Lecture 03

8 Rigidity

Suppose that k is an algebraically closed field and let ℓ be a prime which is distinct from the characteristic of k.

We will be working with the big étale site $(Sch|_k)_{et}$ over the field k throughout this section. Note the abuse: I should have written $(Sch|_{\mathrm{Sp}(k)})_{et}$.

I shall use the notation Gl_n to represent either the algebraic group

$$Gl_n = \mathrm{Sp}(k[X_{ij}]_{det})$$

over k, or the sheaf of groups

$$Gl_n = \hom(\ ,Gl_n)$$

that it represents on the big site $(Sch|_k)_{et}$.

Observe that Gl_1 is the multiplicative group $\mathbb{G}_m$. One sometimes sees the notation $\mu = \mathbb{G}_m$, and one always sees the notation μ_ℓ for its ℓ-torsion part. Since the prime ℓ is distinct from the characteristic of the algebraically closed field k, there is an isomorphism

$$\mu_\ell \cong \Gamma^*\mathbb{Z}/\ell = \mathbb{Z}/\ell,$$

where $\Gamma^*\mathbb{Z}/\ell$ is the constant sheaf on the cyclic group $\mathbb{Z}/\ell$ and the displayed equality is a standard abuse.

In general, the constant sheaf functor $A \mapsto \Gamma^*(A)$ is left adjoint to the global sections functor $X \mapsto \Gamma_* X$, where

$$\Gamma_* X = X(k),$$

and there's a geometric morphism

$$\Gamma : \mathrm{Shv}((Sch|_k)_{et}) \to \mathbf{Set}.$$

This is a special case of a geometric morphism

$$\Gamma : \mathrm{Shv}(\mathcal{C}) \to \mathbf{Set}$$

defined by

$$\Gamma_* X = \varprojlim_{U \in \mathcal{C}} X(U),$$

which is the global sections functor for an arbitrary site $\mathcal{C}$. The general version of Γ_* specializes to the thing above for sheaves on $(Sch|_k)_{et}$ because this site has a terminal object, namely $\mathrm{Sp}(k)$.

Remark 8.1. It's a special feature of étale sites (and some others) that

$$\Gamma^* A(U) = \hom(\pi_0 U, A)$$

where $\pi_0(U)$ is the set of connected components of the k-scheme U, since $\mathrm{Sp}(k)$ is connected. In effect, the k-scheme $\bigsqcup_A \mathrm{Sp}(k)$ represents $\Gamma^* A$, and there is an easily proved isomorphism

$$\hom_k(U, \bigsqcup_A \mathrm{Sp}(k)) \cong \hom(\pi_0 U, A).$$

Note that every k-scheme X represents a sheaf on $(Sch|_k)_{et}$, by the theorem of *faithfully flat descent*. You can find this result in any of the étale cohomology textbooks, such as [7].

In particular, the sheaf of groups Gl_n is defined on affine k-schemes $Sp(R)$ (ie. k-algebras R) by

$$Gl_n(\mathrm{Sp}(R)) = Gl_n(R),$$

where the thing on the right is the group of invertible $n \times n$ matrices with entries in R. There is a standard way to recover the sheaf Gl_n on $(Sch|_k)_{et}$ from the matrix group description for affine schemes, by an equivalence

$$\mathrm{Shv}((Sch|_k)_{et}) \simeq \mathrm{Shv}((\mathrm{Aff}\,|_k)_{et})$$

where $(\mathrm{Aff}\,|_k)_{et}$ is the étale site of affine k-schemes, if you prefer.

The matrix group homomorphisms $Gl_n(R) \to Gl_{n+1}(R)$ defined by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

define a homomorphism $Gl_n \to Gl_{n+1}$ of sheaves of groups. The colimit presheaf

$$Gl = \varinjlim_n \; Gl_n \tag{8.1}$$

has the traditional infinite general linear group $Gl(R)$ in affine sections.

Warning: One typically also writes Gl for the associated sheaf, so that there is a relation of the form (8.1) in the category of sheaves of groups.

Presheaves of groups G have classifying simplicial presheaves BG, with

$$BG(U) = B(G(U))$$

given by the standard simplicial set construction. The object BG is a simplicial sheaf if G is a sheaf, because

$$BG_n = G \times \cdots \times G$$

(n factors) as a presheaf.

