Lecture 02

4 Grothendieck topologies

A *Grothendieck site* is a small category $\mathcal{C}$ equipped with a topology $\mathcal{T}$.

A *Grothendieck topology* $\mathcal{T}$ consists of a collection of subfunctors

$$R \subset \hom(\ ,U), \quad U \in \mathcal{C},$$

called *covering sieves*, such that the following axioms hold:

1) (base change) If $R \subset \hom(\ ,U)$ is covering and $\phi : V \to U$ is a morphism of $\mathcal{C}$, then the subfunctor

$$\phi^{-1}(R) = \{\gamma : W \to V \mid \phi \cdot \gamma \in R\}$$

is covering for V.

2) (local character) Suppose that $R, R' \subset \hom(\ ,U)$ are subfunctors and R is covering. If $\phi^{-1}(R')$ is covering for all $\phi : V \to U$ in R, then R' is covering.

3) $\hom(\ ,U)$ is covering for all $U \in \mathcal{C}$

Typically Grothendieck topologies arise from covering families in sites $\mathcal{C}$ having pullbacks. Covering families are sets of functors which generate covering sieves.

Suppose that $\mathcal{C}$ has pullbacks. A topology $\mathcal{T}$ on $\mathcal{C}$ consists of families of sets of morphisms

$$\{\phi_\alpha : U_\alpha \to U\}, \quad U \in \mathcal{C},$$

called *covering families*, such that the following axioms hold:

1) Suppose that $\phi_\alpha : U_\alpha \to U$ is a covering family and that $\psi : V \to U$ is a morphism of $\mathcal{C}$. Then the collection $V \times_U U_\alpha \to V$ is a covering family for V.

2) If $\{\phi_\alpha : U_\alpha \to V\}$ is covering, and $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \to U_\alpha\}$ is covering for all α, then the family of composites

$$W_{\alpha,\beta} \xrightarrow{\gamma_{\alpha,\beta}} U_\alpha \xrightarrow{\phi_\alpha} U$$

is covering.

3) The family $\{1 : U \to U\}$ is covering for all $U \in \mathcal{C}$.

Examples:

1) X = topological space. $\mathrm{op}\,|_X$ is the poset of open subsets $U \subset X$. A covering family for an open subset U is an open cover $V_\alpha \subset U$.

2) X = topological space. $\mathrm{loc}\,|_X$ is the category of all maps $f : Y \to X$ which are local homeomorphisms. f is a local homeomorphism if each $x \in Y$ has a neighbourhood U such that $f(U)$ is open in X and the restricted map $U \to f(U)$ is a homeomorphism. A morphism of $\mathrm{loc}\,|_X$ is a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

where f and f' are local homeomorphisms. A family $\{\phi_\alpha : Y_\alpha \to Y\}$ of local homeomorphisms (over X) is covering if $Y = \cup\phi_\alpha(Y_\alpha)$.

3) X = a scheme (topological space with sheaf of rings locally isomorphic to affine schemes $\mathrm{Sp}(R)$). The underlying topology on X is the Zariski topology. $Zar|_X$ is the poset with objects all open subschemes $U \subset X$. A family $V_\alpha \subset U$ is covering if $\cup V_\alpha = U$ (as sets).

A scheme homomorphism $\phi : Y \to X$ is *étale at* $y \in Y$ if

a) $\mathcal{O}_y$ is a flat $\mathcal{O}_{f(y)}$-module (ϕ is flat at y).

b) ϕ is unramified at y: $\mathcal{O}_y/\mathcal{M}_{f(y)}\mathcal{O}_y$ is a finite separable field extension of $k(f(y))$.

Say that a map $\phi : Y \to X$ is *étale* if it is étale at every $y \in Y$ (and locally of finite type).

4) $S =$ scheme. The étale site $et|_S$ has as objects all étale maps $\phi : V \to S$ and all diagrams

$$\begin{tikzcd} V \arrow[rr] \arrow[dr, "\phi"'] & & V' \arrow[dl, "\phi'"] \\ & S & \end{tikzcd}$$

for morphisms (with ϕ, ϕ' étale). A covering family for the étale site is a collection of étale morphisms $\phi_\alpha : V_\alpha \to V$ such that $V = \cup\phi_\alpha(V_\alpha)$ as a set. Equivalently every morphism $\mathrm{Sp}(\Omega) \to V$ lifts to some V_α if Ω is a separably closed field.

