Lecture 04

9 Local weak equivalences

Suppose that $\mathcal{C}$ is a small Grothendieck site. Recall that $s\,\mathrm{Pre}(\mathcal{C})$ and $s\,\mathrm{Shv}(\mathcal{C})$ denote the categories of simplicial presheaves and simplicial sheaves on $\mathcal{C}$, respectively.

Recall that a simplicial set map $f : X \to Y$ is a weak equivalence if and only if the induced map $|X| \to |Y|$ is a weak equivalence of topological spaces in the classical sense. This is equivalent to the assertion that all maps

a) $\pi_0 X \to \pi_0 Y$, and

b) $\pi_i(X, x) \to \pi_i(Y, f(x))$, $x \in X_0, i \geq 1$

are bijections. Here $\pi_i(X, x) = \pi_i(|X|, x)$ in general, but

$$\pi_i(X, x) = [(S^i, *), (X, x)] = \pi((S^i, *), (X, x))$$

if X is a Kan complex, by the Milnor theorem. Recall that $S^i = \Delta^i / \partial \Delta^i$ is the simplicial i-sphere, and $\pi((S^i, *), (X, x))$ is pointed simplicial homotopy classes of maps.

There is a different way to organize this: $f : X \to Y$ is a weak equivalence if the following hold:

a) $\pi_0 X \to \pi_0 Y$ is a bijection, and

b) all diagrams

$$\begin{tikzcd} \pi_i X \arrow[r] \arrow[d] & \pi_i Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

are pullbacks for $i \geq 1$.

Here,

$$\pi_i X = \bigsqcup_{x \in X_0} \pi_i(X, x)$$

is the group object over the set X_0 of vertices defined by the groups $\pi_i(X, x)$.

The basic idea behind the homotopy theory of simplicial presheaves is that the topology of the underlying site $\mathcal{C}$ should create the weak equivalences.

It's easy to see how to do this in cases where there are enough points:

Example: A map $f : X \to Y$ of simplicial presheaves on $op|_T$ for some topological space T should be a local weak equivalence if and only if it induces a weak equivalence in stalks $X_x \to Y_x$ for all $x \in T$. In particular f should induce isomorphisms

$$\pi_i(X_x, y) \to \pi_i(Y_x, f(y))$$

for all $i \geq 1$ *and all choices of base point* $y \in X_x$, as well as bijections

$$\pi_0 X_x \xrightarrow{\cong} \pi_0 Y_x.$$

Recall that the stalk

$$X_x = \varinjlim_{x \in U} X(U)$$

is a filtered colimit, and so each base point y comes from somewhere, namely some $z \in X(U)$ for some U. The point z determines a global section of $X|_U$, which is the composite

$$((op|_T)/U)^{op} \to (op|_T)^{op} \xrightarrow{X} s\mathbf{Set}$$

and f restricts to a simplicial presheaf map $f|_U : X|_U \to Y|_U$. The one can show that f is a local weak equivalence if and only if all induced sheaf maps

a) $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$, and

b) $\tilde{\pi}_i(X|_U, z) \to \tilde{\pi}_i(Y|_U, f(z))$, $i \geq 1$, $U \in \mathcal{C}$, $z \in X_0(U)$

are isomorphisms.

This is equivalent to the following: the map $f : X \to Y$ is a local weak equivalence if and only if

a) $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism

b) all presheaf diagrams

$$\begin{tikzcd} \pi_i X \arrow[r] \arrow[d] & \pi_i Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

induce pullback diagrams of associated sheaves.

Both descriptions generalize to equivalent conditions for maps of simplicial presheaves on an arbitrary site $\mathcal{C}$:

Definition A: A map $f : X \to Y$ of $s\,\mathrm{Pre}(\mathcal{C})$ is a *local weak equivalence* if and only if

a) the map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and

b) all maps $\tilde{\pi}_i(X|_U, x) \to \tilde{\pi}_i(Y|_U, f(x))$ are isomorphisms of sheaves on $\mathcal{C}/U$ for all $i \geq 1$, all $U \in \mathcal{C}$, and all $x \in X_0(U)$.

