j} \Omega FX$$

is a stable equivalence.

The natural stable equivalence $\Sigma Y \simeq Y \wedge S^1$ induces a natural stable equivalence

$$\Omega X \simeq \mathbf{hom}_*(S^1, X)$$

for all strictly fibrant spectra X. $\square$

In other words, the suspension and loop functors (real or fake) are equivalent to shift functors, and define equivalences $\mathrm{Ho}(\mathbf{Spt}) \to \mathrm{Ho}(\mathbf{Spt})$ of the stable category.

42 The telescope construction

Observe that a spectrum Y is cofibrant if and only if all bonding maps $\sigma : S^1 \wedge Y^n \to Y^{n+1}$ are cofibrations.

The **telescope** TX for a spectrum X is a natural cofibrant replacement, equipped with a natural strict equivalence $s : TX \to X$.

The construction is an iterated mapping cylinder. We find natural trivial cofibrations

$$X^k \xrightarrow{j_k} CX^k \xrightarrow{\alpha_k} TX^k,\ k \leq n,$$

and $t_k : TX^k \to X^k$ such that $t_k \cdot (\alpha_k \cdot j_k) = 1$ and the maps t_k define a strict weak equivalence of spectra $t : TX \to X$.

- $X^0 = CX^0 = TX^0$ and j_0 and α_0 are identities,
- CX^n is the mapping cylinder for $\sigma : S^1 \wedge X^n \to X^{n+1}$, meaning that there is a pushout diagram

$$\begin{tikzcd}
S^1 \wedge X^n \arrow[r, "\sigma"] \arrow[d, "d^0"'] & X^{n+1} \arrow[d, "j_{n+1}"] \\
(S^1 \wedge X^n) \wedge \Delta^1_+ \arrow[r, "\zeta_{n+1}"'] & CX^{n+1}
\end{tikzcd}$$

  for each n.

  Write σ_* for the composite

$$S^1 \wedge X^n \xrightarrow{d^1} (S^1 \wedge X^n) \wedge \Delta^1_+ \xrightarrow{\zeta_{n+1}} CX^{n+1}$$

  and observe that σ_* is a cofibration.

  The projection map

$$s : (S^1 \wedge X^n) \wedge \Delta^1_+ \to S^1 \wedge X^n$$

  satisfies $s \cdot d^0 = 1$ and induces a map $s_{n+1} : CX^{n+1} \to X^{n+1}$ such that $s_{n+1} \cdot j_{n+1} = 1$. Further $s_{n+1} \cdot \sigma_* = \sigma$.

- Form the pushout diagram

$$\begin{tikzcd}
S^1 \wedge X^n \arrow[r, "\sigma_*"] \arrow[d, "S^1 \wedge j_n"'] & CX^{n+1} \arrow[dd, "\alpha_{n+1}"] \\
S^1 \wedge CX^n \arrow[d, "S^1 \wedge \alpha_n"'] & \\
S^1 \wedge TX^n \arrow[r, "\tilde{\sigma}"'] & TX^{n+1}
\end{tikzcd}$$

Then $\tilde{\sigma}$ is a cofibration, and the maps j_{n+1}, α_{n+1} are trivial cofibrations.

The maps $S^1 \wedge t_n$ and s_{n+1} together induce $t_{n+1} : TX^{n+1} \to X^{n+1}$ such that $t_{n+1} \cdot (\alpha_{n+1} \cdot j_{n+1}) = 1$, and the $t_k : TX^k \to X^k$ define a map of spectra up to level $n+1$.

The projection maps s can be replaced with homotopies $h : (S^1 \wedge X^n) \wedge \Delta^1_+ \to Z^n$ in the construction above, giving the following:

Lemma 42.1. *Suppose X is a spectrum with bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$. Suppose X' is a spectrum with the same objects as X, with bonding maps $\sigma' : S^1 \wedge X^n \to X^{n+1}$. Suppose $j : X' \to Z$*

is a map of spectra such that there are homotopies

$$\begin{tikzcd}
S^1 \wedge X^n \arrow[d, "d^1"'] \arrow[dr, "j\sigma'"] & \\
(S^1 \wedge X^n) \wedge \Delta^1_+ \arrow[r, "h"] & Z^{n+1} \\
S^1 \wedge X^n \arrow[u, "d^0"] \arrow[ur, "j\sigma"'] &
\end{tikzcd}$$

Then the homotopies h define a map $h_* : TX \to Z$, giving a morphism

$$X \underset{\simeq}{\xleftarrow{t}} TX \xrightarrow{h_*} Z$$

from X to Z in the stable category.

