Lecture 05

10 Injective model structure

10.1 The existence theorem

I want to review the classical fibration replacement construction from simplicial homotopy theory, because it is so important.

1) Suppose that $f : X \to Y$ is a map of Kan complexes, and form the diagram

$$\begin{tikzcd} X \times_Y Y^I \arrow[r, "f_*"] \arrow[d, "d_{0*}"'] & Y^I \arrow[r, "d_1"] \arrow[d, "d_0"] & Y \\ X \arrow[r, "f"'] \arrow[ur, "sf"] & Y & \end{tikzcd}$$

Then d_0 is a trivial fibration since Y is a Kan complex, so d_{0*} is a trivial fibration. The section s of d_0 (and d_1) induces a section s_* of d_{0*}. Then

$$(d_1 f_*) s_* = d_1(sf) = f$$

Finally, there is a pullback diagram

$$\begin{tikzcd} X \times_Y Y^I \arrow[r, "f_*"] \arrow[d, "{(d_{0*}, d_1 f_*)}"'] & Y^I \arrow[d, "{(d_0, d_1)}"] \\ X \times Y \arrow[r, "f \times 1"'] & Y \times Y \end{tikzcd}$$

and the map $pr_R : X \times Y \to Y$ is a fibration since X is fibrant, so that $pr_R(d_{0*}, d_1 f_*) = d_1 f_*$ is a fibration.

Write $Z_f = X \times_Y Y^I$ and $\pi_f = d_1 f_*$. Then we have functorial replacement

$$\begin{tikzcd} X \arrow[r, "s_*"] \arrow[dr, "f"'] & Z_f \arrow[r, "d_{0*}"] \arrow[d, "\pi"] & X \\ & Y & \end{tikzcd}$$

of f by a fibration π, where d_{0*} is a trivial fibration such that $d_{0*} s_* = 1$.

2) Suppose that $f : X \to Y$ is a simplicial set map, and form the diagram

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "\theta_f"] \arrow[dd, "f"'] & & \mathrm{Ex}^\infty X \arrow[dr, "s_*"] \arrow[dd] & \\ & \tilde{Z}_f \arrow[rr, "f_*"] \arrow[dl] & & Z_{f_*} \arrow[dl, "\pi"] \\ Y \arrow[rr, "j"'] & & \mathrm{Ex}^\infty Y & \end{tikzcd}$$

where the diagram

$$\begin{tikzcd} \tilde{Z}_f \arrow[r] \arrow[d, "\tilde{\pi}_f"'] & Z_{f_*} \arrow[d, "\pi_{f_*}"] \\ Y \arrow[r, "j"'] & \mathrm{Ex}^\infty Y \end{tikzcd}$$

is a pullback. Then $\tilde{\pi}_f$ is a fibration, and θ_f is a weak equivalence. Furthermore, the construction taking a map f to the factorization

$$\begin{tikzcd} X \arrow[r, "\theta_f"] \arrow[dr, "f"'] & \tilde{Z}_f \arrow[d, "\pi_f"] \\ & Y \end{tikzcd} \tag{10.1}$$

has the following properties:

a) it is natural in f

b) it preserves filtered colimits in f

c) if X and Y are α-bounded where α is some infinite cardinal, then so is $\tilde{Z}_f$

I say that a simplicial set X is *α-bounded* if $|X_n| < \alpha$ for all $n \geq 0$, or in other words if α is an upper bound for the cardinality of all sets of simplices of X.

A simplicial presheaf Y is α-bounded if all of the simplicial sets $Y(U)$, $U \in \mathcal{C}$, are α-bounded.

This construction (10.1) carries over to simplicial presheaves, giving a natural factorization

$$\begin{tikzcd} X \arrow[r, "\theta_f"] \arrow[dr, "f"'] & \tilde{Z}_f \arrow[d, "\pi_f"] \\ & Y \end{tikzcd} \tag{10.2}$$

of a simplicial presheaf map $f : X \to Y$ such that θ_f is a sectionwise weak equivalence and π_f is a sectionwise fibration. Here are some further properties of this factorization:

a) it preserves filtered colimits in f

b) if X and Y are α-bounded where α is some infinite cardinal, then so is $\tilde{Z}_f$

c) f is a local weak equivalence if and only if π_f has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$.

