# **Lecture 14: Basic properties**

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### 41 Suspensions and shift

The suspension  $X \wedge S^1$  and the fake suspension  $\Sigma X$  of a spectrum X were defined in Section 38 — the constructions differ by a non-trivial twist of bonding maps.

The **loop** spectrum for *X* is the function complex object

**hom**<sub>\*</sub>( $S^1, X$ ).

There is a natural bijection

 $\operatorname{hom}(X \wedge S^1, Y) \cong \operatorname{hom}(X, \operatorname{hom}_*(S^1, Y))$ 

so that the suspension and loop functors are adjoint.

The **fake loop** spectrum  $\Omega Y$  for a spectrum *Y* consists of the pointed spaces  $\Omega Y^n$ ,  $n \ge 0$ , with adjoint

bonding maps

$$\Omega \sigma_* : \Omega Y^n \to \Omega^2 Y^{n+1}$$

There is a natural bijection

 $\operatorname{hom}(\Sigma X, Y) \cong \operatorname{hom}(X, \Omega Y),$ 

so the fake suspension functor is left adjoint to fake loops.

The adjoint bonding maps  $\sigma_*: Y^n \to \Omega Y^{n+1}$  define a natural map

$$\gamma: Y \to \Omega Y[1].$$

for spectra Y.

The map  $\omega: Y \to \Omega^{\infty} Y$  of Section 40 is the filtered colimit of the maps

 $Y \xrightarrow{\gamma} \Omega Y[1] \xrightarrow{\Omega \gamma[1]} \Omega^2 Y[2] \xrightarrow{\Omega^2 \gamma[2]} \dots$ 

Recall the statement of the Freudenthal suspension theorem (Theorem 34.2):

**Theorem 41.1.** Suppose that a pointed space X is *n*-connected, where  $n \ge 0$ .

Then the homotopy fibre F of the canonical map  $\eta: X \to \Omega(X \wedge S^1)$  is 2n-connected.

In particular, the suspension homomorphism

$$\pi_i X \to \pi_i(\Omega(X \wedge S^1)) \cong \pi_{i+1}(X \wedge S^1)$$

is an isomorphism for  $i \le 2n$  and is an epimorphism for i = 2n + 1, provided that X is *n*-connected.

In general (ie. with no connectivity assumptions on *Y*), the space  $S^n \wedge Y$  is (n-1)-connected, by Lemma 31.5 and Corollary 34.1.

Thus, the suspension homomorphism

$$\pi_{i+k}(S^{n+k}\wedge Y)\to\pi_{i+k+1}(S^{n+k+1}\wedge Y)$$

is an isomorphism if  $i \le 2n - 2 + k$ , and it follows that the map

$$\pi_i(S^n \wedge Y) \to \pi^s_{i-n}(\Sigma^{\infty}Y)$$

is an isomorphism for  $i \leq 2(n-1)$ .

Here's an easy observation:

**Lemma 41.2.** *The natural map*  $\gamma : X \to \Omega X[1]$  *is a stable equivalence if* X *is strictly fibrant.* 

*Proof.* This is a cofinality argument, which uses the fact that  $\Omega^{\infty}X$  is the filtered colimit of the system

$$X \to \Omega X[1] \to \Omega^2 X[2] \to \dots$$

**Lemma 41.3.** *Suppose that Y is a pointed space. Then the canonical map* 

 $\eta: \Sigma^{\infty}Y \to \Omega\Sigma(\Sigma^{\infty}Y)$ 

is a stable homotopy equivalence.

Proof. The map

$$\pi_k(S^n \wedge Y) \to \pi^s_{k-n}(\Sigma^{\infty}Y)$$

is an isomorphism for  $k \leq 2(n-1)$ .

Similarly (exercise), the map

$$\pi_k(\Omega(S^{n+1}\wedge X))\to \pi^s_{k-n}(\Omega\Sigma(\Sigma^\infty X))$$

is an isomorphism for  $k + 1 \le 2n$  or  $k \le 2n - 1$ . There is a commutative diagram

in which the indicated maps are isomorphisms for  $k \le 2(n-1)$ .