If $j : X' \to Z$ is a strict weak equivalence then the map h_* is a strict weak equivalence.

Remarks:

1) The construction of Lemma 42.1 is natural, and hence applies to diagrams of spectra.

Suppose that $i \mapsto X_i$ and $i \mapsto X'_i$ are spectrum valued functors defined on an index category I such that $X_i^n = X_i'^n$ for all $i \in I$. Let $j : X' \to Z$ be a natural choice of strict fibrant model for the diagram X' and suppose finally that there are natural homotopies

$$\begin{tikzcd} S^1 \wedge X_i^n \arrow[d, "d^1"'] \arrow[dr, "j\sigma'"] & \\ (S^1 \wedge X_i^n) \wedge \Delta^1_+ \arrow[r, "h"] & Z_i^n \\ S^1 \wedge X_i^n \arrow[u, "d^0"] \arrow[ur, "j\sigma"'] & \end{tikzcd}$$

where σ and σ' are the bonding maps for X and X' respectively.

Then the homotopies h canonically determine a natural strict equivalence $h_* : TX \to Z$, and there are natural strict equivalences

$$X \leftarrow TX \xrightarrow{h_*} Z \xleftarrow{j} X'.$$

2) Suppose given S^2-spectra $X(1)$ and $X(2)$ having objects $S^1 \wedge X^{2n}$ and bonding maps

$$\sigma_1, \sigma_2 : S^2 \wedge S^1 \wedge X^{2n} = S^3 \wedge X^{2n} \to S^1 \wedge X^{2n+2}$$

respectively, such that the diagram

$$\begin{tikzcd} S^3 \wedge X^{2n} \arrow[d, "c \wedge 1"'] \arrow[dr, "\sigma_1"] & \\ & S^1 \wedge X^{2n+2} \\ S^3 \wedge X^{2n} \arrow[ur, "\sigma_2"'] & \end{tikzcd}$$

commutes, where c is induced by the cyclic permutation $(3,2,1)$.

The map c has degree 1 and is therefore the identity in the homotopy category.

Choose a strict fibrant model $j : X(2) \to FX(2)$ in S^2-spectra for $X(2)$. Then

$$j \cdot \sigma_1 \simeq j \cdot \sigma_2 : S^3 \wedge X^{2n} \to F(S^1 \wedge X^{2n+2}),$$

and it follows that there are strict equivalences

$$X(1) \xleftarrow{t} TX(1) \xrightarrow{h_*} FX(2) \xleftarrow{j} X(2).$$

If $X(1)$ and $X(2)$ are the outputs of functors defined on spectra (eg. the comparison of fake and real suspension in Theorem 41.6), then these equivalences are natural.

43 Fibrations and cofibrations

Suppose $i : A \to X$ is a levelwise cofibration of spectra with cofibre $\pi : X \to X/A$.

Suppose $\alpha : S^r \to X^n$ represents a homotopy element such that the composite

$$S^r \xrightarrow{\alpha} X^n \xrightarrow{\pi} X^n/A^n$$

represents $0 \in \pi_r(X/A)^n$.

Comparing cofibre sequences gives a diagram

$$
\begin{tikzcd}
S^r \arrow[r] \arrow[d, "\alpha"'] & CS^r \arrow[r] \arrow[d] & S^1 \wedge S^r \arrow[r, "\simeq"] \arrow[d] & S^1 \wedge S^r \arrow[d, "S^1\wedge\alpha"] \\
X^n \arrow[r, "\pi"'] & (X/A)^n \arrow[r] & S^1 \wedge A^n \arrow[r, "S^1\wedge i"] \arrow[d, "\sigma"'] & S^1 \wedge X^n \arrow[d, "\sigma"] \\
 & & A^{n+1} \arrow[r, "i"'] & X^{n+1}
\end{tikzcd}
$$

where $CS^r \simeq *$ is the cone on S^r.