Here, $X|_U$ is the composite

$$(\mathcal{C}/U)^{op} \to \mathcal{C}^{op} \xrightarrow{X} s\mathbf{Set}.$$

Definition B: A map $f : X \to Y$ of $s\,\mathrm{Pre}(\mathcal{C})$ is a *local weak equivalence* if and only if

a) the map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and

b) all diagrams

$$\begin{tikzcd} \pi_i X \arrow[r] \arrow[d] & \pi_i Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

induce pullback diagrams of associated sheaves.

Exercise: Show that Definition A is equivalent to Definition B.

Here's a first example:

Lemma 9.1. *Suppose that $f : X \to Y$ is a sectionwise weak equivalence in the sense that all $X(U) \to Y(U)$ are weak equivalences of simplicial sets. Then f is a local weak equivalence.*

Proof. The map $\pi_0 X \to \pi_0 Y$ is an isomorphism of presheaves and all diagrams

$$\begin{tikzcd} \pi_i X \arrow[r] \arrow[d] & \pi_i Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

are pullbacks of presheaves. Sheafify. □

Suppose that $i : K \subset L$ is a cofibration of finite simplicial sets and that $f : X \to Y$ is a map of simplicial presheaves. We say that f has the *local right lifting property* with respect to i if for every diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & X(U) \arrow[d, "f"] \\ L \arrow[r] & Y(U) \end{tikzcd}$$

there is a covering sieve $R \subset \hom(\ ,U)$ such that the lift exists in the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & X(U) \arrow[r, "\phi^*"] & X(V) \arrow[d, "f"] \\ L \arrow[r] \arrow[urr, dotted] & Y(U) \arrow[r, "\phi^*"'] & Y(V) \end{tikzcd}$$

for every $\phi : V \to U$ in R.

Remark 9.2. There is no requirement for consistency between the lifts along the various members of R. Thus, if R is generated by a covering family $\phi_i : V_i \to U$, we just require liftings

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & X(U) \arrow[r, "\phi_i^*"] & X(V_i) \arrow[d, "f"] \\ L \arrow[r] \arrow[urr, dotted] & Y(U) \arrow[r, "\phi_i^*"'] & Y(V_i) \end{tikzcd}$$

Write X^K for the presheaf defined in sections by the simplicial function complexes

$$X^K(U) = \mathbf{hom}(K, X(U))$$

Lemma 9.3. *A map $f : X \to Y$ has the local right lifting property with respect to $i : K \to L$ if and only if the simplicial presheaf map*

$$X^L \xrightarrow{(i^*,f_*)} X^K \times_{Y^K} Y^L$$

is a local epimorphism in degree 0.

Proof. Exercise. $\square$

The condition on the map $f : X \to Y$ of Lemma 9.3 is the requirement that the presheaf map

$$\hom(L,X) \xrightarrow{(i^*,f_*)} \hom(K,X)\times_{\hom(K,Y)}\hom(L,Y) \tag{9.1}$$

is a local epimorphism, where $\hom(K,X)$ is the presheaf which is specified in sections by

$$\hom(K,X)(U) = \hom(K,X(U))$$

or the simplicial set morphisms $K \to X(U)$.

If K is a finite simplicial set, then $\hom(K,X)$ is a finite limit of the presheaves of simplices X_m, and it is a sheaf if X is a simplicial sheaf (exercise).

The local right lifting property for f with respect to i boils down to the requirement that the map above is a sheaf epimorphism if $f : X \to Y$ is a morphism of simplicial sheaves.

It follows that if $f : X \to Y$ is a simplicial sheaf map which has the local right lifting property with respect to an inclusion $i : K \subset L$ of finite simplicial sets, and if $p : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ is a geometric morphism, then the induced map $p^*f : p^*X \to p^*Y$ has the local right lifting property with respect to $i : K \subset L$.

