# Lecture 01

## 1 Simplicial sets

The finite ordinal number $\mathbf{n}$ is the set of counting numbers

$$\mathbf{n} = \{0, 1, \dots, n\}.$$

There is an obvious ordering on this set which gives it the structure of a poset, and hence a (tiny) category.

**Fact**: If $C$ is a category then the functors $\alpha : \mathbf{n} \to C$ can be identified with strings of arrows

$$\alpha(0) \to \alpha(1) \to \cdots \to \alpha(n)$$

of length $n$.

The collection of all finite ordinal numbers and all order-preserving functions between them (aka. poset morphisms, or functors) form the ordinal number category $\mathbf{\Delta}$.

**Examples**:

1) The ordinal number monomorphisms $d^i : \mathbf{n-1} \to \mathbf{n}$ are defined by the strings of relations

$$0 \leq 1 \leq \cdots \leq i-1 \leq i+1 \leq \cdots \leq n$$

for $0 \leq i \leq n$. These morphisms are called cofaces.

2) The ordinal number epimorphisms $s^j : \mathbf{n+1} \to \mathbf{n}$ are defined by the strings

$$0 \leq 1 \leq \cdots \leq j \leq j \leq \cdots \leq n$$

for $0 \leq j \leq n$. These are the codegeneracies.

The cofaces and codegeneracies together satisfy the following relations

$$d^j d^i = d^i d^{j-1} \text{ if } i < j,$$
$$s^j s^i = s^i s^{j+1} \text{ if } i \leq j$$
$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i = j, j+1, \\ d^{i-1} s^j & \text{if } i > j+1. \end{cases}$$

The ordinal number category $\mathbf{\Delta}$ is the category which is generated by the cofaces and codegeneracies, subject to the cosimplicial identities [4].

Every ordinal number morphism has a unique epi-monic factorization, and in fact has a canonical form defined in terms of strings of codegeneracies and strings of cofaces.

A *simplicial set* is a functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$, ie. a contravariant set-valued functor on the ordinal number category $\mathbf{\Delta}$. Such things are usually written $\mathbf{n} \mapsto X_n$, and $X_n$ is called the set of *$n$-simplices* of $X$.

A *simplicial set map* (or simplicial map) $f : X \to Y$ is a natural transformation of such functors. The simplicial sets and simplicial set maps form the category of simplicial sets, which is denoted by $s\mathbf{Set}$.

A simplicial set is a *simplicial object* in the set category.

Generally, $s\mathcal{A}$ denotes the category of simplicial objects in a category $\mathcal{A}$. Examples include the categories $s\mathbf{Gr}$ of simplicial groups, $s(R-\mathbf{Mod})$ of simplicial $R$-modules, $s(s\mathbf{Set}) = s^2\mathbf{Set}$ of bisimplicial sets, and so on.

**Examples**:

1) The topological standard $n$-simplex is the space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum t_i = 1\}$$

The assignment $\mathbf{n} \mapsto |\Delta^n|$ is a cosimplicial space.

If $X$ is a space, then the *singular complex* $S(X)$ is defined by

$$S(X)_n = \hom(|\Delta^n|, X).$$

The assignment $X \mapsto S(X)$ defines a covariant functor

$$S : \mathbf{CGHaus} \to s\mathbf{Set}$$

in the obvious way, called the *singular functor*.

2) The ordinal number $\mathbf{n}$ represents a contravariant functor

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n}),$$

which is called the *standard $n$-simplex*. Write

$$\iota_n = 1_{\mathbf{n}} \in \hom_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{n}).$$

The $n$-simplex $\iota_n$ is often called the *classifying $n$-simplex*, because the Yoneda Lemma implies that there is a natural bijection

$$\hom_{s\mathbf{Set}}(\Delta^n, Y) \cong Y_n$$

defined by sending the map $\sigma : \Delta^n \to Y$ to the element $\sigma(\iota_n) \in Y_n$. I usually say that a map $\Delta^n \to Y$ is an $n$-simplex of $Y$.

In general, if $\sigma : \Delta^n \to X$ is a simplex of $X$, then the $i^{th}$ *face* $d_i(\sigma)$ is the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X,$$

while the $j^{th}$ *degeneracy* $s_j(\sigma)$ is the composite

$$\Delta^{n+1} \xrightarrow{s^j} \Delta^n \xrightarrow{\sigma} X.$$

3) $\partial\Delta^n$ is the subobject of $\Delta^n$ which is generated by the $(n-1)$-simplices $d^i$, $0 \leq i \leq n$, and let $\Lambda^n_k$ be the subobject of $\partial\Delta^n$ which is generated by the simplices $d^i$, $i \neq k$. $\partial\Delta^n$ is called the *boundary* of $\Delta^n$, and $\Lambda^n_k$ is called the $k^{th}$ *horn*.

