Lecture 07

14 Cocycles

Let $\mathcal{M}$ be a closed model category such that

1) $\mathcal{M}$ is right proper in the sense that weak equivalences pull back to weak equivalences along fibrations, and

2) the class of weak equivalences is closed under products: if $f : X \to Y$ is a weak equivalence, so is any map $f \times 1 : X \times Z \to Y \times Z$

Examples include any of the model structures on $s\,\mathrm{Pre}$, $s\,\mathrm{Shv}$, $s\,\mathrm{Pre}_R$ or $s\,\mathrm{Shv}_R$ that we've seen, where the weak equivalences are local weak equivalences. This can be verified by using Boolean localization arguments.

Suppose that X, Y are objects of $\mathcal{M}$, and write $H(X, Y)$ for the category whose objects are all pairs of maps (f, g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism

$$\alpha : (f, g) \to (f', g')$$

of $H(X,Y)$ is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

I say that $H(X,Y)$ is the *category of cocycles*, or *cocycle category* from X to Y.

Example: Every set X has an associated (homotopically) trivial groupoid whose objects are the elements of X and whose morphisms are pairs of elements of X. Suppose that a presheaf map $U \to *$ is a local epimorphism. Then the canonical simplicial presheaf map $BC(U) \to *$ is a local weak equivalence (in fact, it's a local trivial fibration), and $BC(U)$ is called the *Čech resolution* associated to the covering $U \to *$.

Given a covering $U \to *$ and a (pre)sheaf of groups G, a normalized cocycle on U with values in G is, precisely, either a groupoid morphism $C(U) \to G$ or a simplicial presheaf map $BC(U) \to BG$. Such a map defines a cocycle

$$* \xleftarrow{\simeq} BC(U) \to BG$$

in the sense described above. Normalized cocycles were the original examples of such cocycles.

Write $\pi_0 H(X,Y)$ for the class of path components of $H(X,Y)$. There is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y]$$

defined by $(f,g) \mapsto g \cdot f^{-1}$.

Lemma 14.1. *Suppose that $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then the function*

$$(\alpha,\beta)_* : \pi_0 H(X,Y) \to \pi_0 H(X',Y')$$

is a bijection.

Proof. An object (f,g) of $H(X',Y')$ is a map $(f,g) : Z \to X' \times Y'$ such that f is a weak equivalence. There is a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

such that j is a trivial cofibration and $(p_{X'},p_{Y'})$ is a fibration. The map $p_{X'}$ is a weak equivalence. Form the pullback

$$\begin{tikzcd} W_* \arrow[r, "(\alpha\times\beta)_*"] \arrow[d, "(p_X^*,p_Y^*)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ X \times Y \arrow[r, "\alpha\times\beta"'] & X' \times Y' \end{tikzcd}$$

Then the map (p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a local weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). The map p_X^* is also a weak equivalence.

The assignment $(f, g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \to \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. $\square$

Lemma 14.2. *Suppose that Y is fibrant and X is cofibrant. Then the canonical map*

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

is a bijection.

Proof. The function $\pi(X, Y) \to [X, Y]$ relating naive homotopy classes to morphisms in the homotopy category is a bijection since X is cofibrant and Y is fibrant.

If $f, g : X \to Y$ are homotopic, there is a diagram

$$\begin{tikzcd}
 & X \arrow[dl, "1"'] \arrow[d, "d_0"'] \arrow[dr, "f"] & \\
X & X \otimes I \arrow[l, "s"'] \arrow[r, "h"] & Y \\
 & X \arrow[ul, "1"] \arrow[u, "d_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

where h is the homotopy. Thus, sending $f : X \to Y$ to the class of $(1_X, f)$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

and there is a diagram

$$\begin{tikzcd}
\pi(X,Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 H(X,Y) \arrow[d, "\phi"] \\
& {[X,Y]}
\end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

The weak equivalence f has a factorization

$$\begin{tikzcd}
Z \arrow[r, "j"] \arrow[dr, "f"'] & V \arrow[d, "p"] \\
& X
\end{tikzcd}$$

where j is a trivial cofibration and p is a trivial fibration. The object Y is fibrant, so the dotted arrow θ exists in the diagram