To fix notation, suppose that $\mathcal{C}$ is a small Grothendieck site.

Suppose that α is a regular cardinal such that $\alpha > |\operatorname{Mor}(\mathcal{C})|$.

Remark 10.1. Regular cardinals are used throughout these notes, so that the size of filtered colimits works out correctly, as in the proof of the next Lemma.

Specifically, if α is a regular cardinal and $F = \varinjlim_{i\in I} F_i$ is a filtered colimit of sets F_i such that $|I| < \alpha$ and all $|F_i| < \alpha$, then $|F| < \alpha$. One could take this condition to be the definition of a regular cardinal.

It is easy to see that if β is an infinite cardinal, then the successor cardinal $\beta + 1$ is regular, so that regular cardinals abound in nature. There are well known examples of limit cardinals that are not regular.

Lemma 10.2. *Suppose that $i : X \to Y$ is a cofibration and a local weak equivalence of $s\,\mathrm{Pre}(\mathcal{C})$. Suppose further that $A \subset Y$ is an α-bounded subobject of Y. Then there is an α-bounded subobject of Y such that $A \subset B$ and the map $B \cap X \to B$ is a local weak equivalence.*

Proof. Write $\pi_B : Z_B \to B$ for the natural pointwise Kan fibration replacement for the cofibration $B \cap X \to B$. The map $\pi_Y : Z_Y \to Y$ has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$.

Suppose given a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z_A(U) \arrow[d, "\pi_A"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & A(U) \end{tikzcd}$$

where A is α-bounded. The lifting problem can be solved locally over Y along some covering sieve for U having at most α elements. $Z_Y = \varinjlim_{|B|<\alpha} Z_B$

since Y is a filtered colimit of its α-bounded subobjects. It follows that there is an α-bounded subobject $A' \subset Y$ with $A \subset A'$ such that the original lifting problem can be solved over A'. The list of all such lifting problems is α-bounded, so there is an α-bounded subobject $B_1 \subset Y$ with $A \subset B_1$ so that all lifting problems as above over A can be solved locally over B_1. Repeat this procedure countably many times to produce an ascending family

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of α-bounded subobjects of Y such that all lifting local lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z_{B_i}(U) \arrow[d, "\pi_{B_i}"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & B_i(U) \end{tikzcd}$$

over B_i can be solved over B_{i+1}. Set $B = \cup_i B_i$. $\square$

Say that a map $p : X \to Y$ of $s\,\mathrm{Pre}(\mathcal{C})$ is an *injective fibration* if p has the right lifting property with respect to all maps $A \to B$ which are cofibrations and local weak equivalences.

Lemma 10.3. *The map $p : X \to Y$ is an injective fibration if and only if it has the right lift-*

ing property with respect to all α-bounded trivial cofibrations.

Proof. Suppose that $p : X \to Y$ has the right lifting property with respect to all α-bounded trivial cofibrations, and suppose given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

where i is a trivial cofibration. Consider the poset of partial lifts

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[dd, "p"] \\ A' \arrow[ur] \arrow[d] & \\ B \arrow[r] & Y \end{tikzcd}$$

This poset is non-empty: given $x \in B(U) - A(U)$ there is an α-bounded subcomplex $C \subset B$ with $x \in C(U)$, and there is an α-bounded subcomplex $C' \subset B$ with $C \subset C'$ and $i_* : C' \cap A \to C'$ a trivial cofibration. Then $x \in C' \cup A$, and there is a diagram

$$\begin{tikzcd} C' \cap A \arrow[r] \arrow[d, "i_*"'] & A \arrow[r] \arrow[d] & X \arrow[dd, "p"] \\ C' \arrow[r] & C' \cup A \arrow[ur] \arrow[d] & \\ & B \arrow[r] & Y \end{tikzcd}$$

where the indicated lift exists because p has the right lifting property with respect to the α-bounded trivial cofibration i_*.