The classifying space construction commutes with filtered colimits, so we are entitled to a classifying simplicial sheaf (or presheaf) BGl with

$$BGl = \varinjlim_n BGl_n.$$

In general, simplicial sheaves (or presheaves) X have cohomology groups and homology sheaves.

The *homology sheaves* $\tilde{H}_n(X, A)$ are easier to define: form the presheaf of chain complexes

$$\mathbb{Z}(X) \otimes A,$$

with

$$(\mathbb{Z}(X) \otimes A)(U) = \mathbb{Z}(X(U)) \otimes A(U),$$

where $\mathbb{Z}(X(U))$ is the standard (functorial) Moore chain complex for the simplicial set $X(U)$. Then the sheaf $\tilde{H}_n(X, A)$ is the sheaf which is associated to the presheaf $H_n(\mathbb{Z}(X) \otimes A)$.

Example: The sheaf $\tilde{H}_n(X, \mathbb{Z}/\ell)$ is the sheaf associated to the presheaf $H_n(\mathbb{Z}/\ell(X))$.

Cohomology is more interesting to define: the n^{th} (étale) *cohomology group* $H^n(X, A)$ of the simplicial presheaf X with coefficients in the abelian presheaf A is defined by

$$H^n(X, A) = [X, K(A, n)],$$

where the thing on the right is morphisms in the local homotopy category of simplicial presheaves on the étale site.

There is a model structure on simplicial presheaves (respectively, and Quillen equivalently, simplicial sheaves) on the site $(Sch|_k)_{et}$, for which the weak equivalences are those maps $X \to Y$ which induce weak equivalences of simplicial sets in all stalks — I call these *local weak equivalences*, and for which the cofibrations are the monomorphisms. This is a special case of a construction which holds for arbitrary Grothendieck sites, which we'll discuss later.

Example: The canonical map $\eta : X \to \tilde{X}$ from a simplicial presheaf to its associated simplicial sheaf is a local weak equivalence.

One way to think about the simplicial presheaf $K(A, n)$ is that it should be the diagonal of the multi-simplicial presheaf $B^n(A)$. Alternatively, it's the presheaf $\Gamma(A[-n])$, where Γ is the Dold-Kan functor from chain complexes to simplicial abelian groups, and $A[-n]$ is the presheaf of chain complexes which consists of a copy of A concentrated in degree n.

Remark 8.2. 1) If X is represented by a (simplicial) scheme having the same name, and A is a sheaf of abelian groups, then $H^n(X, A)$ coincides up to isomorphism with the étale cohomology group $H^n_{et}(X, A)$ of X, as it is normally defined. Again, this will be proved later.

In particular, if X is a k-scheme, and $A \to I^*$ is an injective resolution of A in sheaves of abelian groups, then there is an isomorphism

$$H^n(X, A) \cong H^n(I^*(X)) \cong \mathrm{Ext}^n(\tilde{\mathbb{Z}}(X), A).$$

We have, in effect, generalized the standard definition of étale cohomology groups of schemes to arbitrary simplicial presheaves.

2) There is a spectral sequence [4] relating homology sheaves and cohomology groups, with

$$E_2^{p,q} = \mathrm{Ext}^p(\tilde{H}_q(X), A) \Rightarrow H^{p+q}(X, A).$$

There is also an ℓ-torsion version, with

$$E_2^{p,q} = \mathrm{Ext}^p(\tilde{H}_q(X, \mathbb{Z}/\ell), A) \Rightarrow H^{p+q}(X, A) \tag{8.2}$$

if A is an ℓ-torsion sheaf.

It follows that if $f : X \to Y$ is a map of simplicial presheaves which induces homology sheaf isomorphisms

$$f_* : \tilde{H}_n(X, \mathbb{Z}/\ell) \xrightarrow{\cong} \tilde{H}_n(Y, \mathbb{Z}/\ell), \ n \geq 0,$$

then f induces isomorphisms

$$f^* : H^n(Y, \mathbb{Z}/\ell) \xrightarrow{\cong} H^n(X, \mathbb{Z}/\ell)$$

in étale cohomology groups for all $n \geq 0$.