Definition: A *local fibration* is a map which has the local right lifting property with respect to all $\Lambda^n_k \subset \Delta^n$. A simplicial presheaf X is *locally fibrant* if the map $X \to *$ is a local fibration.

Lemma 9.4. *Suppose that X and Y are presheaves of Kan complexes. Then a map $p : X \to Y$ is a local fibration and a local weak equivalence if and only if it has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

Say that a map $p : X \to Y$ which has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$ is a *local trivial fibration*. Such a map is also called a *hypercover*. This is the natural generalization, to simplicial presheaves, of the concept of a hypercover of a scheme (for the étale topology) which was introduced by Artin and Mazur [1].

Suppose that X is a simplicial sheaf. Then the map $X \to *$ is a hypercover if the maps

$$\begin{aligned} & X_0 \to *, \\ & \hom(\Delta^n, X) \to \hom(\partial\Delta^n, X),\ n \geq 1, \end{aligned} \tag{9.2}$$

are sheaf epimorphisms. There is a standard definition

$$\operatorname{cosk}_m(X)_n = \hom(\operatorname{sk}_m \Delta^n, X),$$

so that the second map of (9.2) can be written as

$$X_n \to \operatorname{cosk}_{n-1}(X)_n,$$

which is the way that it's displayed in [1].

It will be shown (Corollary 9.17) that a map $p : X \to Y$ of simplicial presheaves is a local weak equivalence and a local fibration if and only if it is a local trivial fibration.

Proof of Lemma 9.4. Suppose that p is a local fibration and a local weak equivalence, and that we have a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X(U) \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y(U) \end{tikzcd}$$

The idea is to show that this diagram is locally homotopic to diagrams

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X(V) \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y(V) \end{tikzcd}$$

for which the local lift exists. This means that there are homotopies

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r] \arrow[d] & X(V) \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r] & Y(V) \end{tikzcd}$$

from the diagrams

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X(U) \arrow[r, "\phi^*"] & X(V) \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y(U) \arrow[r, "\phi^*"'] & Y(V) \end{tikzcd}$$

to the corresponding diagrams above for all $\phi : V \to U$ in a covering for U. If such local homotopies exist, then solutions to the lifting problems

$$\begin{tikzcd} (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r] \arrow[d] & X(V) \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r] & Y(V) \end{tikzcd}$$

have local solutions for each V, and so the original lifting problem is solved on the refined covering of

U. The local homotopies are created by arguments similar to the proof of the corresponding result in the simplicial set case [2, I.7.10].

For the converse, show that the induced presheaf maps

$$\pi_0 X \to \pi_0 Y,$$
$$\pi_i(X|_U, x) \to \pi_i(Y|_U, p(x))$$

are local epis and monics — use presheaves of simplicial homotopy groups for this. □

Kan's Ex^∞ construction, which we now describe, gives a natural combinatorial method of replacing a simplicial set by a Kan complex up to weak equivalence. The naturality means that the construction can be imported to the categories of simplicial presheaves and simplicial sheaves, and the combinatorial nature of the Ex^∞ construction means that it is preserved by inverse image functors, up to isomorphism.

The functor $\mathrm{Ex} : s\mathbf{Set} \to s\mathbf{Set}$ is defined by

$$\mathrm{Ex}(X)_n = \hom(\mathrm{sd}\,\Delta^n, X).$$

$\mathrm{sd}\,\Delta^n = BN\Delta^n$, where $N\Delta^n$ is the poset of non-degenerate simplices of Δ^n (subsets of $\{0, 1, \dots, n\}$). Any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induces a functor $N\Delta^m \to N\Delta^n$, and hence induces a simplicial set map $\operatorname{sd}\Delta^m \to \operatorname{sd}\Delta^n$. Precomposition with this map gives the simplicial structure of $\operatorname{Ex}(X)$. There is a last vertex functor $N\Delta^n \to \mathbf{n}$, which is natural in $\mathbf{n}$; the collection of such functors determines a natural simplicial set map

$$\eta : X \to \operatorname{Ex}(X).$$

Observe that $\operatorname{Ex}(X)_0 = X_0$, and that η induces a bijection on vertices.