The faces $d^i : \Delta^{n-1} \to \Delta^n$ determine a *covering*

$$\bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n,$$

and for each $i < j$ there are pullback diagrams

$$\begin{array}{ccc} \Delta^{n-2} & \xrightarrow{d^{j-1}} & \Delta^{n-1} \\ {\scriptstyle d^i}\downarrow & & \downarrow{\scriptstyle d^i} \\ \Delta^{n-1} & \xrightarrow[d^j]{} & \Delta^n \end{array}$$

It follows that there is a coequalizer

$$\bigsqcup_{i<j,0\leq i,j\leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{0\leq i\leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

in $s\mathbf{Set}$. Similarly, there is a coequalizer

$$\bigsqcup_{i<j,i,j\neq k} \Delta^{n-2} \rightrightarrows \bigsqcup_{0\leq i\leq n,i\neq k} \Delta^{n-1} \longrightarrow \Lambda^n_k.$$

4) Suppose that a category $C$ is *small* in the sense that the morphisms $\mathrm{Mor}(C)$ and objects $\mathrm{Ob}(C)$ are sets. Examples of such things include all finite ordinal numbers $\mathbf{n}$, all monoids (small categories having one object), and all groups.

If $C$ is a small category there is a simplicial set $BC$ with

$$BC_n = \hom(\mathbf{n}, C),$$

meaning the functors $\mathbf{n} \to C$. A functor $\alpha : \mathbf{n} \to C$ can be identified with a string of arrows

$$\alpha(0) \to \alpha(1) \to \cdots \to \alpha(n)$$

of length $n$ in $C$.

The simplicial structure on $BC$ is defined by precomposition with ordinal number maps. The object $BC$ is called, variously, the *classifying space* or *nerve* of $C$.

Note that $B\mathbf{n} = \Delta^n$ in this notation.

5) Suppose that $I$ is a small category, and that $X : I \to \mathbf{Set}$ is a set-valued functor. The *category of elements* (or *translation category*, or *slice category*)

$$*/X = E_I(X)$$

associated to $X$ has as objects all pairs $(i, x)$ with $x \in X(i)$, or equivalently all functions

$$* \xrightarrow{x} X(i).$$

A morphism $\alpha : (i, x) \to (j, y)$ is a morphism $\alpha : i \to j$ of $I$ such that $\alpha_*(x) = y$, or equivalently a commutative diagram

$$\begin{tikzcd} & X(i) \arrow[d, "\alpha_*"] \\ * \arrow[ur, "x"] \arrow[r, "y"'] & X(j) \end{tikzcd}$$

The simplicial set $B(E_I X)$ is often called the *homotopy colimit* for the functor $X$, and one writes

$$\underrightarrow{\mathrm{holim}}_I X = B(E_I X).$$

**Example**: $BI = \underrightarrow{\mathrm{holim}}_I *$.

There is a canonical functor $E_I X \to I$ which is defined by the assignment $(i, x) \mapsto i$, which induces a canonical simplicial set map

$$\pi : B(E_I X) = \underrightarrow{\mathrm{holim}}_I X \to BI.$$

The functors $\mathbf{n} \to E_I X$ can be identified with strings

$$(i_0, x_0) \xrightarrow{\alpha_1} (i_1, x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n, x_n).$$

Note that such a string is uniquely specified by the underlying string $i_0 \to \cdots \to i_n$ in the index category $Y$ and $x_0 \in X(i_0)$. It follows that there is an identification

$$(\underrightarrow{\mathrm{holim}}_I\, X)_n = B(E_I X)_n = \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0).$$

The construction is natural with respect to natural transformations in $X$. Thus a diagram $X : I \to s\mathbf{Set}$ in simplicial sets determines a bisimplicial set with $(n,m)$ simplices

$$B(E_I X)_m = \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0)_m.$$

The *diagonal* $d(Y)$ of a bisimplicial set $Y$ is the simplicial set with $n$-simplices $Y_{n,n}$. Equivalently, $d(Y)$ is the composite functor

$$\mathbf{\Delta}^{op} \xrightarrow{\Delta} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{Y} \mathbf{Set}$$

where $\mathbf{\Delta}$ is the diagonal functor.