The poset of partial lifts has maximal elements by Zorn's Lemma, and one uses the same argument as above to show that the maximal elements of the poset must have the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

$\square$

Lemma 10.4. *Suppose that $q : Z \to W$ has the right lifting property with respect to all cofibrations. Then q is an injective fibration and a local weak equivalence.*

Proof. The map q is obviously an injective fibration, and it has the right lifting property with respect to all cofibrations $L_U\partial\Delta^n \to L_U\Delta^n$, so that all maps $q : Z(U) \to W(U)$ are trivial Kan fibrations. But then q is a local weak equivalence. $\square$

One defines

$$L_U K = \hom(\ ,U) \times K$$

for $U \in \mathcal{C}$ and simplicial sets K. The functor $K \mapsto L_U K$ is left adjoint to the U-sections functor $X \mapsto X(U)$.

Lemma 10.5. *A map $q : Z \to W$ has the right lifting property with respect to all cofibrations if and only if it has the right lifting property with respect to all α-bounded cofibrations.*

Proof. Exercise. $\square$

Lemma 10.6. *Any simplicial presheaf map $f : X \to Y$ has factorizations*

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where

1) *the map i is a cofibration and a local weak equivalence, and p is an injective fibration,*
2) *the map j is a cofibration and p has the right lifting property with respect to all cofibrations (and is therefore an injective fibration and a local weak equivalence)*

Proof. For the first factorization, choose a cardinal $\lambda > 2^{\alpha}$ and do a transfinite small object argument of size λ to solve all lifting problems

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

arising from locally trivial cofibrations i which are α-bounded. We need to know that locally trivial cofibrations are closed under pushout, but we proved this in Lemma 9.14 with a Boolean localization argument. The small object argument stops on account of the condition on the size of the cardinal λ.

The second factorization is similar, and uses Lemma 10.5. □

Theorem 10.7. *Suppose that $\mathcal{C}$ is a small Grothendieck site. The category $s\,\mathrm{Pre}(\mathcal{C})$ with local weak equivalences, cofibrations and injective fibrations, satisfies the axioms for a proper closed simplicial model category.*

Proof. The simplicial presheaf category $s\,\mathrm{Pre}(\mathcal{C})$ has all small limits and colimits, giving **CM1**. The weak equivalence axiom **CM2** was proved in Lemma 9.13 with a Boolean localization argument.

The retract axiom **CM3** is trivial to verify — exercise. The factorization axiom **CM5** is Lemma 10.6.

Suppose that $\pi : X \to Y$ is an injective fibration and a local weak equivalence. Then by the proof of Lemma 10.6, π has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\pi"'] & W \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where p has the right lifting property with respect to all cofibrations and is therefore a local weak equivalence. Then j is a local weak equivalence, and so π is a retract of p (exercise). Thus π has the right lifting property with respect to all cofibrations, giving **CM4**.

The simplicial model structure comes from the function complex

$$\mathbf{hom}(X, Y)_n = \hom_{s\,\mathrm{Pre}(\mathcal{C})}(X \times \Delta^n, Y).$$

Quillen's axiom **SM7** is a consequence of the fact that local weak equivalences are closed under finite products: if $f : X \to Y$ and $f' : X' \to Y'$ are local weak equivalences, then the map

$$f \times f' : X \times X' \to Y \times Y'$$

is a local weak equivalence. The proof of this statement is a Boolean localization argument (exercise).

Properness is also proved with a Boolean localization argument (exercise). □

Remark 10.8. Every injective fibration (respectively trivial injective fibration) $p : X \to Y$ is a sectionwise Kan fibration (respectively sectionwise trivial Kan fibration). In effect, if $p : X \to Y$ is an injective fibration then it has the right lifting property with respect to the trivial cofibrations $L_U\Lambda^n_k \to L_U\Delta^n$, and if p is a trivial injective fibration then it has the right lifting property with respect to the cofibrations $L_U\partial\Delta^n \to L_U\Delta^n$. It follows, in particular, that every injective fibration is a local fibration.

10.2 Injective fibrant models and descent

We start with an example.

Suppose that A is a sheaf, and let $K(A,0)$ be the constant simplicial object associated to A. There is a bijection

$$\hom(X, K(A,0)) \cong \hom(\tilde{\pi}_0(X), A)$$

It follows that the simplicial sheaf $K(A,0)$ is injective fibrant.