5) The Nisnevich site $Nis|_S$ has the same underlying category as the étale site, namely all étale maps $V \to S$ and morphisms between them. A Nisnevich cover is a family of étale maps $V_\alpha \to V$ such that every morphism $\mathrm{Sp}(K) \to V$ lifts to some V_α where K is any field.

6) A flat covering family of a scheme T is a set of flat morphisms $\phi_\alpha : T_\alpha \to T$ (ie. morphisms which are flat at each point) such that $T = \cup \phi_\alpha(T_\alpha)$ as a set (equivalently $\sqcup T_\alpha \to T$ is faithfully flat).

   $(Sch|_S)_{fl}$ is the "big" flat site. Pick a large cardinal κ; then $(Sch|_S)$ is the category of S-schemes $X \to S$ such that the cardinality of both the underlying point set of X and all sections $\mathcal{O}_X(U)$ of its sheaf of rings are bounded above by κ.

7) There are corresponding big sites $(Sch|_S)_{Zar}$, $(Sch|_S)_{et}$, $(Sch|_S)_{Nis}$, ... and you can play similar games with topological spaces.

8) Suppose that $G = \{G_i\}$ is profinite group such that all $G_j \to G_i$ are surjective group homomorphisms. Write also $G = \varprojlim G_i$. A discrete G-set is a set X with G-action which factors through an action of G_i for some i. Write $G - \mathbf{Set}_{df}$ for the category of G-sets which are both discrete and finite. A family $U_\alpha \to X$ in this category is covering if and only if $\bigsqcup U_\alpha \to X$ is surjective.

9) Suppose that $\mathcal{C}$ is any small category. Say that $R \subset \hom(\ , x)$ is covering if and only if $1_x \in R$. This is the *chaotic topology* on $\mathcal{C}$.

10) Suppose that $\mathcal{C}$ is a site and that $U \in \mathcal{C}$. Then the slice category $\mathcal{C}/U$ inherits a topology from $\mathcal{C}$: a collection of maps $V_\alpha \to V \to U$ is covering if and only if the family $V_\alpha \to V$ covers V.

Definitions: Suppose that $\mathcal{C}$ is a Grothendieck site.

1) A *presheaf* (of sets) on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to$ **Set**. If $\mathcal{A}$ is a category, an $\mathcal{A}$-valued presheaf on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathcal{A}$.

The set-valued presheaves on $\mathcal{C}$ form a category (morphisms are natural transformation), written $\operatorname{Pre}(\mathcal{C})$. One can talk about presheaves taking values in any category: I write $s\operatorname{Pre}(\mathcal{C})$ for presheaves on $\mathcal{C}$ taking values in simplicial sets — this is the category of simplicial presheaves on $\mathcal{C}$.

2) A *sheaf* (of sets) on $\mathcal{C}$ is a presheaf $F : \mathcal{C}^{op} \to$ **Set** such that the canonical map

$$F(U) \to \varprojlim_{V \to U \in R} F(V)$$

is an isomorphism for each covering sieve $R \subset \hom(\ , U)$.

Morphisms of sheaves are natural transformations: write $\mathrm{Shv}(\mathcal{C})$ for the corresponding category. The sheaf category $\mathrm{Shv}(\mathcal{C})$ is a full subcategory of $\mathrm{Pre}(\mathcal{C})$. One can also speak of sheaves in any complete category, such as simplicial sets: $s\,\mathrm{Shv}(\mathcal{C})$ denotes the category of simplicial sheaves on the site $\mathcal{C}$.

Exercise: If the topology on $\mathcal{C}$ is defined by a pretopology (so that $\mathcal{C}$ has all pullbacks), then F is a sheaf if and only if all pictures

$$F(U) \to \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta)$$

arising from covering families $U_\alpha \to U$ are equalizers.