It follows that the image of $[\alpha]$ under the suspension map

$$\pi_r X^n \to \pi_{r+1} X^{n+1}$$

is in the image of the map $\pi_{r+1} A^{n+1} \to \pi_{r+1} X^{n+1}$.

We have proved the following:

Lemma 43.1. *Suppose $A \to X \to X/A$ is a level cofibre sequence of spectra.*

Then the sequence

$$\pi_k^s A \to \pi_k^s X \to \pi_k^s (X/A)$$

is exact.

Corollary 43.2. *Any levelwise cofibre sequence*

$$A \to X \to X/A$$

induces a long exact sequence

$$\dots \xrightarrow{\partial} \pi_k^s A \to \pi_k^s X \to \pi_k^s (X/A) \xrightarrow{\partial} \pi_{k-1}^s A \to \dots \tag{1}$$

The sequence (1) is the **long exact sequence** in stable homotopy groups for a level cofibre sequence of spectra.

Proof. The map $X/A \to A \wedge S^1$ in the Puppe sequence induces the boundary map

$$\pi_k^s(X/A) \to \pi_k^s(A \wedge S^1) \cong \pi_k^s(A[1]) \cong \pi_{k-1}^s A,$$

since $A \wedge S^1$ is naturally stably equivalent to the shifted spectrum $A[1]$ by Corollary 41.7. $\square$

Corollary 43.3. *Suppose that X and Y are spectra. Then the inclusion*

$$X \vee Y \to X \times Y$$

is a natural stable equivalence.

Proof. The sequence

$$0 \to \pi_k^s X \to \pi_k^s(X \vee Y) \to \pi_k^s Y \to 0$$

arising from the level cofibration $X \subset X \vee Y$ is split exact, as is the sequence

$$0 \to \pi_k^s X \to \pi_k^s(X \times Y) \to \pi_k^s Y \to 0$$

arising from the fibre sequence $X \to X \times Y \to Y$.

It follows that the map $X \vee Y \to X \times Y$ induces an isomorphism in all stable homotopy groups. $\square$

Corollary 43.4. *The stable homotopy category* $\mathrm{Ho}(\mathbf{Spt})$ *is additive.*

Proof. The sum of two maps $f,g : X \to Y$ is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} Y \times Y \xleftarrow{\simeq} Y \vee Y \xrightarrow{\nabla} Y.$$

□

Corollary 43.5. *Suppose that*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$

is a pushout in **Spt** *where i is a levelwise cofibration. Then there is a long exact sequence in stable homotopy groups*

$$\dots \xrightarrow{\partial} \pi_k^s A \xrightarrow{(i,\alpha)} \pi_k^s C \oplus \pi_k^s B \xrightarrow{j-\beta} \pi_k^s D \xrightarrow{\partial} \pi_{k-1}^s A \to \dots \tag{2}$$

The sequence (2) is the **Mayer-Vietoris sequence** for the cofibre square.

The boundary map $\partial : \pi_k^s D \to \pi_{k-1}^s A$ is the composite

$$\pi_k^s D \to \pi_k^s(D/C) = \pi_k^s(B/A) \xrightarrow{\partial} \pi_{k-1}^s A.$$

Lemma 43.6. *Suppose*

$$A \xrightarrow{i} X \xrightarrow{\pi} X/A$$

is a level cofibre sequence in **Spt**, *and let F be the strict homotopy fibre of the map π.*

Then the map $i_* : A \to F$ is a stable equivalence.

Proof. Choose a strict fibration $p : Z \to X/A$ such that $Z \to *$ is a strict weak equivalence.