It follows that the map

$$\pi_p^s(\Sigma^{\infty}Y) \to \pi_p^s(\Omega\Sigma(\Sigma^{\infty}Y))$$

is an isomorphism for  $p \le n-2$ . Finish by letting *n* vary.

**Remark:** What we've really shown in Lemma 41.3 is that the composite

$$\Sigma^{\infty} X \xrightarrow{\eta} \Omega \Sigma(\Sigma^{\infty} X) \xrightarrow{\Omega j} \Omega F(\Sigma(\Sigma^{\infty} X))$$

is a natural stable equivalence.

**Lemma 41.4.** *Suppose that Y is a spectrum. Then the composite* 

$$Y \xrightarrow{\eta} \Omega \Sigma Y \xrightarrow{\Omega j} \Omega F(\Sigma Y)$$

is a stable equivalence.

*Proof.* We show that the maps

$$L_n Y \xrightarrow{\eta} \Omega \Sigma L_n Y \xrightarrow{\Omega j} \Omega F(\Sigma L_n Y)$$

arising from the layer filtration for *Y* are stable equivalences.

In the layer filtration

$$L_nY: Y^0,\ldots,Y^n,S^1\wedge Y^n,S^2\wedge Y^n,\ldots$$

the maps

$$(\Sigma^{\infty}Y^{n}[-n])^{r} \to L_{n}Y^{r}$$

are isomorphisms for  $r \ge n$ .

Thus, the maps

$$(\Omega F(\Sigma(\Sigma^{\infty}Y^{n}[-n])))^{r} \to \Omega F(\Sigma(L_{n}Y))^{r}$$

are weak equivalences for  $r \ge n$ , so that

$$\Omega F(\Sigma(\Sigma^{\infty}Y^{n}[-n])) \to \Omega F(\Sigma(L_{n}Y))$$

is a stable equivalence.

The map  $\eta : X \to \Omega \Sigma X$  respects shifts, so Lemma 41.3 implies that the composite

$$\Sigma^{\infty}Y^{n}[-n] \to \Omega\Sigma(\Sigma^{\infty}Y^{n}[-n]) \to \Omega F(\Sigma(\Sigma^{\infty}Y^{n}[-n]))$$

is a stable equivalence.

**Theorem 41.5.** *Suppose that X is a spectrum.* 

Then the canonical map

$$\sigma: \Sigma X \to X[1]$$

is a stable equivalence.

*Proof.* The map  $\sigma$  is adjoint to the map  $\sigma_* : X \to \Omega X[1]$ , so that there is a commutative diagram

where  $j: \Sigma X \to F(\Sigma X)$  is a strictly fibrant model.

The composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega(FX)[1]$$

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is a stable equivalence by Lemma 41.2, and the shifted map  $j[1]: X[1] \rightarrow (FX)[1]$  is a strictly fibrant model of X[1].

It follows that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j} \Omega F(X[1])$$

is a stable equivalence.

The composite

$$X \xrightarrow{\eta} \Omega \Sigma X \xrightarrow{\Omega j} \Omega F(\Sigma X)$$

is a stable equivalence by Lemma 41.4.

The map  $\Omega F \sigma$  is therefore a stable equivalence, so Lemma 41.2 implies that

$$F\sigma: F(\Sigma X) \to F(X[1])$$

is a stable equivalence.

Here's another, still elementary but fussier result:

**Theorem 41.6.** *The functors*  $X \mapsto X \wedge S^1$  *and*  $X \mapsto \Sigma X$  *are naturally stably equivalent.* 

**Sketch Proof:** ([1], Lemma 1.9, p.7) The isomorphisms  $\tau: S^1 \wedge X^n \to X^n \wedge S^1$  and the bonding maps  $\sigma \wedge S^1$  together define a spectrum with the space

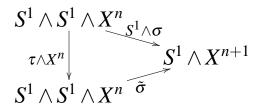
 $S^1 \wedge X^n$  in level *n*, and with bonding maps  $\tilde{\sigma}$  defined by the diagrams

$$S^{1} \wedge S^{1} \wedge X^{n} \xrightarrow{\tilde{\sigma}} S^{1} \wedge X^{n+1}$$