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, "j"] \arrow[dr, "g"] & \\
X & V \arrow[l, "p"] \arrow[r, dotted, "\theta"'] & Y
\end{tikzcd}$$

Since X is cofibrant, the trivial fibration p has a section σ, and so there is a commutative diagram

$$
\begin{tikzcd}
& X \arrow[dl, "1"'] \arrow[dr, "\theta\sigma"] \arrow[dd, "\sigma"] & \\
X & & Y \\
& V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}
$$

Then the composite $\theta\sigma$ is the required map k. $\square$

Theorem 14.3. *Suppose that the model category has the properties 1) and 2) listed above, and that X, Y are objects of $\mathcal{M}$. Then the canonical map*

$$\phi : \pi_0 H(X,Y) \to [X,Y]$$

is a bijection for all X and Y.

Proof. There are weak equivalences $\pi : X' \to X$ and $j : Y \to Y'$ such that X' and Y' are cofibrant and fibrant, respectively, and there is a commutative diagram

$$
\begin{tikzcd}
\pi_0 H(X,Y) \arrow[r, "\phi"] \arrow[d, "(1,j)_*"', "\cong"] & {[X,Y]} \arrow[d, "\cong"', "j_*"] \\
\pi_0 H(X,Y') \arrow[r, "\phi"] & {[X,Y']} \arrow[d, "\cong"', "\pi^*"] \\
\pi_0 H(X',Y') \arrow[u, "(\pi,1)_*", "\cong"'] \arrow[r, "\cong", "\phi"'] & {[X',Y']}
\end{tikzcd}
$$

The functions $(1,j)_*$ and $(\pi,1)_*$ are bijections by the first Lemma, and the bottom map ϕ is a bijection by the second Lemma. $\square$

Remark 14.4. Cocycle categories have appeared before, in the context of Dwyer-Kan hammock localizations [3], [2]. One of the main results in the area, which holds for arbitrary model categories $\mathcal{M}$, says roughly that the nerve $BH(X,Y)$ is a model for the function space of maps from X to Y if Y is fibrant. This result implies Theorem 14.3 if the target object Y is fibrant. On the other hand, I shall demonstrate below that the most powerful applications of Theorem 14.3 involve target objects Y which are not fibrant in general.

15 Sheaf cohomology

Suppose that A is a sheaf of abelian groups, and let $A \to J$ be an injective resolution of A, thought of as a $\mathbb{Z}$-graded chain complex, concentrated in negative degrees.

Write $A[-n]$ for the chain complex consisting of A concentrated in degree n, and consider the chain map $A[-n] \to J[-n]$.

Recall that $K(A,n) = \Gamma A[-n]$ defines the Eilenberg-Mac Lane simplicial sheaf associated to A. Let

$$K(J,n) = \Gamma\,\mathrm{Tr}_0(J[-n])$$

where $\mathrm{Tr}_0(J[-n])$ is the good truncation of $J[-n]$ in non-negative degrees.

Suppose that C is an ordinary chain complex and that I is an unbounded chain complex which is 0 in non-negative degrees. Form the bicomplex

$$\hom(C, I)_{p,q} = \hom(C_{-p}, I_q)$$

with the obvious induced differentials:

$$\partial' = \partial_C^* : \hom(C_{-p}, I_q) \to \hom(C_{-p-1}, I_q)$$
$$\partial'' = (-1)^p \partial_{I*} : \hom(C_{-p}, I_q) \to \hom(C_{-p}, I_{q-1}).$$

Then $\hom(C, I)$ is a third quadrant bicomplex with associated total complex

$$\begin{aligned} \mathrm{Tot}_{-n} \hom(C, I) &= \bigoplus_{p+q=-n} \hom(C_{-p}, I_q) \\ &= \bigoplus_{0 \leq p \leq n} \hom(C_p, I_{-n+p}), \end{aligned}$$

for $n \geq 0$, which is concentrated in negative degrees.

Exercise: Show that there are natural isomorphisms

$$\begin{aligned} H_{-n}(\mathrm{Tot} \hom(C, I)) &\cong \pi(C(0), I[-n]) \\ &\cong \pi(C, \mathrm{Tr}_0 I[-n]), \end{aligned}$$

where $\pi(C(0), I[-n])$ denotes chain homotopy classes of maps from the unbounded complex $C(0)$ canonically associated to C to the shifted complex $I[-n]$, and $\pi(C, \mathrm{Tr}_0 I[-n])$ is chain homotopy classes of maps in the bounded complex category.