Lemma 4.1. *1) If $R \subset R' \subset \hom(\ , U)$ and R is covering then R' is covering.*

2) If $R, R' \subset \hom(\ , U)$ are covering then $R \cap R'$ is covering.

Proof. 1) $\phi^{-1}(R) = \phi^{-1}(R')$ for all $\phi \in R$.

2) $\phi^{-1}(R \cap R') = \phi^{-1}(R')$ for all $\phi \in R$. □

Suppose that $R \subset \hom(\ ,U)$ is a sieve, and F is a presheaf on $\mathcal{C}$. Write

$$F(U)_R = \varprojlim_{V \to U \in R} F(V)$$

I say that $F(U)_R$ is the set of *R-compatible families* in U. If $S \subset R$ then there is an obvious map

$$F(U)_R \to F(U)_S$$

Write

$$LF(U) = \varinjlim_{R} F(U)_R$$

where the colimit is indexed over the filtering diagram of all covering sieves $R \subset \hom(\ ,U)$. Then $x \mapsto LF(U)$ is a presheaf and there is a natural presheaf map

$$\eta : F \to LF$$

Say that a presheaf G is *separated* if (equivalently)

1) the map $\eta : G \to LG$ is monic in each section, ie. all functions $G(U) \to LG(U)$ are injective, or

2) Given $x, y \in G(U)$, if there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(x) = \phi^*(y)$ for all $\phi \in R$, then $x = y$.

Lemma 4.2. *1) LF is separated, for all presheaves F.*

2) If G is separated then LG is a sheaf.

3) If $f : F \to G$ is a presheaf map and G is a sheaf, then f factors uniquely through a presheaf map $f_* : LF \to G$.

The object L^2F is a sheaf for every presheaf F, and the functor $F \mapsto L^2F$ is left adjoint to the inclusion $\mathrm{Shv}(\mathcal{C}) \subset \mathrm{Pre}(\mathcal{C})$. The unit of the adjunction is the composite

$$F \xrightarrow{\eta} LF \xrightarrow{\eta} L^2F$$

One often writes $\eta : F \to L^2F = \tilde{F}$ for this composite.

5 Exactness properties

Lemma 5.1. *1) The associated sheaf functor preserves all finite limits.*

2) $\mathrm{Shv}(\mathcal{C})$ is complete and co-complete. Limits are formed sectionwise.

3) Every monic is an equalizer.

4) If $\theta : F \to G$ in $\mathrm{Shv}(\mathcal{C})$ is both monic and epi, then θ is an isomorphism.

Proof. 1) LF is defined by filtered colimits, and finite limits commute with filtered colimits.

2) If $X : I \to \mathrm{Shv}(\mathcal{C})$ is a diagram of sheaves, then the colimit in the sheaf category is $L^2(\varinjlim X)$, where $\varinjlim X$ is the presheaf colimit.

3) If $A \subset X$ is a subset, then there is an equalizer

$$A \longrightarrow X \overset{p}{\underset{*}{\rightrightarrows}} X/A$$

The same holds for subobjects $A \subset X$ of presheaves, and hence for subobjects of sheaves, since L^2 is exact.

4) The map θ appears in an equalizer

$$F \xrightarrow{\theta} G \overset{f}{\underset{g}{\rightrightarrows}} K$$

since θ is monic. θ is an epi, so $f = g$. But then $1_G : G \to G$ factors through θ, giving a section $\sigma : G \to F$. Finally, $\theta\sigma\theta = \theta$ and θ is monic, so $\sigma\theta = 1$. $\square$

Definitions:

1) A presheaf map $f : F \to G$ is a *local epimorphism* if for each $\alpha \in G(x)$ there is a covering $R \subset \hom(\ ,x)$ such that $\phi^*(x) = f(y_\phi)$ for all $\phi \in R$.

2) $f : F \to G$ is a *local monic* if given $\alpha, \beta \in F(x)$ such that $f(\alpha) = f(\beta)$, there is a covering $R \subset \hom(\ , x)$ such that $\phi^*(\alpha) = \phi^*(\beta)$ for all $\phi \in R$.

3) A presheaf map $f : F \to G$ which is both a local epi and a local monic is a *local isomorphism*.