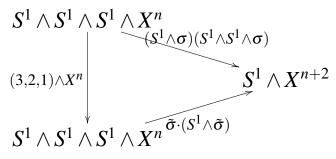
$$S^{1} \wedge \tau \downarrow \cong \qquad \cong \downarrow \tau$$

$$S^{1} \wedge X^{n} \wedge S^{1} \xrightarrow{\sigma \wedge S^{1}} X^{n+1} \wedge S^{1}$$

There are commutative diagrams



Composing then gives a diagram



where (3,2,1) is induced on the smash factors making up  $S^3$  by the corresponding cyclic permutation of order 3.

The spaces  $S^1 \wedge X^0, S^1 \wedge X^2, \ldots$  and the respective composite bonding maps  $(S^1 \wedge \sigma)(S^1 \wedge S^1 \wedge \sigma)$  and  $\tilde{\sigma} \cdot (S^1 \wedge \tilde{\sigma})$  define "partial" spectrum structures from which the stable homotopy types of the original spectra can be recovered.

The self map (3,2,1) of the 3-sphere  $S^3$  has degree 1 and is therefore homotopic to the identity.

This homotopy can be used to describe a telescope construction (see [1], p.11-15, and the next section) which is stably equivalent to both of these partial spectra.  $\Box$ 

**Remark**: The proof of Theorem 41.6 that is sketched here is essentially classical. See Prop. 10.53 of [2] for an alternative.

**Corollary 41.7.** 1) The functors  $X \mapsto X[1]$ ,  $X \mapsto \Sigma X$  and  $X \mapsto X \wedge S^1$  are naturally stably equivalent.

2) The functors  $X \mapsto X[-1]$ ,  $X \mapsto \Omega X$  and  $X \mapsto \mathbf{hom}_*(S^1, X)$  are naturally stably equivalent.

Proof. Lemma 41.2 implies that the composite

 $X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega F X[1]$ 

is a stable equivalence for all spectra X, where  $j: X \rightarrow FX$  is a strictly fibrant model.

Shift preserves stable equivalences, so the induced composite

$$X[-1] \xrightarrow{\sigma_*[-1]} \Omega X \xrightarrow{\Omega j} \Omega F X$$

is a stable equivalence.

The natural stable equivalence  $\Sigma Y \simeq Y \wedge S^1$  induces a natural stable equivalence

$$\Omega X \simeq \mathbf{hom}_*(S^1, X)$$

for all strictly fibrant spectra *X*.

In other words, the suspension and loop functors (real or fake) are equivalent to shift functors, and define equivalences  $Ho(Spt) \rightarrow Ho(Spt)$  of the stable category.

### 42 The telescope construction

Observe that a spectrum *Y* is cofibrant if and only if all bonding maps  $\sigma : S^1 \wedge Y^n \to Y^{n+1}$  are cofibrations.

The **telescope** *TX* for a spectrum *X* is a natural cofibrant replacement, equipped with a natural strict equivalence  $s: TX \rightarrow X$ .

The construction is an iterated mapping cylinder. We find natural trivial cofibrations

$$X^k \xrightarrow{j_k} CX^k \xrightarrow{\alpha_k} TX^k, \ k \leq n,$$

and  $t_k : TX^k \to X^k$  such that  $t_k \cdot (\alpha_k \cdot j_k) = 1$  and the maps  $t_k$  define a strict weak equivalence of spectra  $t : TX \to X$ .

- $X^0 = CX^0 = TX^0$  and  $j_0$  and  $\alpha_0$  are identities,
- $CX^n$  is the mapping cylinder for  $\sigma: S^1 \wedge X^n \rightarrow X^{n+1}$ , meaning that there is a pushout diagram

for each *n*.

Write  $\sigma_*$  for the composite

$$S^1 \wedge X^n \xrightarrow{d^1} (S^1 \wedge X^n) \wedge \Delta^1_+ \xrightarrow{\zeta_{n+1}} CX^{n+1}$$

and observe that  $\sigma_*$  is a cofibration.