Form the pullback

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi_*"] \arrow[d, "p_*"'] & Z \arrow[d, "p"] \\ X \arrow[r, "\pi"'] & X/A \end{tikzcd}$$

Then $\tilde{X}$ is the homotopy fibre of π and the maps $i : A \to X$ and $* : A \to Z$ together determine a map $i_* : A \to \tilde{X}$. We show that i_* is a stable equivalence.

Pull back the cofibre square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & * \arrow[d] \\ X \arrow[r, "\pi"'] & X/A \end{tikzcd}$$

along the fibration p to find a (levelwise) cofibre square

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d, "\tilde{i}"'] & U \arrow[d] \\ \tilde{X} \arrow[r] & Z \end{tikzcd}$$

A Mayer-Vietoris sequence argument (Corollary 43.5) implies that the map $\tilde{A} \to \tilde{X} \times U$ is a stable equivalence.

From the fibre square

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & U \arrow[d] \\ A \arrow[r] & * \end{tikzcd}$$

we see that the map $\tilde{A} \to A \times U$ is a stable equivalence.

The map $i_* : A \to \tilde{X}$ induces a section $\theta : A \to \tilde{A}$ of the map $\tilde{A} \to A$ which composes with the projection $\tilde{A} \to U$ to give the trivial map $* : A \to U$.

Thus, there is a commutative diagram

$$\begin{tikzcd} & A \arrow[r, "i_*"] \arrow[dl, "(1_A,*)"'] \arrow[d, "\theta"] & \tilde{X} \arrow[d, "(1_{\tilde{X}},*)"] \\ A \times U \arrow[dr, "pr"'] & \tilde{A} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & \tilde{X} \times U \arrow[dl] \\ & U & \end{tikzcd}$$

and it follows that A is the stable fibre of the map $\tilde{A} \to U$, so i_* is a stable equivalence. $\square$

Lemma 43.7. *Suppose that*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

is a strict fibre sequence, where i is a level cofibration.

Then the map $E/F \to B$ is a stable equivalence.

Proof. There is a diagram

$$\begin{tikzcd}
F \arrow[rr, "i"] \arrow[dr, "j'_*"] \arrow[dd, "="'] & & E \arrow[rr, "\pi"] \arrow[dr, "j'"] \arrow[dd, "=" near start] & & E/F \arrow[dr, "="] \arrow[dd, "\gamma" near start] & \\
& F' \arrow[rr, "i'" near start] \arrow[dl, "\theta_*"] & & U \arrow[rr, "p'" near start] \arrow[dl, "\theta"] & & E/F \arrow[dl, "\gamma"] \\
F \arrow[rr, "i"'] & & E \arrow[rr, "p"'] & & B &
\end{tikzcd}$$

where p' is a strict fibration, j' is a cofibration and a strict equivalence, and θ exists by a lifting property:

$$\begin{tikzcd}
E \arrow[r, "="] \arrow[d, "j'"'] & E \arrow[d, "p"] \\
U \arrow[r, "\gamma p'"'] \arrow[ur, dotted, "\theta"] & B
\end{tikzcd}$$

The map j'_* is a stable equivalence by Lemma 43.6, so θ_* is a stable equivalence.

The map θ is a strict equivalence, and a comparison of long exact sequences shows that γ is a stable equivalence. $\square$

Remark: Lemma 43.6 and Lemma 43.7 together say that fibre and cofibre sequences coincide in the stable category.

44 Cofibrant generation

We will show that the stable model structure on the category **Spt** of spectra is cofibrantly generated.

This means that there are sets I and J of stably trivial cofibrations and cofibrations, such that $p : X \to Y$ is a stable fibration (resp. stably trivial fibration) if and only if it has the RLP wrt all members of the set I (resp. all members of J).

Recall that a map $p : X \to Y$ is a stably trivial fibration if and only if it is a strict fibration and a strict weak equivalence.

Thus p is a stably trivial fibration if and only if it has the RLP wrt all maps

$$\Sigma^\infty \partial\Delta^n_+[m] \to \Sigma^\infty \Delta^n_+[m].$$

We have found our set of maps J.

It remains to find a set of stably trivial cofibrations I which generates the full class of stably trivial cofibrations. We do this in a sequence of lemmas.