Suppose that X is a simplicial presheaf such that all higher local homotopy groups vanish in the sense that $\tilde{\pi}_n(X) \to \tilde{X}_0$ is an isomorphism for $n \geq 1$. Then the map $X \to K(\pi_0(X),0)$ is a local weak equivalence. It follows that the composite

$$X \to K(\pi_0(X),0) \to K(\tilde{\pi}_0(X),0)$$

is a local weak equivalence, and therefore gives an "injective fibrant model" for X. Note that all higher homotopy groups

$$\pi_n(K(\tilde{\pi}_0(X),0)(U),x),\ n \geq 1,$$

vanish in all sections.

Remark 10.9. This observation is a special case of (and the starting point for) a result which asserts that if X is a simplicial presheaf such that $\tilde{\pi}_n(X) \to \tilde{X}_0$ is an isomorphism for $n \geq k$, then any injective fibrant model $X \to Y$ has the same property in all sections: $\pi_n(Y(U),x) = 0$ for $n \geq k$, for all $x \in Y(U)$ and all $U \in \mathcal{C}$ — see [7, Prop. 6.11]. This result is particular to simplicial presheaves — it does not hold in motivic homotopy theory, where every motivic homotopy type is representable by a presheaf (see the appendix of [8]).

An *injective fibrant model* for a simplicial presheaf X is a local weak equivalence $f : X \to Z$ such that Z is injective fibrant.

Any two injective fibrant models for a simplicial presheaf X are equivalent in a rather strong sense: given models $f : X \to Z$ and $f' : X \to Z'$, f has a factorization $f = p \cdot j$ where p is a injective fibration and j is a cofibration and both are local weak equivalences, and there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] \arrow[dd, "f'"'] & & Z \\ & W \arrow[ur, "p"'] \arrow[dl, dotted] & \\ Z' & & \end{tikzcd}$$

where the dotted arrow exists since j is a trivial cofibration and Z' is injective fibrant. Note that all morphisms in the picture are local weak equivalences, and we have the following:

Lemma 10.10. *Suppose that $f : Z \to W$ is a weak equivalence of injective fibrant objects. Then all maps $f : Z(U) \to W(U)$ are weak equivalences of simplicial sets.*

Proof. The map f is a simplicial homotopy equivalence since Z and W are cofibrant and injective fibrant. In other words, there is a map $g : W \to Z$ and homotopies $Z \times \Delta^1 \to Z$ from gf to 1_Z and $W \times \Delta^1 \to W$ from fg to 1_W. The map g restricts to $g : W(U) \to Z(U)$ in each section, and the homotopies restrict to simplicial set maps $Z(U) \times \Delta^1 \to Z(U)$ and $W(U) \times \Delta^1 \to W(U)$. In particular $f : Z(U) \to W(U)$ is a simplicial homotopy equivalence with homotopy inverse $g : W(U) \to Z(U)$, for each $U \in \mathcal{C}$. □

Corollary 10.11. *Any two injective fibrant models for a simplicial presheaf X are sectionwise homotopy equivalent.*

Here's the idea that pervades most applications of local homotopy theory:

I say that a simplicial presheaf X *satisfies descent* if some (hence any) injective fibrant model $j : X \to Z$ is a sectionwise weak equivalence.

Injective fibrant objects satisfy descent, in view of Lemma 10.10.

The question of whether or not a fixed simplicial presheaf (or later, presheaf of spectra) satisfies descent is called a *descent problem*, and the assertion that it does is usually a serious result which is often called a descent theorem.

The descent concept has a long history — see [11], [13].

Examples include the Brown-Gersten descent theorem for the algebraic K-theory presheaf of spectra and the Zariski topology, Thomason's étale descent theorem for Bott periodic algebraic K-theory with torsion coefficients, and the Nisnevich descent theorem for torsion K-theory with respect to the Nisnevich (or *cdh*) topology. The Lichtenbaum-Quillen conjecture is a type of descent problem for algebraic K-theory with torsion coefficients and the étale topology. These issues are discussed, in these terms, in [7], and more recently in [10].

We now know that a stack can be characterized as a sheaf or presheaf of groupoids which satisfies descent for some ambient topology — see, most recently, [10]. This phenomenon will be explored later in these notes.

Generally, one is very to happy to know that a fixed simplicial presheaf or presheaf of spectra satisfies descent, because one then has techniques for computing its homotopy groups in all sections from sheaf cohomological information, most often through a descent spectral sequence.