Example: If $A \to J$ is an injective resolution of an abelian sheaf A, then the bicomplex $\hom(C, J)$ determines a spectral sequence with

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(C), A) \Rightarrow \pi(C, \mathrm{Tr}_0 J[-p-q]).$$

Lemma 15.1. *Every local weak equivalence $f : X \to Y$ induces an isomorphism*

$$\pi_{ch}(N\tilde{\mathbb{Z}}Y, \mathrm{Tr}_0 J[-n]) \xrightarrow{\cong} \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n])$$

in chain homotopy classes for all $n \geq 0$.

Proof. The map f induces a homology sheaf isomorphism $N\tilde{\mathbb{Z}}X \to N\tilde{\mathbb{Z}}Y$, and then a comparison of spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X), A) \Rightarrow \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-p-q])$$

gives the desired result. □

If two chain maps $f, g : N\tilde{\mathbb{Z}}X \to \mathrm{Tr}_0 J[-n]$ are chain homotopic, then there is a right homotopy

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur] \arrow[r, "(f_*, g_*)"'] & K(J,n) \times K(J,n) \end{tikzcd}$$

for some path object Z over $K(J,n)$ in the projective model structure for $\mathcal{C}^{op}$-diagrams of simplicial sets (see Section 11). Choose a sectionwise trivial fibration $\pi : W \to X$ such that W is projective cofibrant. Then $f_*\pi$ is left homotopic to $g_*\pi$ for some choice of cylinder object $W \otimes I$ for W, again in the projective structure. This means that there is a diagram

$$\begin{tikzcd} & & W \arrow[dl, "1"'] \arrow[d, "i_0"] \arrow[r, "\pi"] & X \arrow[dr, "f_*"] & \\ X & W \arrow[l, "\pi"'] & W \otimes I \arrow[l, "s"'] \arrow[rr, "h"] & & K(J,n) \\ & & W \arrow[ul, "1"] \arrow[u, "i_1"'] \arrow[r, "\pi"'] & X \arrow[ur, "g_*"'] & \end{tikzcd}$$

where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences. It follows that

$$(1, f_*) \sim (\pi, f_*\pi) \sim (\pi s, h) \sim (\pi, g_*\pi) \sim (1, g_*)$$

in $\pi_0 H(X, K(J,n))$. It follows that there is a well defined abelian group homomorphism

$$\phi : \pi_{ch}(N\tilde{\mathbb{Z}}X, \operatorname{Tr}_0 J[-n]) \to \pi_0 H(X, K(J,n)).$$

This map is natural in X.

Lemma 15.2. *The map*

$$\phi : \pi_{ch}(N\tilde{\mathbb{Z}}X, \operatorname{Tr}_0 J[-n]) \to \pi_0 H(X, K(J,n)).$$

is an isomorphism.

Proof. Suppose that $X \xleftarrow{f} Z \xrightarrow{g} K(J,n)$ is an object of $H(X, K(J,n))$. Then there is a unique chain homotopy class $[v] : N\tilde{\mathbb{Z}}X \to J[-n]$ such that $[v_* f] = [g]$ since f is a local weak equivalence. This chain homotopy class $[v]$ is also independent of choice of representative for the component of (f,g). We therefore have a well defined function

$$\psi : \pi_0 H(X, K(J,n)) \to \pi_{ch}(N\tilde{\mathbb{Z}}X, \operatorname{Tr}_0 J[-n]).$$

Then the composites $\psi \cdot \phi$ and $\phi \cdot \psi$ are identity morphisms. □

We have proved

Theorem 15.3. *Suppose that A is a sheaf of abelian groups on $\mathcal{C}$, and let $A \to J$ be an injective resolution of A in the category of abelian sheaves. Let X be a simplicial presheaf on $\mathcal{C}$.*

Then there is an isomorphism

$$\pi_{ch}(N\tilde{\mathbb{Z}}X, \operatorname{Tr}_0 J[-n]) \cong [X, K(A,n)].$$

This isomorphism is natural in X.