Say that a spectrum A is **countable** if all consituent simplicial sets A^n are countable in the sense that they have countably many simplices in each degree — see Section 11.

It follows from Lemma 11.2 that a countable spectrum A has countable stable homotopy groups.

The following "bounded cofibration lemma" is the analogue of Lemma 11.3 for the category of spectra.

Lemma 44.1. *Suppose given level cofibrations of spectra*

$$\begin{tikzcd} & X \arrow[d, "j"] \\ A \arrow[r, "i"'] & Y \end{tikzcd}$$

such that A is countable and j is a stable equivalence.

Then there is a countable subobject $B \subset Y$ such that $A \subset B \subset Y$ and the map $B \cap X \to B$ is a stable equivalence.

Proof. The map $B \cap X \to B$ is a stable equivalence if and only if all stable homotopy groups

$$\pi_n^s(B/(B \cap X))$$

vanish, by Corollary 43.2.

Write $A_0 = A$. Y is a filtered colimit of its countable subobjects, and the countable set of elements of the homotopy groups $\pi_n^s(A_0/(A_0 \cap X))$ vanish in $\pi_n^s(A_1/(A_1 \cap X))$ for some countable subobject $A_1 \subset X$ with $A_0 \subset A_1$.

Repeat the construction inductively to find countable subcomplexes

$$A = A_0 \subset A_1 \subset A_2 \subset \dots$$

of Y such that all induced maps

$$\pi_n^s(A_i/(A_i \cap X)) \to \pi_n^s(A_{i+1}/(A_{i+1} \cap X))$$

are 0. Set $B = \cup_i A_i$. Then B is countable and all groups $\pi_n^s(B/(B \cap X))$ vanish. $\square$

Consider the set of all stably trivial level cofibrations $j : C \to D$ with D countable, and find a factorization

$$\begin{tikzcd} C \arrow[r, "in_j"] \arrow[dr, "j"'] & E_j \arrow[d, "p_j"] \\ & D \end{tikzcd}$$

for each such j such that in_j is a stably trivial cofibration and p_j is a stably trivial fibration.

Make fixed choices of the factorizations $j = p_j \cdot in_j$, and let I be the set of all stably trivial cofibrations in_j.

Lemma 44.2. *The set I generates the class of stably trivial cofibrations.*

Proof. Suppose given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where j is a cofibration, f is a stable equivalence and B is countable.

Then f has a factorization $f = q \cdot i$ where i is a stably trivial cofibration and q is a stably trivial fibration.

There is a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "j"'] & X \arrow[d, "i"] \\ & Z \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

where the lift θ exists since j is a cofibration and q is a stably trivial fibration.

The image $\theta(B)$ of B is a countable subobject of Z, so Lemma 44.1 says that there is a subobject $D \subset Z$ such that D is countable and the level cofibration $j : D \cap X \to D$ is a stable equivalence.

What we have, then, is a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & D \cap X \arrow[r] \arrow[d, "j"] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

of the original diagram, such that j is a countable, stably trivial level countable.

We can further assume (by lifting to E_j) that the original diagram has a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & D \cap X \arrow[r] \arrow[d, "in_j"] & X \arrow[d, "f"] \\ B \arrow[r] & E_j \arrow[r] & Y \end{tikzcd}$$

where the map in_j is a member of the set I.

Now suppose that $i : U \to V$ is a stably trivial cofibration. Then i has a factorization

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[dr, "i"'] & W \arrow[d, "q"] \\ & V \end{tikzcd}$$

where α is a member of the saturation of I and q has the RLP wrt all members of I.

But then q has the RLP wrt all countable cofibrations by the construction above, so that q has the RLP wrt all cofibrations.

In particular, there is a diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "i"'] & W \arrow[d, "q"] \\ V \arrow[r, "1"'] \arrow[ur, dotted] & V \end{tikzcd}$$

so that i is a retract of j. $\square$

Remark: Compare the proof of Lemma 44.2 with the proof of Lemma 11.5 — they are the same.

References

[1] J. F. Jardine. *Generalized Étale Cohomology Theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[2] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.