Lemma 5.2. *1) The natural map $\eta : F \to LF$ is a local monomorphism and a local epimorphism.*

2) Suppose that $f : F \to G$ is a presheaf morphism. Then f induces an isomorphism of associated sheaves if and only if f is both a local epi and a local monic.

Proof. For 2) observe that, given a commutative diagram

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[dr, "h"'] & F' \arrow[d, "f"] \\ & F'' \end{tikzcd}$$

of presheaf morphisms, if any two of f, g and h are local isomorphisms, then so is the third.

A sheaf map $g : E \to E'$ is a monic (respectively epi) if and only if it is a local monic (respectively local epi). $\square$

A *Grothendieck topos* is a category $\mathcal{E}$ which is equivalent to a sheaf category $\mathrm{Shv}(\mathcal{C})$ on some Grothendieck site $\mathcal{C}$.

Grothendieck toposes are characterized by exactness properties:

Theorem 5.3 (Giraud)**.** *A category $\mathcal{E}$ having all finite limits is a Grothendieck topos if and only if it has the following properties:*

1) $\mathcal{E}$ has all small coproducts; they are disjoint and stable under pullback

2) every epimorphism of $\mathcal{E}$ is a coequalizer

3) every equivalence relation $R \rightrightarrows E$ in $\mathcal{E}$ is a kernel pair and has a quotient

4) every coequalizer $R \rightrightarrows E \to Q$ is stably exact

5) there is a (small) set of objects which generates $\mathcal{E}$.

A sketch proof of Giraud's Theorem appears below, but the result is proved in many places — see, for example, [2], [3], [1].

Here are the definitions of the terms appearing in the statement of Giraud's Theorem:

1) The coproduct $\bigsqcup_i A_i$ is *disjoint* if all diagrams

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A_j \arrow[d] \\ A_i \arrow[r] & \bigsqcup_i A_i \end{tikzcd}$$

are pullbacks for $i \neq j$. $\bigsqcup_i A_i$ is *stable under pullback* if all diagrams

$$\begin{tikzcd} \bigsqcup_i B' \times_B A_i \arrow[r] \arrow[d] & \bigsqcup_i A_i \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

are pullbacks.

3) An *equivalence relation* is a monomorphism $m = (m_0, m_1) : R \to E \times E$ such that

a) the diagonal $\Delta : E \to E \times E$ factors through m (ie. $a \sim a$)

b) the composite $R \xrightarrow{m} E \times E \xrightarrow{\tau} E \times E$ factors through m (ie. $a \sim b \Rightarrow b \sim a$).

c) the map

$$(m_0 m_{0*}, m_1 m_{1*}) : R \times_E R \to E \times E$$

factors through m (this is transitivity) where the pullback is defined by

$$\begin{tikzcd} R \times_E R \arrow[r, "m_{1*}"] \arrow[d, "m_{0*}"'] & R \arrow[d, "m_0"] \\ R \arrow[r, "m_1"'] & E \end{tikzcd}$$

The *kernel pair* of a morphism $u : E \to D$ is a pullback

$$\begin{tikzcd} R \arrow[r, "m_1"] \arrow[d, "m_0"'] & E \arrow[d, "u"] \\ E \arrow[r, "u"'] & D \end{tikzcd}$$

(Exercise: every kernel pair is an equivalence relation).

A *quotient* for an equivalence relation $(m_0, m_1) : R \to E \times E$ is a coequalizer

$$R \underset{m_1}{\overset{m_0}{\rightrightarrows}} E \longrightarrow E/R$$

4) A coequalizer $R \rightrightarrows E \to Q$ is *stably exact* if the diagram

$$R \times_Q Q' \rightrightarrows E \times_Q Q' \to Q'$$

is a coequalizer for all morphisms $Q' \to Q$.

5) A *generating set* is a set $\{A_i\}$ which detects non-trivial monomorphisms: if a monomorphism

$m : P \to Q$ induces bijections $\hom(A_i, P) \to \hom(A_i, Q)$ for all i, then m is an isomorphism.