Suppose that A is an abelian (pre)sheaf on $\mathcal{C}$ and that X is a simplicial presheaf. Write

$$H^n(X, A) = [X, K(A,n)],$$

and say that this group is the n^{th} *cohomology group* of X with coeffients in A.

The following basic result is then an immediate consequence of Theorem 15.3 (but has another, simpler proof — exxercise):

Corollary 15.4. *Suppose that $f : X \to Y$ induces an isomorphism*

$$\tilde{H}_*(X) \cong \tilde{H}_*(Y)$$

in all homology sheaves. Then the induced map in cohomology

$$H^*(Y, A) \to H^*(X, A)$$

is an isomorphism for all coefficient presheaves A.

Proof. The induced map $\mathbb{Z}(X) \to \mathbb{Z}(Y)$ is a local weak equivalence. □

There is also a torsion coeffients version:

Corollary 15.5. *If $f : X \to Y$ induces a homology sheaf isomorphism*

$$\tilde{H}_*(X, \mathbb{Z}/n) \cong \tilde{H}_*(Y, \mathbb{Z}/n)$$

then f induces an isomorphism

$$H^*(Y, A) \to H^*(X, A)$$

for all n-torsion presheaves A.

Remark 15.6. 1) The associated sheaf map

$$K(A, n) \to K(\tilde{A}, n)$$

is a local weak equivalence, so that

$$H^n(X, A) \cong H^n(X, \tilde{A}).$$

2) One can (and does) define sheaf cohomology $H^n(\mathcal{C}, A)$ for an abelian sheaf A on a site $\mathcal{C}$ by

$$H^n(\mathcal{C}, A) = H_{-n}(\Gamma_* J)$$

where $A \to J$ is an injective resolution of A concentrated in negative degrees and Γ_* is global sections (ie. inverse limit). But $\Gamma_* Y = \hom(*, Y)$ for any Y, and so

$$H^n(\mathcal{C}, A) \cong \pi_{ch}(\tilde{\mathbb{Z}}*, \operatorname{Tr}_0 J[-n]) \cong [*, K(A, n)]$$

by Theorem 15.3.

3) There is a *universal coefficients spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X), \tilde{A}) \Rightarrow H^{p+q}(X, A)$$

for abelian presheaves A and simplicial presheaves X. There is a corresponding spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X, \mathbb{Z}/n), \tilde{A}) \Rightarrow H^{p+q}(X, A)$$

for n-torsion presheaves A.

Cup products

Suppose that

$$X \xleftarrow{\simeq} X' \to K(A,n), \quad Y \xleftarrow{\simeq} Y' \to K(B,m)$$

are cocycles. Then the adjoint simplicial abelian presheaf maps

$$\mathbb{Z}X' \to K(A,n), \quad \mathbb{Z}Y' \to K(B,m)$$

have a (simplicial abelian group) tensor product

$$\mathbb{Z}(X' \times Y') \cong \mathbb{Z}X' \otimes \mathbb{Z}Y' \to K(A,n) \otimes K(B,m)$$

and there is a natural weak equivalence

$$K(A,n) \otimes K(B,m) \simeq K(A \otimes B, n+m).$$

in simplicial abelian groups, hence in simplicial abelian presheaves.

Proving this last claim is an exercise. Use the weak equivalence

$$K(A,n) \simeq \mathbb{Z}_*(S^1)^{\wedge n}) \otimes A$$

where $\mathbb{Z}_*(K)$ denotes the reduced complex of a pointed simplicial set K.

The adjoint

$$X \times Y \xleftarrow{\simeq} X' \times Y' \to K(A \otimes B, n+m)$$

represents the external cup product of the classes represented by the two cocycles. We have defined an *external cup product*

$$H^n(X,A) \times H^m(Y,B) \to H^{n+m}(X \times Y, A \otimes B).$$

If A happens to be a presheaf of rings this construction specializes to the cup product pairing

$$\begin{aligned} H^n(X,A) \times H^m(X,A) &\to H^{n+m}(X \times X, A) \\ &\xrightarrow{\Delta^*} H^{n+m}(X,A). \end{aligned}$$

where $\Delta : X \to X \times X$ is the diagonal map.

Cohomology operations

A *cohomology operation* is a map

$$K(A,n) \to K(B,m)$$

in the homotopy category.