11 Other model structures

11.1 Injective model structure for simplicial sheaves

Write $s\,\mathrm{Shv}(\mathcal{C})$ for the category of simplicial sheaves on $\mathcal{C}$. Say that a map $f : X \to Y$ is a *local weak equivalence* of simplicial sheaves if it is a local weak equivalence of simplicial presheaves. A *cofibration* of simplicial sheaves is a monomorphism, and an *injective fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

Theorem 11.1. *Let $\mathcal{C}$ be a small Grothendieck site.*

1) The category $s\,\mathrm{Shv}(\mathcal{C})$ with local weak equivalences, cofibrations and injective fibrations, satisfies the axioms for a proper closed simplicial model category.

2) The inclusion i of sheaves in presheaves and the associated sheaf functor L^2 together induce a Quillen equivalence of homotopy categories

$$L^2 : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})) \simeq \mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C})) : i.$$

Proof. The associated sheaf functor L^2 preserves and reflects local weak equivalences. The inclusion functor i preserves injective fibrations and L^2 preserves cofibrations. The associated sheaf map $\eta : X \to L^2X$ is a local weak equivalence, while the counit of the adjunction is an isomorphism. Thus, we get 2) if we can prove 1).

The completeness axiom **CM1** follows from completeness and cocompleteness for $s\,\mathrm{Shv}(\mathcal{C})$. The weak equivalence axiom **CM2** follows from the corresponding statement for simplicial presheaves. The retract axiom **CM3** is trivial, and **CM4** follows from the corresponding statement for simplicial presheaves.

A map $p : X \to Y$ is an injective fibration (resp. trivial injective fibration) of $s\,\mathrm{Shv}(\mathcal{C})$ if and only if it is an injective fibration (resp. trivial injective fibration) of $s\,\mathrm{Pre}(\mathcal{C})$ (exercise).

Choose a regular cardinal β such that $\beta > |\tilde{B}|$ for all α-bounded simplicial presheaves B. Then the β-bounded trivial cofibrations of simplicial sheaves generate the trivial cofibrations of simplicial sheaves, and the β-bounded cofibrations of simplicial sheaves generate the cofibrations of simplicial sheaves.

The factorization axiom **CM5** is then proved by transfinite small object arguments of size λ where $\lambda > 2^\beta$.

The simplicial model structure (aka. function complexes) is inherited from simplicial presheaves, as is properness. $\square$

The injective model structure for simplicial sheaves (part 1) of Theorem 11.1) first appeared in a letter of Joyal to Grothendieck [12], while the injective model structure for simplicial presheaves first appeared in [6].

Example: The category $s\operatorname{Pre}(\mathcal{C})$ of simplicial presheaves is also the category of simplicial sheaves for the "chaotic" Grothendieck topology on $\mathcal{C}$ whose covering sieves are the representable functors

$$\hom(\ ,U),\ U \in \mathcal{C}.$$

The injective model structures, for simplicial presheaves or simplicial sheaves, therefore specialize to injective model structures for categories of diagrams of simplicial sets. The injective model structure for diagrams is the good setting for describing homotopy inverse limits — see [4, VIII.2]. The existence of this model structure is usually attributed to Heller [5].

11.2 Intermediate model structures

There is a *projective* model structure on $s\,\mathrm{Pre}(\mathcal{C})$, for which the fibrations are sectionwise Kan fibrations and the weak equivalences are sectionwise weak equivalences. The cofibrations for this theory are the projective cofibrations, and this class of maps has a generating set S_0 consisting of all maps $L_U(\partial\Delta^n) \to L_U(\Delta^n)$. This model structure first appeared in [3].

Write $\mathbf{C}_P$ for the class of projective cofibrations, and write $\mathbf{C}$ for the full class of cofibrations. Obviously $\mathbf{C}_P \subset \mathbf{C}$.

Let S be any *set* of cofibrations which contains S_0. Let $\mathbf{C}_S$ be the saturation of the set of all cofibrations of the form

$$(B \times \partial\Delta^n) \cup_{(A\times\partial\Delta^n)} (A \times \Delta^n) \subset B \times \Delta^n$$

which are induced by members $A \to B$ of the set S. "Saturation" means the smallest class of cofibrations containing the list above which is contains all isomorphisms, and is closed under pushout and all transfinite compositions. $\mathbf{C}_S$ is the class of *S-cofibrations.*

An *S-fibration* is a map which has the right lifting property with respect to all S-cofibrations which are local weak equivalences.