Exercise: Show that any category $\mathrm{Shv}(C)$ on a site $\mathcal{C}$ satisfies the conditions of Giraud's theorem. The family $L^2 \hom(\ , U)$, $U \in \mathcal{C}$ is a set of generators.

Sketch proof of Giraud's Theorem. The key is to show that the category $\mathcal{E}$ has coequalizers, and is therefore cocomplete — see [2], [1].

If A is the set of generators for $\mathcal{E}$ prescribed by Giraud's theorem, let $\mathcal{C}$ be the full subcategory of $\mathcal{E}$ on the set of objects A. A subfunctor $R \subset \hom(\ , x)$ on $\mathcal{C}$ is covering if the map

$$\bigsqcup_{y \to x \in R} y \to x$$

is an epimorphism of $\mathcal{E}$.

Every object $E \in \mathcal{E}$ represents a sheaf $\hom(\ , E)$ on $\mathcal{C}$, and a sheaf F on $\mathcal{C}$ determines an object

$$\varinjlim_{\hom(\ ,y) \to F} y$$

of $\mathcal{E}$.

The adjunction

$$\hom(\varinjlim_{\hom(\ ,y)\to F} y, E) \cong \hom(F, \hom(\ ,E))$$

determines an adjoint equivalence between $\mathcal{E}$ and $\mathrm{Shv}(\mathcal{C})$. $\square$

The proof of Giraud's Theorem is arguably more important than the statement of the Theorem itself. Here are some examples of the use of the basic ideas:

1) Suppose that G is a sheaf of groups, and let $G - \mathrm{Shv}(\mathcal{C})$ denote the category of all sheaves X admitting G-action, with equivariant maps between them. The objects $G \times \hom(\ ,x)$ form a generating set. By Giraud's Theorem, $G - \mathrm{Shv}(\mathcal{C})$ is a Grothendieck topos, and is called the *classifying topos* for G.

2) If $G = \{G_i\}$ is a profinite group with all transition maps $G_i \to G_j$ epi, then the category $G - \mathbf{Set}_d$ of discrete G-sets is a Grothendieck topos. The finite discrete G-sets form a generating set for this topos, and the site of finite discrete G-sets is a small fattening of the site prescribed by Giraud's Theorem. The site that is specified by Giraud's Theorem is the orbit category.

6 Geometric morphisms

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck sites. A *geometric morphism* $f : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ consists of functors $f_* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ and $f^* : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ such that f^* is left adjoint to f_* and f^* preserves finite limits.

The left adjoint f^* is called the *inverse image* functor, while f_* is called the *direct image*.

The functor f^* is left and right exact in the sense that it preserves all finite limits and colimits; f_* is usually not left exact (does not preserve finite colimits), and hence has higher derived functors.

Examples

1) Suppose $f : X \to Y$ is a continuous map of topological spaces. Pullback along f induces a functor $\mathrm{op}\,|_Y \to \mathrm{op}\,|_X$: $U \subset Y \mapsto f^{-1}(U)$. Open covers pull back to open covers, so if F is a sheaf on X then composition with the pullback gives a sheaf f_*F on Y with $f_*F(U) = F(f^{-1}(U))$. The resulting functor $f_* : \mathrm{Shv}(\mathrm{op}\,|_X) \to \mathrm{Shv}(\mathrm{op}\,|_Y)$ is the direct image

The left Kan extension $f^p : \mathrm{Pre}(\mathrm{op}\,|_Y) \to \mathrm{Pre}(\mathrm{op}\,|_X)$

is defined by

$$f^p G(V) = \varinjlim G(U)$$

where the colimit is indexed over all diagrams

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The category $\mathrm{op}\,|_Y$ has all products (ie. intersections), so the colimit is filtered. The functor $G \mapsto f^p G$ therefore commutes with finite limits. The inverse image functor

$$f^* : \mathrm{Shv}(\mathrm{op}\,|_Y) \to \mathrm{Shv}(\mathrm{op}\,|_X)$$

is defined by $f^*(G) = L^2 f^p(G)$. The resulting pair of functors forms a geometric morphism $f : \mathrm{Shv}(\mathrm{op}\,|_X) \to \mathrm{Shv}(\mathrm{op}\,|_Y)$.