The diagonal $dB(E_I X)$ of the bisimplicial set $B(E_I X)$ is the *homotopy colimit* $\underrightarrow{\mathrm{holim}}_I\, X$ of the functor $X : I \to s\mathbf{Set}$, and there is a natural simplicial set map

$$\pi : \underrightarrow{\mathrm{holim}}_I\, X \to BI.$$

6) Suppose that $X$ and $Y$ are simplicial sets. There is a simplicial set $\mathbf{hom}(X, Y)$ with $n$-simplices

$$\mathbf{hom}(X, Y)_n = \hom(X \times \Delta^n, Y),$$

called the *function complex*.

There is a natural simplicial set map

$$ev : X \times \mathbf{hom}(X, Y) \to Y$$

which sends the pair $(x, f : X \times \Delta^n \to Y)$ to the simplex $f(x, \iota_n)$.

Suppose that $K$ is another simplicial set. The function

$$ev_* : \hom(K, \mathbf{hom}(X, Y)) \to \hom(X \times K, Y),$$

which is defined by sending the map $g : K \to \mathbf{hom}(X, Y)$ to the composite

$$X \times K \xrightarrow{1 \times g} X \times \mathbf{hom}(X, Y) \xrightarrow{ev} Y,$$

is a natural bijection, giving the exponential law

$$\hom(K, \mathbf{hom}(X, Y)) \cong \hom(X \times K, Y).$$

This natural isomorphism gives $s\mathbf{Set}$ the structure of a cartesian closed category. The function complexes also give $s\mathbf{Set}$ the structure of a category enriched in simplicial sets.

## 2 The simplex category and realization

The *simplex category* $\mathbf{\Delta}/X$ for a simplicial set $X$ has for objects all simplices $\Delta^n \to X$; its morphisms are the incidence relations between the simplices, meaning all commutative diagrams

$$\begin{tikzcd} \Delta^m \arrow[d, "\theta"'] \arrow[dr, "\tau"] & \\ \Delta^n \arrow[r, "\sigma"'] & X \end{tikzcd} \tag{2.1}$$

**Fact**: Every simplicial set $X$ is a colimit of its simplices, in that the simplices $\Delta^n \to X$ define a simplicial set map

$$\varinjlim_{\Delta^n \to X} \Delta^n \to X$$

which is an isomorphism.

The *realization* $|X|$ of a simplicial set $X$ is defined by

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

The assignment $X \mapsto |X|$ defines a functor

$$|\ | : s\mathbf{Set} \to \mathbf{CGHaus}.$$

**Lemma 2.1.** *The realization functor is left adjoint to the singular functor $S : \mathbf{CGHaus} \to s\mathbf{Set}$.*

**Examples**:

1) $|\Delta^n| = |\Delta^n|$, since the simplex category $\mathbf{\Delta}/\Delta^n$ has a terminal object, namely $1 : \Delta^n \to \Delta^n$.

2) $|\partial\Delta^n| = \partial|\Delta^n|$, and $|\Lambda^n_k|$ is the part of the boundary $\partial|\Delta^n|$ with the face opposite the vertex $k$ removed, since the realization functor is a left adjoint and therefore preserves coequalizers and coproducts.

The $n^{th}$ *skeleton* $\operatorname{sk}_n X$ of a simplicial set $X$ is the subobject generated by the simplices $X_i$, $0 \leq i \leq n$. The ascending sequence of subcomplexes

$$\operatorname{sk}_0 X \subset \operatorname{sk}_1 X \subset \operatorname{sk}_2 X \subset \dots$$

defines a filtration of $X$, and there are pushout diagrams

$$\begin{tikzcd}
\bigsqcup_{x \in NX_n} \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} X \arrow[d] \\
\bigsqcup_{x \in NX_n} \Delta^n \arrow[r] & \operatorname{sk}_n X
\end{tikzcd}$$

Here, $NX_n$ denotes the set of non-degenerate $n$-simplices of $X$.

**Facts**: 1) The realization of a simplicial set is a $CW$-complex.

2) Every monomorphism $A \to B$ of simplicial sets induces a cofibration $|A| \to |B|$ of spaces. In fact, $|B|$ is constructed from $|A|$ by attaching cells.

**Lemma 2.2.** *The realization functor preserves finite limits. Equivalently, it preserves finite products and equalizers.*

## 3 Model structure for simplicial sets

This section summarizes material which is presented in some detail in [3].

Say that a map $f : X \to Y$ of simplicial sets is a *weak equivalence* if the induced map $f_* : |X| \to |Y|$ is a weak equivalence of **CGHaus**.

A map $i : A \to B$ of simplicial sets is a *cofibration* if and only if it is a monomorphism, meaning that all functions $i : A_n \to B_n$ are injective.

A simplicial set map $p : X \to Y$ is a *fibration* if and only if it has the right lifting property with respect to all trivial cofibrations.