Theorem 11.2. *The category $s\,\mathrm{Pre}(\mathcal{C})$ and the classes of S-cofibrations, local weak equivalences, and S-fibrations, satisfies the axioms for a proper closed simplicial model category.*

Proof. The axioms **CM1** – **CM3** are trivial to verify.

Any $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where $j \in \mathbf{C}_S$ and p has the right lifting property with respect to all members of $\mathbf{C}_S$. Then p is an S-fibration and is a sectionwise hence local weak equivalence.
The map f also has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where q is an injective fibration and i is a cofibration and local weak equivalence. Then q is an S-fibration. Factorize i as $i = p \cdot j$ where $j \in \mathbf{C}_S$

and p is an S-fibration and a local weak equivalence (as above). Then j is a local weak equivalence, so $f = (qp) \cdot j$ factorizes f as an S-fibration following a map which is an S-cofibration and a local weak equivalence.

Exercise: Prove **CM4**.

The simplicial model structure is the usual one.

Exercise: Prove that the structure is proper. $\square$

The case $S = S_0$ gives the local projective structure of Blander [2].

The model structure of Theorem 11.2 is cofibrantly generated. This was originally proved by Beke [1], by verifying a solution set condition. Beke's argument was deconstructed in [9], in the form of a basic and useful trick for verifying cofibrant generation in the presence of some kind of cardinality calculus, and that trick is reprised here, in the proof of Lemma 11.3 below.

Suppose that α is a regular cardinal such that $|\operatorname{Mor}(\mathcal{C})| < \alpha$ and $|D| < \alpha$ for all members $C \to D$ of the set of cofibrations generating $\mathbf{C}_S$.

Every α-bounded trivial cofibration $i : A \to B$

has a factorization

$$\begin{tikzcd} A \arrow[r, "j_i"] \arrow[dr, "i"'] & C \arrow[d, "p_i"] \\ & B \end{tikzcd}$$

such that j_i is an S-cofibration, p_i is an S-fibration and both maps are local weak equivalences. Write I for the set of all of the trivial S-cofibrations j_i which are constructed in this way.

Lemma 11.3. *The set I generates the class of trivial S-cofibrations.*

Proof. Suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

such that i is an α-bounded member of $\mathbf{C}_S$ and f is a local weak equivalence. Then, since B is α-bounded, this diagram has a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[r] \arrow[d, "j"] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

where j is a member of the set of cofibrations I.

In effect, by factorizing $f = p \cdot u$ where u is a trivial S-cofibration and q is a trivial S-fibration, we can assume that f is a trivial cofibration. The bounded cofibration property then implies that there is a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[r] \arrow[d, "v"] & X \arrow[d, "f"] \\ B \arrow[r] & F \arrow[r] & Y \end{tikzcd}$$

with v an α-bounded trivial cofibration. Factorize $v = p_v j_v$ as above, again with p_v a trivial S-fibration and j_v a trivial S-cofibration. Then p_v has the right lifting property with respect to i since it is a trivial S-fibration, and j_v is the desired member of the set I.

Every trivial S-cofibration $j : A' \to B'$ has a factorization

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[dr, "j"'] & C' \arrow[d, "q"] \\ & B' \end{tikzcd}$$

such that α is an S-cofibration in the saturation of the set I and q has the right lifting property with respect to all members of I. Then q is also a local weak equivalence, and therefore has the right lifting property with respect to all members i of the class $\mathbf{C}_S$ of S-cofibrations by the previous paragraph, since all generators of $\mathbf{C}_S$ are α-bounded.

It follows that the lifting problem

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "j"'] & C' \arrow[d, "q"] \\ B' \arrow[r, "1"'] \arrow[ur, dotted] & B' \end{tikzcd}$$

has a solution, so that j is a retract of α. $\square$

Corollary 11.4. *A map $p : X \to Y$ is an S-fibration if and only if it has the right lifting property with respect to all members of the set I.*

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