2) Suppose that $f : X \to Y$ is a morphism of schemes. Etale maps (resp. covers) are stable under pullback, and so there is a functor $\mathrm{et}\,|_Y \to \mathrm{et}\,|_X$ defined by pullback, and if F is a sheaf on $\mathrm{et}\,|_X$ then there is a sheaf f_*F on $\mathrm{et}\,|_Y$ defined by $f_*F(V \to Y) = F(X \times_Y V \to X)$.

The restriction functor $f_* : \mathrm{Pre}(\mathrm{et}\,|_X) \to \mathrm{Pre}(\mathrm{et}\,|_Y)$ has a left adjoint f^p defined by

$$f^p G(U \to X) = \varinjlim G(V)$$

where the colimit is indexed over all diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where both vertical maps are étale. The colimit is filtered, essentially because étale maps are stable under pullback and composition. The inverse image functor

$$f^* : \mathrm{Shv}(\mathrm{et}\,|_Y) \to \mathrm{Shv}(et|_X)$$

is defined by $f^*F = L^2 f^p F$, and so f induces a geometric morphism $f : \mathrm{Shv}(\mathrm{et}\,|_X) \to \mathrm{Shv}(\mathrm{et}\,|_Y)$.

A morphism of schemes $f : X \to Y$ induces a geometric morphism $f : \mathrm{Shv}(?|_X) \to \mathrm{Shv}(?|_Y)$ and/or $f : (Sch|_X)_? \to (Sch|_Y)_?$ for all of the geometric topologies (eg. Zariski, flat, Nisnevich, qfh, ...), by similar arguments.

3) A *point* of $\mathrm{Shv}(\mathcal{C})$ is a geometric morphism $\mathbf{Set} \to \mathrm{Shv}(\mathcal{C})$. Every point $x \in X$ of a topological space X determines a continuous map $\{x\} \subset X$ and hence a geometric morphism

$$\mathbf{Set} \cong \mathrm{Shv}(\mathrm{op}\,|_{\{x\}}) \xrightarrow{x} \mathrm{Shv}(\mathrm{op}\,|_X)$$

The set

$$x^*F = \varinjlim_{x \in U} F(U)$$

is the *stalk* of F at x

4) Suppose that k is a field. Any scheme map $x : \mathrm{Sp}(k) \to X$ induces a geometric morphism

$$\mathrm{Shv}(et|_k) \to \mathrm{Shv}(et|_X)$$

If k happens to be separably closed, then there is an equivalence $\mathrm{Shv}(et|_k) \simeq \mathbf{Set}$ and the resulting geometric morphism $x : \mathbf{Set} \to \mathrm{Shv}(\mathrm{et}\,|_X)$ is called a geometric point of X. The inverse image functor

$$F \mapsto f^*F = \varinjlim_{\begin{tikzcd}[ampersand replacement=\&] \& U \arrow[d] \\ \mathrm{Sp}(k) \arrow[ur] \arrow[r, "x"'] \& X \end{tikzcd}} F(U)$$

is the stalk of F at x.

5) Suppose that S and T are topologies on a site $\mathcal{C}$ so that $S \subset T$. In other words, T has more covers than S and hence refines S. Then every sheaf for T is a sheaf for S; write

$$\pi_* : \mathrm{Shv}(\mathcal{C}, T) \subset \mathrm{Shv}(\mathcal{C}, S)$$

for the corresponding inclusion. The associated sheaf functor for the topology T gives a left adjoint π^* for the inclusion functor π_*, and of course π^* preserves finite limits.

Here's an example: there is a geometric morphism

$$\mathrm{Shv}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C})$$

determined by the inclusion of the sheaf category in the presheaf category and the associated sheaf functor.

7 Points

Say that a Grothendieck topos $\mathrm{Shv}(\mathcal{C})$ has *enough points* if there is a set of geometric morphisms $x_i : \mathbf{Set} \to \mathrm{Shv}(\mathcal{C})$ such that the induced morphism

$$\mathrm{Shv}(\mathcal{C}) \xrightarrow{(x_i^*)} \prod_i \mathbf{Set}$$

is faithful.