Exercise: Show that if $p : F \to F'$ is a local epimorphism of presheaves on $(Sch|_k)_{et}$, then the induced map $F(k) \to F'(k)$ in global sections is surjective, since k is an algebraically closed field.

It follows that the associated sheaf map $\eta : F \to \tilde{F}$ induces a bijection $F(k) \xrightarrow{\cong} \tilde{F}(k)$ in global sections.

It also follows that the global sections functor on $\mathrm{Shv}((Sch|_k)_{et})$ is exact on abelian sheaves. In particular, there are isomorphisms

$$H^n_{et}(k, A) \cong \begin{cases} A(k) & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

More generally, the map $A \to I^*$ of chain complexes defined by an injective resolution with I^* is in negative degrees induces a natural isomorphism

$$H^n(X, A(k)) \cong H^n(\Gamma^* X, A)$$

for any simplicial set X and sheaf of abelian groups A.

It follows that the canonical map

$$\epsilon : \Gamma^*\Gamma_* BGl \to BGl$$

has the form

$$\epsilon : \Gamma^* BGl(k) \to BGl$$

up to isomorphism, and that the induced map

$$\epsilon^* : H^n(BGl, \mathbb{Z}/\ell) \to H^n(\Gamma^* BGl(k), \mathbb{Z}/\ell)$$

can be written as

$$\epsilon^* : H^n_{et}(BGl, \mathbb{Z}/\ell) \to H^n(BGl(k), \mathbb{Z}/\ell), \quad (8.3)$$

where the object on the right is a standard cohomology group of the simplicial set $BGl(k)$ with coefficients in the abelian group $\mathbb{Z}/\ell$.
The map (8.3) is a comparison map of étale with discrete cohomology for the group Gl.

Theorem 8.3. *Suppose that k is an algebraically closed field, and that ℓ is prime which is distinct from the characteristic of k. Then the comparison map*

$$\epsilon^* : H^n_{et}(BGl, \mathbb{Z}/\ell) \to H^n(BGl(k), \mathbb{Z}/\ell)$$

is an isomorphism.

Remark 8.4. This theorem gives a calculation

$$H^*(BGl(k), \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, c_2, \dots],$$

since standard results in étale cohomology theory imply that $H^*_{et}(BGl, \mathbb{Z}/\ell)$ is a polynomial ring in Chern classes c_i, with $deg(c_i) = 2i$.

Proof of Theorem 8.3. The idea is to show that the map ϵ induces isomorphisms

$$\tilde{H}_n(\Gamma^* BGl(k), \mathbb{Z}/\ell) \xrightarrow{\cong} \tilde{H}_n(BGl, \mathbb{Z}/\ell)$$

in all homology sheaves, and then invoke a comparison of spectral sequences (8.2).

The category $\mathrm{Shv}((Sch|_k)_{et})$ has a good theory of stalks, and it's enough to compare stalks at all closed points $x \in U$ of all k-schemes U (which are locally of finite type over k). The map ϵ_* at the stalk for such a point x is the map

$$H_n(BGl(k), \mathbb{Z}/\ell) \to H_n(BGl(\mathcal{O}_x^{sh}), \mathbb{Z}/\ell),$$

where $\mathcal{O}_x^{sh}$ is the strict Henselization of the local ring $\mathcal{O}_x$ of U at x, and the indicated map is induced by the k-algebra structure map $k \to \mathcal{O}_x^{sh}$.

The Gabber Rigidity Theorem [2], [3] asserts that the residue field homomorphism $\pi : \mathcal{O}_x^{sh} \to k$ induces an isomorphism

$$\pi_* : H_n(BGl(\mathcal{O}_x^{sh}), \mathbb{Z}/\ell) \xrightarrow{\cong} H_n(BGl(k), \mathbb{Z}/\ell).$$

The Theorem follows. $\square$

The Gabber Rigidity Theorem is a consequence of a mod ℓ K-theory rigidity statement, namely that the residue map induces isomorphisms

$$\pi_* : K_*(\mathcal{O}_x^{sh}, \mathbb{Z}/\ell) \xrightarrow{\cong} K_*(k, \mathbb{Z}/\ell)$$

As such, it is an essentially stable statement that very much depends on the existence of the K-theory transfer, as well as the homotopy property for algebraic K-theory ($K_*(A) \cong K_*(A[t])$ for regular rings A).