The projection map

$$s: (S^1 \wedge X^n) \wedge \Delta^1_+ \to S^1 \wedge X^n$$

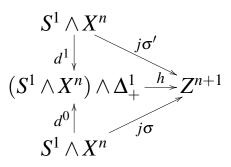
satisfies  $s \cdot d^0 = 1$  and induces a map  $s_{n+1}$ :  $CX^{n+1} \to X^{n+1}$  such that  $s_{n+1} \cdot j_{n+1} = 1$ . Further  $s_{n+1} \cdot \sigma_* = \sigma$ . • Form the pushout diagram

Then  $\tilde{\sigma}$  is a cofibration, and the maps  $j_{n+1}$ ,  $\alpha_{n+1}$  are trivial cofibrations.

The maps  $S^1 \wedge t_n$  and  $s_{n+1}$  together induce  $t_{n+1}$ :  $TX^{n+1} \rightarrow X^{n+1}$  such that  $t_{n+1} \cdot (\alpha_{n+1} \cdot j_{n+1}) = 1$ , and the  $t_k : TX^k \rightarrow X^k$  define a map of spectra up to level n + 1.

The projection maps *s* can be replaced with homotopies  $h: (S^1 \wedge X^n) \wedge \Delta^1_+ \to Z^n$  in the construction above, giving the following:

**Lemma 42.1.** Suppose X is a spectrum with bonding maps  $\sigma : S^1 \wedge X^n \to X^{n+1}$ . Suppose X' is a spectrum with the same objects as X, with bonding maps  $\sigma' : S^1 \wedge X^n \to X^{n+1}$ . Suppose  $j : X' \to Z$  is a map of spectra such that there are homotopies



Then the homotopies h define a map  $h_*: TX \to Z$ , giving a morphism

$$X \xleftarrow{t} TX \xrightarrow{h_*} Z$$

from X to Z in the stable category.

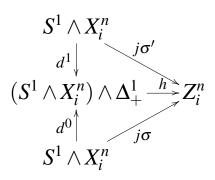
If  $j: X' \to Z$  is a strict weak equivalence then the map  $h_*$  is a strict weak equivalence.

# **Remarks**:

1) The construction of Lemma 42.1 is natural, and hence applies to diagrams of spectra.

Suppose that  $i \mapsto X_i$  and  $i \mapsto X'_i$  are spectrum valued functors defined on an index category I such that  $X_i^n = X_i'^n$  for all  $i \in I$ . Let  $j : X' \to Z$  be a natural choice of strict fibrant model for the diagram X' and suppose finally that there are natural

homotopies



where  $\sigma$  and  $\sigma'$  are the bonding maps for X and X' respectively.

Then the homotopies h canonically determine a natural strict equivalence  $h_*: TX \to Z$ , and there are natural strict equivalences

$$X \leftarrow TX \xrightarrow{h_*} Z \xleftarrow{j} X'.$$

2) Suppose given  $S^2$ -spectra X(1) and X(2) having objects  $S^1 \wedge X^{2n}$  and bonding maps

$$\sigma_1, \sigma_2: S^2 \wedge S^1 \wedge X^{2n} = S^3 \wedge X^{2n} \to S^1 \wedge X^{2n+2}$$

respectively, such that the diagram

$$S^{3} \wedge X^{2n}$$

$$S^{1} \wedge X^{2n+2}$$

$$S^{3} \wedge X^{2n}$$

commutes, where c is induced by the cyclic permutation (3,2,1).

The map c has degree 1 and is therefore the identity in the homotopy category.

Choose a strict fibrant model  $j: X(2) \rightarrow FX(2)$  in  $S^2$ -spectra for X(2). Then

$$j \cdot \sigma_1 \simeq j \cdot \sigma_2 : S^3 \wedge X^{2n} \to F(S^1 \wedge X^{2n+2}),$$

and it follows that there are strict equivalences

$$X(1) \xleftarrow{t} TX(1) \xrightarrow{h_*} FX(2) \xleftarrow{j} X(2).$$

If X(1) and X(2) are the outputs of functors defined on spectra (eg. the comparison of fake and real suspension in Theorem 41.6), then these equivalences are natural.