Lemma 7.1. *Suppose that $f : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ is a geometric morphism. Then the following are equivalent:*

a) $f^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ is faithful.

b) f^* reflects isomorphisms

c) f^* reflects epimorphisms

d) f^* reflects monomorphisms

Proof. Suppose that f^* is faithful, ie. that $f^*(g_1) = f^*(g_2)$ implies that $g_1 = g_2$. Suppose that $m : F \to G$ is a morphism of $\mathrm{Shv}(\mathcal{C})$ such that $f^*(m)$ is monic. If $m \cdot f_1 = m \cdot f_2$ then $f^*(f_1) = f^*(f_2)$ so $f_1 = f_2$. The map m is therefore monic. Similarly f^* reflects epimorphisms and hence isomorphisms.

Suppose that f^* reflects epimorphisms and suppose given $g_1, g_2 : F \to G$ such that $f^*(g_1) = f^*(g_2)$. $g_1 = g_2$ if and only if their equalizer $e : E \to F$ is an epimorphism. But f^* preserves equalizers and reflects epimorphisms, so e is an epi and $g_1 = g_2$. The other arguments are similar. $\square$

Here are some basic definitions:

1) A *lattice* L is a partially ordered set which has all finite coproducts $x \vee y$ and all finite products $x \wedge y$.

NB: The collection of finite coproducts includes the empty coproduct, which is an initial object 0. Similarly, the empty product, which is finite product, is a terminal object 1. Every lattice L, defined as above, has both an initial object 0 and a terminal object 1.

2) A lattice L is said to be *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all x, y, z.

3) A *complement* for x in a lattice L with 0 and 1 is an element a such that $x \vee a = 1$ and $x \wedge a = 0$, if it exists.

If L is also distibutive, then the complement, if it exists, is unique: if b is another complement for x, then

$$\begin{aligned} b &= b \wedge 1 = b \wedge (x \vee a) = (b \wedge x) \vee (b \wedge a) \\ &= (x \wedge a) \vee (b \wedge a) = (x \vee b) \wedge a = a \end{aligned}$$

One usually writes $\neg x$ for the complement of x.

4) A *Boolean algebra* $\mathcal{B}$ is a distributive lattice in which every element has a complement.

5) A lattice L is said to be *complete* if it has all small limits and colimits (aka. all small meets and joins).

6) A *frame* P is a lattice which has all small joins (and all finite meets) and which satisfies an infinite distributive law

$$U \wedge (\bigvee_i V_i) = \bigvee_i (U \wedge V_i)$$

Remark: There is a frame category whose objects are the frames and the morphisms are the poset maps which preserve structure. The category of locales is the opposite category of the frame category. I tend to use the term "locale" instead of frame.

Examples:

1) The poset $\mathcal{O}(T)$ of open subsets of a topological space T is a frame. Every continuous map $f : S \to T$ induces a morphism of frames $f^{-1} : \mathcal{O}(T) \to \mathcal{O}(S)$, defined by $U \mapsto F^{-1}(U)$.

2) The power set $\mathcal{P}(I)$ of a set I is a complete Boolean algebra.

3) Every complete Boolean algebra $\mathcal{B}$ is a frame. For the infinite distributive law, observe that every join is a filtered colimit of finite joins.

Every frame A has a canonical Grothendieck topology: a family $y_i \leq x$ is covering if $\bigvee_i y_i = x$. Write $\mathrm{Shv}(A)$ for the corresponding sheaf category. Every complete Boolean algebra $\mathcal{B}$ is a frame, and therefore has an associated sheaf category $\mathrm{Shv}(\mathcal{B})$.