Iterating gives

$$\operatorname{Ex}^\infty(X) = \varinjlim \operatorname{Ex}^n(X).$$

The salient features of the construction are the following (see [2, III.4]):

1) the map $\eta : X \to \operatorname{Ex}(X)$ is a weak equivalence,
2) the functor $X \mapsto \operatorname{Ex}(X)$ preserves Kan fibrations
3) $\operatorname{Ex}^\infty(X)$ is a Kan complex, and the natural map $j : X \to \operatorname{Ex}^\infty(X)$ is a weak equivalence.

The Ex^∞ construction extends naturally to a construction for simplicial presheaves, which construction preserves and reflects local weak equivalences:

Lemma 9.5. *A map $f : X \to Y$ of simplicial presheaves is a local weak equivalence if and only if the induced map $\operatorname{Ex}^\infty X \to \operatorname{Ex}^\infty Y$ is a local weak equivalence.*

Proof. The natural simplicial set map $j : X \to \operatorname{Ex}^\infty X$ restricts to a natural bijection

$$X_0 \xrightarrow{\cong} \operatorname{Ex}^\infty X_0$$

of vertices for all simplicial sets X, and the horizontal arrows in the natural pullback diagrams

$$\begin{tikzcd} \pi_n X \arrow[r] \arrow[d] & \pi_n \operatorname{Ex}^\infty X \arrow[d] \\ X_0 \arrow[r] & \operatorname{Ex}^\infty X_0 \end{tikzcd}$$

are bijections.

It follows that the diagram of sheaf homomorphisms

$$\begin{tikzcd} \tilde{\pi}_n X \arrow[r] \arrow[d] & \tilde{\pi}_n Y \arrow[d] \\ \tilde{X}_0 \arrow[r] & \tilde{Y}_0 \end{tikzcd}$$

is a pullback if and only if the diagram

$$\begin{tikzcd} \tilde{\pi}_n \operatorname{Ex}^\infty X \arrow[r] \arrow[d] & \tilde{\pi}_n \operatorname{Ex}^\infty Y \arrow[d] \\ \widetilde{\operatorname{Ex}^\infty X}_0 \arrow[r] & \widetilde{\operatorname{Ex}^\infty Y}_0 \end{tikzcd}$$

is a pullback. $\square$

Lemma 9.6. *Suppose that a simplicial presheaf map $f : X \to Y$ has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$. Then f is a local fibration and a local weak equivalence.*

Proof. The local fibration part is trivial: the map f has the right lifting property with respect to all inclusions of finite simplicial sets.

The induced map

$$f : \mathrm{Ex}(X) \to \mathrm{Ex}(Y)$$

has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, since f has the local right lifting property with respect to all $\mathrm{sd}\,\partial\Delta^n \to \mathrm{sd}\,\Delta^n$. Thus, the map

$$f : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$$

has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$ and is a map of presheaves of Kan complexes. Finish by using Lemma 9.4 and Lemma 9.5. □

Corollary 9.7. *The maps $\eta : X \to LX$ and $\eta : X \to L^2X$ are local fibrations and local weak equivalences.*

Proof. Show that $\eta : X \to LX$ has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$: the

map
$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{LX^{\partial\Delta^n}} LX^{\Delta^n}$$
is a local epi in degree 0 if and only if the map of associated sheaves is a sheaf epi. But the map of associated sheaves is an isomorphism. □

Corollary 9.8. *A map $f : X \to Y$ of simplicial presheaves is a local weak equivalence if and only if the induced map $f_* : LX \to LY$ is a local weak equivalence.*