**Theorem 3.1.** *With these definitions of weak equivalence, cofibration and fibration, the category $s$**Set** of simplicial sets satisfies the axioms for a closed model category.*

Here are the basic ingredients of the proof:

**Lemma 3.2.** *A map $p : X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

The proof of this Lemma is formal.

The following can be proved with simplicial approximation techniques [2]:

**Lemma 3.3.** *Suppose that a simplicial set $X$ has at most countably many non-degenerate simplices. Then the set of path components $\pi_0|X|$ and all homotopy groups $\pi_n(|X|, x)$ are countable.*

Here's a consequence:

**Lemma 3.4** (Bounded cofibration lemma)**.** *Suppose that $i : X \to Y$ is a trivial cofibration and that $A \subset Y$ is a countable subcomplex. Then there is a countable subcomplex $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a trivial cofibration.*

Lemma 3.4 implies that the set of countable trivial cofibrations generates the class of all trivial cofibrations, while the Lemma 3.2 implies that the set of all inclusions $\partial\Delta^n \subset \Delta^n$ generates the class of all cofibrations. Theorem 3.1 follows from small object arguments.

**Remark 3.5.** The realization functor preserves cofibrations and trivial cofibrations. It's an immediate consequence that the singular functor $S$ preserves fibrations and trivial fibrations. It also follows that the adjoint pair

$$| \, | : s\mathbf{Set} \leftrightarrows \mathbf{CGHaus} : S,$$

is a *Quillen adjunction.*

A *Kan fibration* is a map $p : X \to Y$ of simplicial sets which has the right lifting property with respect to all inclusions $\Lambda^n_k \subset \Delta^n$. A *Kan complex* is a simplicial set $X$ for which the canonical map $X \to *$ is a Kan fibration.

Every fibration is a Kan fibration. Every fibrant simplicial set is a Kan complex.

Kan complexes $Y$ have combinatorially defined homotopy groups: if $x \in Y_0$ is a vertex of $Y$, then

$$\pi_n(Y, x) = \pi((\Delta^n, \partial\Delta^n), (Y, x))$$

where $\pi(\ ,\ )$ denotes simplicial homotopy classes of maps. The path components of any simplicial set $X$ are defined by the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the maps $X_1 \to X_0$ are the face maps $d_0, d_1$.

Say that a map $f : Y \to Y'$ of Kan complexes is a *combinatorial weak equivalence* if it induces isomorphisms

$$\pi_n(Y, x) \xrightarrow{\cong} \pi_n(Y', f(x))$$

for all $x \in Y_0$, and

$$\pi_0(Y) \xrightarrow{\cong} \pi_0(Y').$$

Going further requires the following major theorem, due to Quillen:

**Theorem 3.6.** *The realization of a Kan fibration is a Serre fibration.*

The proof of this result requires much of the classical homotopy theory of Kan complexes (in particular the theory of minimal fibrations), and will not be discussed here.

Here are the consequences:

**Theorem 3.7** (Milnor theorem)**.** *Suppose that $Y$ is a Kan complex and that $\eta : Y \to S(|Y|)$ is the adjunction homomorphism. Then $\eta$ is a combinatorial weak equivalence.*

It follows that the combinatorial homotopy groups of $\pi_n(Y, x)$ coincide up to natural isomorphism with the ordinary homotopy groups $\pi_n(|Y|, x)$ of the realization, for all Kan complexes $Y$.

The proof is an inductive long exact sequence argument using path-loop fibre sequences in simplicial sets. These are Kan fibre sequences, and the key is to know that their realizations are fibre sequences.

**Theorem 3.8.** *Every Kan fibration is a fibration.*

The key step in the proof of Theorem 3.8 is to show, using Theorem 3.7, that every map which is a Kan fibration and a weak equivalence is a trivial fibration. This is used to show that every trivial cofibration has the left lifting property with respect to all Kan fibrations. It follows that every Kan fibration has the right lifting property with respect to all trivial cofibrations.

**Remark 3.9.** Theorem 3.8 implies that the model structure of Theorem 3.1 consists of cofibrations, Kan fibrations and weak equivalences. This is the standard, classical model structure for simplicial sets. The identification of the fibrations with Kan fibrations is the "hard" part of its construction.

**Theorem 3.10.** *The adjunction maps* $\eta : X \to S(|X|)$ *and* $\epsilon : |S(Y)| \to Y$ *are weak equivalences, for all simplicial sets* $X$ *and spaces* $Y$*, respectively.*

In particular, the standard model structures on $s$**Set** and **CGHaus** are Quillen equivalent.

## References


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