### 43 Fibrations and cofibrations

Suppose  $i : A \to X$  is a levelwise cofibration of spectra with cofibre  $\pi : X \to X/A$ .

Suppose  $\alpha : S^r \to X^n$  represents a homotopy element such that the composite

$$S^r \xrightarrow{\alpha} X^n \xrightarrow{\pi} X^n / A^n$$

represents  $0 \in \pi_r(X/A)^n$ .

Comparing cofibre sequences gives a diagram

where  $CS^r \simeq *$  is the cone on  $S^r$ .

It follows that the image of  $[\alpha]$  under the suspension map

$$\pi_r X^n \to \pi_{r+1} X^{n+1}$$

is in the image of the map  $\pi_{r+1}A^{n+1} \rightarrow \pi_{r+1}X^{n+1}$ . We have proved the following:

**Lemma 43.1.** Suppose  $A \rightarrow X \rightarrow X/A$  is a level cofibre sequence of spectra.

Then the sequence

$$\pi_k^s A o \pi_k^s X o \pi_k^s (X/A)$$

is exact.

**Corollary 43.2.** Any levelwise cofibre sequence

$$A \to X \to X/A$$

induces a long exact sequence

$$\dots \xrightarrow{\partial} \pi_k^s A \to \pi_k^s X \to \pi_k^s (X/A) \xrightarrow{\partial} \pi_{k-1}^s A \to \dots$$
(1)

The sequence (1) is the **long exact sequence** in stable homotopy groups for a level cofibre sequence of spectra.

*Proof.* The map  $X/A \rightarrow A \wedge S^1$  in the Puppe sequence induces the boundary map

$$\pi_k^s(X/A) \to \pi_k^s(A \wedge S^1) \cong \pi_k^s(A[1]) \cong \pi_{k-1}^sA,$$

since  $A \wedge S^1$  is naturally stably equivalent to the shifted spectrum A[1] by Corollary 41.7.

**Corollary 43.3.** *Suppose that X and Y are spectra. Then the inclusion* 

$$X \lor Y \to X \times Y$$

is a natural stable equivalence.

*Proof.* The sequence

$$0 \to \pi_k^s X \to \pi_k^s (X \lor Y) \to \pi_k^s Y \to 0$$

arising from the level cofibration  $X \subset X \lor Y$  is split exact, as is the sequence

$$0 o \pi_k^s X o \pi_k^s (X imes Y) o \pi_k^s Y o 0$$

arising from the fibre sequence  $X \to X \times Y \to Y$ . It follows that the map  $X \lor Y \to X \times Y$  induces an isomorphism in all stable homotopy groups. **Corollary 43.4.** *The stable homotopy category* Ho(**Spt**) *is additive.* 

*Proof.* The sum of two maps  $f, g: X \rightarrow Y$  is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xleftarrow{\simeq} Y \vee Y \xrightarrow{\nabla} Y.$$

**Corollary 43.5.** Suppose that

$$\begin{array}{c} A \xrightarrow{i} B \\ \alpha \downarrow \qquad \qquad \downarrow \beta \\ C \xrightarrow{j} D \end{array}$$

is a pushout in **Spt** where *i* is a levelwise cofibration. Then there is a long exact sequence in stable homotopy groups

$$\dots \xrightarrow{\partial} \pi_k^s A \xrightarrow{(i,\alpha)} \pi_k^s C \oplus \pi_k^s B \xrightarrow{j-\beta} \pi_k^s D \xrightarrow{\partial} \pi_{k-1}^s A \to \dots$$
(2)

The sequence (2) is the **Mayer-Vietoris sequence** for the cofibre square.

The boundary map  $\partial : \pi_k^s D \to \pi_{k-1}^s A$  is the composite

$$\pi_k^s D o \pi_k^s (D/C) = \pi_k^s (B/A) \stackrel{\partial}{ o} \pi_{k-1}^s A.$$

Lemma 43.6. Suppose

$$A \xrightarrow{i} X \xrightarrow{\pi} X/A$$

is a level cofibre sequence in **Spt**, and let *F* be the strict homotopy fibre of the map  $\pi$ .