Proof. The map $\eta : X \to LX$ induces a natural isomorphism $\tilde{\pi}_0 X \xrightarrow{\cong} \tilde{\pi}_0 LX$, and the horizontal morphisms in the pullback diagrams

$$\begin{tikzcd} \tilde{\pi}_n X \arrow[r] \arrow[d] & \tilde{\pi}_n LX \arrow[d] \\ \tilde{X}_0 \arrow[r] & \widetilde{LX}_0 \end{tikzcd}$$

of sheaves are isomorphisms by Corollary 9.7. Now use the same argument as for Lemma 9.5. □

These concepts for have very special interpretations for simplicial sheaves on a complete Boolean algebra $\mathcal{B}$:

Lemma 9.9. *Suppose that $\mathcal{B}$ is a complete Boolean algebra.*

1) *A map $p : X \to Y$ of simplicial sheaves on $\mathcal{B}$ is a local (resp. local trivial) fibration if and only if all maps $p : X(b) \to Y(b)$ are Kan fibrations (resp. trivial Kan fibrations).*
2) *A map $f : X \to Y$ of locally fibrant simplicial sheaves on $\mathcal{B}$ is a local weak equivalence if and only if all maps $f : X(b) \to Y(b)$ are weak equivalences of simplicial sets.*

Proof. An induced map

$$X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}$$

is a sheaf epi in degree 0 if and only if it is a sectionwise epi in degree 0, since $\mathrm{Shv}(\mathcal{B})$ satisfies the Axiom of Choice (Lemma 7.3). The local fibration statement is similar.

Suppose that f is a local weak equivalence. The objects X and Y are sheaves of Kan complexes, so the map f has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & X \times_Y Y^{\Delta^1} \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where p is a sectionwise Kan fibration and j is right inverse to a sectionwise trivial Kan fibration (all objects are sheaves of Kan complexes). The map p is a local weak equivalence and a local fibration, and is therefore a sectionwise weak equivalence by Lemma 9.4. But then f is a sectionwise weak equivalence. □

Lemma 9.10. *Suppose that*

$$p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$$

is a Boolean localization. A map $f : X \to Y$ in $s\,\mathrm{Shv}(\mathcal{C})$ is a local fibration (resp. local trivial fibration) if and only if the induced map

$$p^*X \to p^*Y$$

is a sectionwise Kan fibration (resp. sectionwise trivial Kan fibration) in $s\,\mathrm{Shv}(\mathcal{B})$.

Proof. The simplicial sheaf map

$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

is a sheaf epi in degree zero if and only if the induced map

$$p^*X^{\Delta^n} \to p^*X^{\partial\Delta^n} \times_{p^*Y^{\partial\Delta^n}} p^*Y^{\Delta^n}$$

is a sheaf epi in degree 0 (note: $p^*(Y^K) \cong (p^*Y)^K$ if K is a finite simplicial set). Now use Lemma 9.9. □

Proposition 9.11. *Suppose that*

$$p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$$

is a Boolean localization, and that $f : X \to Y$ is a map of $s\,\mathrm{Pre}(\mathcal{C})$. Then f is a local weak equivalence if and only if the map

$$f_* : p^* L^2 X \to p^* L^2 Y$$

is a local weak equivalence of $s\,\mathrm{Shv}(\mathcal{B})$.

Proof. The map $f : X \to Y$ is a local weak equivalence if and only if the map $L^2 X \to L^2 Y$ of associated simplicial sheaves is a local weak equivalence, by Corollary 9.8. Thus it suffices to show that a map $f : X \to Y$ of $s\,\mathrm{Shv}(\mathcal{C})$ is a local weak equivalence if and only the induced map $p^* X \to p^* Y$ is a local weak equivalence of $s\,\mathrm{Shv}(\mathcal{B})$.

The map f is a local weak equivalence if and only the induced map $L^2 \mathrm{Ex}^\infty X \to L^2 \mathrm{Ex}^\infty Y$ is a weak equivalence of locally fibrant simplicial sheaves, by Lemma 9.5 and Corollary 9.8.