An axiomatic approach to rigidity has evolved in the intervening years, which first appeared in [9], and achieved its modern form for torsion presheaves with transfers satisfying the homotopy property in [10].

Theorem 8.3 implies that an inclusion of algebraically closed fields $k \to L$ of characteristic away from ℓ induces an isomorphism

$$i^* : H^*(BGl(L), \mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell), \tag{8.4}$$

since there is an isomorphism of the corresponding étale cohomology rings by a smooth base change argument. The map i^* is an isomorphism if and only if the map

$$i_* : K_*(k, \mathbb{Z}/\ell) \to K_*(L, \mathbb{Z}/\ell)$$

is an isomorphism, by H-space tricks, so that Theorem 8.3 implies Suslin's first rigidity theorem [8].

The proof of Suslin's second rigidity theorem, for local fields [11], uses Gabber rigidity explicitly. The outcome of that result, that there are isomorphisms

$$K_n(\mathbb{C}, \mathbb{Z}/\ell) \cong \pi_n KU/\ell$$

for $n \geq 0$, is also a consequence of Theorem 8.3.

The comparison map

$$\epsilon^* : H^n_{et}(BGl, \mathbb{Z}/\ell) \to H^n(BGl(k), \mathbb{Z}/\ell)$$

is a special case of a natural comparison map

$$\epsilon^* : H^n(X, \mathbb{Z}/\ell) \to H^n(X(k), \mathbb{Z}/\ell)$$

which one can can construct for an arbitrary simplicial presheaf X on the big site $(Sch|_k)_{et}$.

There are versions of Theorem 8.3 for all of the classical infinite families of algebraic groups. In particular, there are comparison isomorphisms

$$\epsilon^* : H^*_{et}(BSl, \mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BSl(k), \mathbb{Z}/\ell),$$
$$\epsilon^* : H^*_{et}(BSp, \mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BSp(k), \mathbb{Z}/\ell),$$
$$\epsilon^* : H^*_{et}(BO, \mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BO(k), \mathbb{Z}/\ell),$$

for the infinite special linear, symplectic and orthogonal groups, respectively. The special linear case follows from Theorem 8.3, by a fibre sequence argument. The symplectic and orthogonal group statements follow from a rigidity statement for Karoubi L-theory which is deduced from Gabber rigidity with a Karoubi peridicity argument [5].

There is also a comparison map

$$\epsilon^* : H^n_{et}(BG, \mathbb{Z}/\ell) \to H^n(BG(k), \mathbb{Z}/\ell) \quad (8.5)$$

for an arbitrary algebraic group G over k. The Friedlander-Milnor conjecture asserts that this comparison map is an isomorphism if G is reductive. This conjecture specializes to a conjecture of Milnor when the underlying field is the complex numbers, in which case the étale cohomology groups $H^n(BG, \mathbb{Z}/\ell)$ correspond with the ordinary singular cohomology groups of the (simplicial analytic) classifying space $BG(\mathbb{C})$.

The isomorphism conjecture holds when $k = \overline{\mathbb{F}}_p$ is the algebraic closure of the finite field $\mathbb{F}_p$ with $p \neq \ell$ — this is a result of Friedlander and Mislin [1] which depends strongly on the Lang isomorphism for algebraic groups defined over $\mathbb{F}_p$. The isomorphism conjecture is not known to hold, in general, for any other algebraically closed field. It is not even known to hold for any of the general linear groups Gl_n outside of a stable range in homology. See Kevin Knudson's book [6] for a description of the current state of the problem.

This conjecture is perhaps the most important unsolved classical problem of algebraic K-theory. It was known since the 1970s that a calculation of the form

$$H^*(BGl_n(k), \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, \dots, c_n]$$

would imply the Lichtenbaum conjecture that

$$K_*(k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$$

where $\beta \in K_2(k, \mathbb{Z}/\ell)$ is the Bott element. Suslin proved this conjecture with the stable calculations of [8], [11] which were referred to above, but the unstable problem remains open.

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