Then the map  $i_*: A \to F$  is a stable equivalence.

*Proof.* Choose a strict fibration  $p: Z \rightarrow X/A$  such that  $Z \rightarrow *$  is a strict weak equivalence.

Form the pullback

Then  $\tilde{X}$  is the homotopy fibre of  $\pi$  and the maps  $i: A \to X$  and  $*: A \to Z$  together determine a map  $i_*: A \to \tilde{X}$ . We show that  $i_*$  is a stable equivalence.

Pull back the cofibre square

$$\begin{array}{c}
A \longrightarrow * \\
\downarrow & \downarrow \\
X \longrightarrow X/A
\end{array}$$

along the fibration p to find a (levelwise) cofibre

square

$$\begin{array}{c} \tilde{A} \longrightarrow U \\ {}_{\tilde{i}} \downarrow & \downarrow \\ \tilde{X} \longrightarrow Z \end{array}$$

A Mayer-Vietoris sequence argument (Corollary 43.5) implies that the map  $\tilde{A} \rightarrow \tilde{X} \times U$  is a stable equivalence.

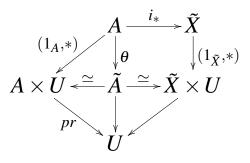
From the fibre square



we see that the map  $\tilde{A} \rightarrow A \times U$  is a stable equivalence.

The map  $i_*: A \to \tilde{X}$  induces a section  $\theta: A \to \tilde{A}$ of the map  $\tilde{A} \to A$  which composes with the projection  $\tilde{A} \to U$  to give the trivial map  $*: A \to U$ .

Thus, there is a commutative diagram



and it follows that *A* is the stable fibre of the map  $\tilde{A} \rightarrow U$ , so  $i_*$  is a stable equivalence.

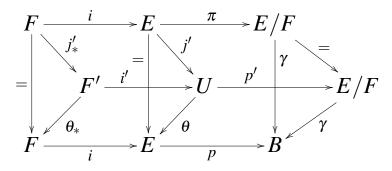
Lemma 43.7. Suppose that

 $F \xrightarrow{i} E \xrightarrow{p} B$ 

is a strict fibre sequence, where i is a level cofibration.

Then the map  $E/F \rightarrow B$  is a stable equivalence.

Proof. There is a diagram



where p' is a strict fibration, j' is a cofibration and a strict equivalence, and  $\theta$  exists by a lifting property:



The map  $j'_*$  is a stable equivalence by Lemma 43.6, so  $\theta_*$  is a stable equivalence.

The map  $\theta$  is a strict equivalence, and a comparison of long exact sequences shows that  $\gamma$  is a stable equivalence.

**Remark:** Lemma 43.6 and Lemma 43.7 together say that fibre and cofibre sequences coincide in the stable category.

### 44 Cofibrant generation

We will show that the stable model structure on the category **Spt** of spectra is cofibrantly generated.

This means that there are sets *I* and *J* of stably trivial cofibrations and cofibrations, such that  $p: X \rightarrow Y$  is a stable fibration (resp. stably trivial fibration) if and only if it has the RLP wrt all members of the set *I* (resp. all members of *J*).

Recall that a map  $p: X \to Y$  is a stably trivial fibration if and only if it is a strict fibration and a strict weak equivalence.

Thus *p* is a stably trivial fibration if and only if it has the RLP wrt all maps

$$\Sigma^{\infty}\partial \Delta^n_+[m] \to \Sigma^{\infty}\Delta^n_+[m].$$

We have found our set of maps J.

It remains to find a set of stably trivial cofibrations *I* which generates the full class of stably trivial cofibrations. We do this in a sequence of lemmas.

Say that a spectrum A is **countable** if all consituent simplicial sets  $A^n$  are countable in the sense that they have countably many simplices in each degree — see Section 11.