The map $f_* : \mathrm{Ex}^\infty X \to \mathrm{Ex}^\infty Y$ of presheaves of Kan complexes has a factorization

$$\begin{tikzcd} \mathrm{Ex}^\infty X \arrow[r, "j"] \arrow[dr, "f_*"'] & Z \arrow[d, "q"] \\ & \mathrm{Ex}^\infty Y \end{tikzcd}$$

where q is a sectionwise Kan fibration and j is a section of a sectionwise trivial Kan fibration $\pi : Z \to \operatorname{Ex}^\infty X$. Then $j_* : L^2 \operatorname{Ex}^\infty X \to L^2 Z$ is a section of a local trivial fibration $\pi_* : L^2 Z \to L^2 \operatorname{Ex}^\infty X$, and the induced map $q_* : L^2 Z \to L^2 \operatorname{Ex}^\infty Y$ is a local fibration between locally fibrant simplicial sheaves. It follows that $f : X \to Y$ is a local weak equivalence if and only if q_* is a local trivial fibration. But this is so if and only if $p^* q_*$ is a sectionwise trivial fibration, by Lemma 9.10. Thus, $f : X \to Y$ is a local weak equivalence if and only if the induced map $f_* : p^* L^2 \operatorname{Ex}^\infty X \to p^* L^2 \operatorname{Ex}^\infty Y$ is a sectionwise weak equivalence of simplicial sheaves on $\mathcal{B}$.

Finally, by exactness of p^* and L^2, there is a natural isomorphism

$$p^* L^2 \operatorname{Ex}^\infty X \cong L^2 \operatorname{Ex}^\infty p^* X$$

for simplicial sheaves X. Thus $f : X \to Y$ is a local weak equivalence of simplicial sheaves on $\mathcal{C}$ if and only if $f_* : p^* X \to p^* Y$ is a local weak equivalence of simplicial sheaves on $\mathcal{B}$. $\square$

The following result is a corollary of the proof of Proposition 9.11:

Corollary 9.12. *Suppose that*

$$p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$$

is a Boolean localization. Then a simplicial presheaf map $f : X \to Y$ is a local weak equivalence if and only if the induced map

$$p^* L^2 \operatorname{Ex}^\infty X \to p^* L^2 \operatorname{Ex}^\infty Y$$

is a sectionwise weak equivalence of simplicial sheaves on $\mathcal{B}$.

Now for some applications:

Lemma 9.13. *Suppose given a commutative diagram of simplicial presheaf maps*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

on a Grothendieck site $\mathcal{C}$. If any two of f, g or h are local weak equivalences then so is the third.

Proof. Apply $p^* L^2 \operatorname{Ex}^\infty$. □

Say that a simplicial presheaf map $i : A \to B$ is a *cofibration* if it is a monomorphism in all sections and in all simplicial degrees.

Lemma 9.14. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & D \end{tikzcd}$$

in the category $s\,\mathrm{Shv}(\mathcal{B})$ *such that* i *is a cofibration and a local weak equivalence. Then* i_* *is a cofibration and a local weak equivalence.*

Proof. Form the diagram of simplicial presheaf maps

$$\begin{tikzcd} \mathrm{Ex}^\infty A \arrow[r] \arrow[d, "i_*"'] & \mathrm{Ex}^\infty C \arrow[d] \\ \mathrm{Ex}^\infty B \arrow[r] & E \end{tikzcd}$$

where i_* is a cofibration. Then the induced map $D \to E$ is a sectionwise weak equivalence. Sheafifying gives a pushout diagram of simplicial sheaves

$$\begin{tikzcd} L^2\,\mathrm{Ex}^\infty A \arrow[r] \arrow[d, "i_*"'] & L^2\,\mathrm{Ex}^\infty C \arrow[d] \\ L^2\,\mathrm{Ex}^\infty B \arrow[r] & L^2 E \end{tikzcd}$$

which is locally equivalent to the original. We can therefore assume that the simplicial sheaves A, B and C are locally fibrant.