Example: Suppose that I is a set. Then there is an equivalence

$$\mathrm{Shv}(\mathcal{P}(I)) \simeq \prod_{i \in I} \mathbf{Set}$$

If F is a sheaf on $\mathcal{P}(I)$ and $A \subset I$, then

$$F(A) \cong \prod_{x \in A} F(\{x\}).$$

Any set I of points $x_j : \mathbf{Set} \to \mathrm{Shv}(\mathcal{C})$ assembles to give a geometric morphism

$$x : \mathrm{Shv}(\mathcal{P}(I)) \to \mathrm{Shv}(\mathcal{C}).$$

Lemma 7.2. *Suppose that F is a sheaf of sets on a complete Boolean algebra $\mathcal{B}$. Then the poset $Sub(F)$ of subobjects of F is a complete Boolean algebra.*

Proof. $Sub(F)$ is a frame, by an argument on the presheaf level. It remains to show that every object $G \in Sub(F)$ is complemented. The obvious candidate for $\neg G$ is

$$\neg G = \bigvee_{H \wedge G = \emptyset} H$$

and we need to show that $G \bigvee \neg G = F$.

Every $K \leq \hom(\ ,A)$ is representable: in effect,

$$K = \varinjlim_{\hom(\ ,B) \to K} \hom(\ ,B) = \hom(\ ,C)$$

where

$$C = \bigvee_{\hom(\ ,B) \to K} B \in \mathcal{B}.$$

It follows that $Sub(\hom(\ ,A)) \cong Sub(A)$ is a complete Boolean algebra.

Consider all diagrams

$$\begin{tikzcd}
\phi^{-1}(G) \arrow[r] \arrow[d] & G \arrow[d] \\
\hom(\ ,A) \arrow[r, "\phi"'] & F
\end{tikzcd}$$

There is an induced pullback

$$\begin{tikzcd}
\phi^{-1}(G) \vee \neg\phi^{-1}(G) \arrow[r] \arrow[d, "\cong"'] & G \vee \neg G \arrow[d] \\
\hom(\ ,A) \arrow[r, "\phi"'] & F
\end{tikzcd}$$

F is a union of its representables (all ϕ are monic since all $\hom(\ ,A)$ are subobjects of the terminal sheaf), so $G \vee \neg G = F$. $\square$

Lemma 7.3. *Suppose that $\mathcal{B}$ is a complete Boolean algebra. Then every epimorphism $\pi : F \to G$ in* $\mathrm{Shv}(\mathcal{B})$ *has a section.*

Remark 7.4. Lemma 7.3 asserts that the sheaf category on a complete Boolean algebra satisfies the Axiom of Choice.

Proof. Consider the family of lifts

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ N \arrow[ur] \arrow[r, "\leq"'] & G \end{tikzcd}$$

This family is non-empty, because every $x \in G(1)$ restricts along some covering $B \leq 1$ to a family of elements x_B which lift to $F(B)$.

All maps $\hom(\ ,B) \to G$ are monic, since all maps $\hom(\ ,B) \to \hom(\ ,1) = *$ are monic. Thus, all such morphisms represent objects of $Sub(G)$, which is a complete Boolean algebra by Lemma 7.2.

Zorn's Lemma implies that the family of lifts has maximal elements.

Suppose that N is maximal and that $\neg N \neq \emptyset$. Then there is an $x \in \neg N(C)$ for some C, and there is a covering $B' \leq C$ such that $x_{B'} \in N(B')$ lifts to $F(B')$ for all members of the cover. Then $N \wedge \hom(\ ,B') = \emptyset$ so the lift extends to a lift on $N \vee \hom(\ ,B')$, contradicting the maximality of N. $\square$

A *Boolean localization* for $\mathrm{Shv}(\mathcal{C})$ is a geometric morphism $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ such that $\mathcal{B}$ is a complete Boolean algebra and p^* is faithful.

Theorem 7.5 (Barr)**.** *Boolean localizations exist for every Grothendieck topos* $\mathrm{Shv}(\mathcal{C})$*.*

Theorem 7.5 is proved in multiple places — see [2], for example. There is a shorter version of the proof in [1].

A Grothendieck topos $\mathrm{Shv}(\mathcal{C})$ does not have enough points, in general (eg. sheaves on the flat site for a scheme), but the result asserts that every Grothendieck topos has a "fat point" given by a Boolean localization. This is of fundamental importance in setting up the general local homotopy theory of simplicial sheaves and presheaves.

References

[1] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.

[2] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[3] Horst Schubert. *Categories*. Springer-Verlag, New York, 1972. Translated from the German by Eva Gray.