It follows from Lemma 11.2 that a countable spectrum *A* has countable stable homotopy groups.

The following "bounded cofibration lemma" is the analogue of Lemma 11.3 for the category of spectra.

**Lemma 44.1.** Suppose given level cofibrations of spectra



such that A is countable and j is a stable equivalence.

Then there is a countable subobject  $B \subset Y$  such that  $A \subset B \subset Y$  and the map  $B \cap X \to B$  is a stable equivalence.

*Proof.* The map  $B \cap X \to B$  is a stable equivalence if and only if all stable homotopy groups

$$\pi_n^s(B/(B\cap X))$$

vanish, by Corollary 43.2.

Write  $A_0 = A$ . *Y* is a filtered colimit of its countable subobjects, and the countable set of elements of the homotopy groups  $\pi_n^s(A_0/(A_0 \cap X))$  vanish in  $\pi_n^s(A_1/(A_1 \cap X))$  for some countable subobject  $A_1 \subset X$  with  $A_0 \subset A_1$ .

Repeat the construction inductively to find countable subcomplexes

 $A = A_0 \subset A_1 \subset A_2 \subset \dots$ 

of *Y* such that all induced maps

$$\pi_n^s(A_i/(A_i\cap X))\to \pi_n^s(A_{i+1}/(A_{i+1}\cap X))$$

are 0. Set  $B = \bigcup_i A_i$ . Then *B* is countable and all groups  $\pi_n^s(B/(B \cap X))$  vanish.

Consider the set of all stably trivial level cofibrations  $j: C \rightarrow D$  with D countable, and find a factorization



for each such *j* such that  $in_j$  is a stably trivial cofibration and  $p_j$  is a stably trivial fibration.

Make fixed choices of the factorizations  $j = p_j \cdot in_j$ , and let *I* be the set of all stably trivial cofibrations  $in_j$ .

**Lemma 44.2.** The set I generates the class of stably trivial cofibrations.

Proof. Suppose given a diagram

$$\begin{array}{c} A \longrightarrow X \\ \downarrow j & \qquad \downarrow f \\ B \longrightarrow Y \end{array}$$

where j is a cofibration, f is a stable equivalence and B is countable.

Then *f* has a factorization  $f = q \cdot i$  where *i* is a stably trivial cofibration and *q* is a stably trivial fibration.

There is a diagram

$$\begin{array}{c|c} A \longrightarrow X \\ \downarrow & \downarrow^i \\ j & Z \\ \theta & \downarrow^q \\ B \longrightarrow Y \end{array}$$

where the lift  $\theta$  exists since *j* is a cofibration and *q* is a stably trivial fibration.

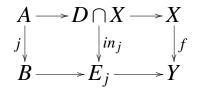
The image  $\theta(B)$  of *B* is a countable subobject of *Z*, so Lemma 44.1 says that there is a subobject  $D \subset$ *Z* such that *D* is countable and the level cofibration  $j: D \cap X \to D$  is a stable equivalence.

What we have, then, is a factorization

$$\begin{array}{c} A \longrightarrow D \cap X \longrightarrow X \\ j \downarrow & \downarrow j & \downarrow f \\ B \longrightarrow D \longrightarrow Y \end{array}$$

of the original diagram, such that *j* is a countable, stably trivial level countable.

We can further assume (by lifting to  $E_j$ ) that the original diagram has a factorization



where the map  $in_i$  is a member of the set *I*.

Now suppose that  $i: U \rightarrow V$  is a stably trivial cofibration. Then *i* has a factorization

$$U \xrightarrow{\alpha} W$$

$$\downarrow q$$

$$V$$

where  $\alpha$  is a member of the saturation of *I* and *q* has the RLP wrt all members of *I*.

But then q has the RLP wrt all countable cofibrations by the construction above, so that q has the RLP wrt all cofibrations.

In particular, there is a diagram



so that *i* is a retract of *j*.

**Remark**: Compare the proof of Lemma 44.2 with the proof of Lemma 11.5 — they are the same.

#### References

- [1] J. F. Jardine. *Generalized Étale Cohomology Theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.
- [2] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.