The map $i : A \to B$ is a local weak equivalence of locally fibrant simplicial sheaves on $\mathcal{B}$ and is therefore a sectionwise weak equivalence. Sectiowise trivial cofibrations are closed under pushout in the simplicial presheaf category, and since $D = L^2(B \cup_A C)$ is the associated sheaf of the presheaf pushout, the map $C \to D$ must then be a local weak equivalence by Lemma 9.13. $\square$

Corollary 9.15. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & D \end{tikzcd}$$

of simplicial presheaves on a Grothendieck site $\mathcal{C}$, and suppose that i is a cofibration and a local weak equivalence. Then i_* is a local weak equivalence.

Proof. Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization. The functor p^*L^2 preserves cofibrations and pushouts, and preserves and reflects local weak equivalences.

The map $p^*L^2A \to p^*L^2B$ induced by i is a local weak equivalence and a cofibration, so the map $p^*L^2C \to p^*L^2D$ induced by i_* is a local weak equivalence by Lemma 9.14. But then i_* must be a local weak equivalence. $\square$

Lemma 9.16. *Suppose that $p : X \to Y$ is a map of $s\,\mathrm{Shv}(\mathcal{B})$ such that p is a sectionwise Kan fibration and is a local weak equivalence. Then p is a sectionwise trivial fibration.*

Proof. The functor $X \mapsto L^2 \operatorname{Ex}^\infty X$ preserves sectionwise Kan fibrations and preserves pullbacks. Also, the sectionwise fibration $p : X \to Y$ is local weak equivalence if and only if the induced map $p_* : L^2 \operatorname{Ex}^\infty X \to L^2 \operatorname{Ex}^\infty Y$ is a sectionwise weak equivalence. It follows that the family of all maps which are simultaneously sectionwise Kan fibrations and local weak equivalences is closed under base change.

Suppose given a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, "i"'] & X(b) \arrow[d, "p"] \\
\Delta^n \arrow[r, "\beta"'] & Y(b)
\end{tikzcd}$$

The simplex Δ^n contracts onto the vertex 0; write $h : \Delta^n \times \Delta^1 \to \Delta^n$ for the contracting homotopy. Let $h' : \partial\Delta^n \times \Delta^1 \to X(b)$ be a choice of lifting

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d] & X(b) \arrow[d, "p"] \\
\partial\Delta^n \times \Delta^1 \arrow[ur, "h'"] \arrow[r, "\beta\cdot h\cdot(i\times 1)"'] & Y(b)
\end{tikzcd}$$

Then the original diagram is homotopic to a diagram of the form

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha'"] \arrow[d, "i"'] & X(b) \arrow[d, "p"] \\
\Delta^n \arrow[r, "x"'] & Y(b)
\end{tikzcd}$$

where $x : \Delta^n \to X(b)$ factors through a vertex $x \in Y(b)$. Consider the induced diagram of sheaf maps

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, "i"'] & (L_b\Delta^0 \times_Y X)(b) \arrow[d, "p_*"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & L_b\Delta^0(b)
\end{tikzcd}$$

Then $L_b\Delta^0$ is a diagram of points as a simplicial presheaf and hence is locally fibrant. Applying the associated sheaf functor therefore gives a sheaf of Kan complexes.

The map of associated sheaves which is induced by the map $p_* : L_b\Delta^0 \times_Y X \to L_b\Delta^0$ is a local fibration and a local weak equivalence between sheaves of Kan complexes and is therefore a sectionwise trivial fibration, so the indicated lift exists. $\square$

Corollary 9.17. *A map $q : X \to Y$ is a local weak equivalence and a local fibration in $s\,\mathrm{Pre}(\mathcal{C})$*

if and only if it has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n, n \geq 0$.

Proof. If q has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$ then it is a local fibration and a local weak equivalence, by Lemma 9.6. We prove the converse statement here.

Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization. Then p^*L^2q is a local weak equivalence and a local fibration, and is therefore a sectionwise trivial fibration by Lemma 9.16. The functor p^*L^2 reflects local epimorphisms, so that the map

$$X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}$$

is a local epi in degree 0. $\square$

References

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