The *Steenrod operation* Sq^i is a morphism

$$K(\mathbb{Z}/2,n) \to K(\mathbb{Z}/2,n+i)$$

in the ordinary homotopy category. The constant presheaf functor preserves weak equivalences, and so Sq^i induces a morphism

$$K(\Gamma^*\mathbb{Z}/2,n) \to K(\Gamma^*\mathbb{Z}/2,n+i)$$

in the homotopy category of simplicial presheaves on an arbitrary small site $\mathcal{C}$. It therefore induces a homomorphism

$$\mathrm{Sq}^i : H^n(X,\mathbb{Z}/2) \to H^{n+i}(X,\mathbb{Z}/2)$$

which is natural in simplicial presheaves X. The collection of Steenrod operations $\{\mathrm{Sq}^i\}$ for simplicial presheaves has the same basic list of properties as the Steenrod operations for ordinary spaces.

Steenrod operations for mod 2 étale cohomology were first introduced by Breen [1]; the definition given here for mod 2 cohomology of arbitrary simplicial presheaves is a vast generalization. The first calculational application was in questions concerning Hasse-Witt classes for non-degenerate symmetric bilinear forms in the mod 2 Galois cohomology of fields — see [5] and [6].

That said, the definition of Steenrod operations which is given here has its uses, but it is now relatively naive. Voevodsky introduced and made very effective use of a much more sophisticated construction for motivic homotopy theory in his proof of the Milnor conjecture [11], [12].

16 Descent spectral sequences

Proposition 16.1. *Suppose that A is a presheaf of abelian groups, and that*

$$j : K(A,n) \to GK(A,n)$$

is an injective fibrant model of $K(A,n)$. Then there are isomorphisms

$$\pi_j GK(A,n)(U) \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

Exercise 16.2. Suppose given a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

where p and p' are local fibrations and f is a local weak equivalence. Suppose that $Z \to Y$ is a map of simplicial presheaves. Show that the induced map

$$Z \times_Y X \xrightarrow{f_*} Z \times_Y X'$$

is a local weak equivalence — use Boolean localization.

Suppose that $U \in \mathcal{C}$ and write $X|_U$ for the restriction of X along the functor

$$\mathcal{C}/U \to \mathcal{C}.$$

Lemma 16.3. *The restriction functor $X \mapsto X|_U$ preserves injective fibrations and local weak equivalences, and therefore preserves injective fibrant models.*

Proof. The restriction functor $X \mapsto X|_U$ has a left adjoint j_U^* where

$$j_U^*(Y)(V) = \bigsqcup_{V \to U} Y(V).$$

Then j_U^* clearly preserves cofibrations and sectionwise weak equivalences. The functor j_U^* also preserves local trivial fibrations (exercise) and therefore preserves local weak equivalences.

Restriction preserves sectionwise equivalences and local trivial fibrations, and therefore preserves local weak equivalences. $\square$

Proof of Proposition 16.1. There are isomorphisms

$$\begin{aligned}\pi_0 GK(A,n)(U) &\cong [*, GK(A,n)(U)] \\ &\cong [*, GK(A|_U, n)]_{\mathcal{C}/U} \\ &\cong H^n(\mathcal{C}/U, \tilde{A}|_U).\end{aligned}$$

Note that $GK(A,n)|_U$ is an injective fibrant model of $K(A|_U, n)$ by Lemma 16.3, giving the second and third isomorphisms.

Observe that the associated sheaf map

$$\eta : K(A,0) \to K(\tilde{A},0)$$

is an injective fibrant model for the constant simplicial presheaf $K(A,0)$, and

$$\pi_j K(\tilde{A},0)(U) = 0$$

for $j > 0$.

There is a sectionwise fibre sequence

$$\begin{aligned}K(A,n-1) \to &WK(A,n-1) \\ &\to \overline{W}K(A,n-1) = K(A,n)\end{aligned}$$

where $WK(A,n-1)$ is sectionwise contractible. Take an injective fibrant model

$$\begin{tikzcd} WK(A,n-1) \arrow[r, "j"] \arrow[d] & GWK(A,n-1) \arrow[d, "p"] \\ K(A,n) \arrow[r, "j"'] & GK(A,n) \end{tikzcd}$$

where the maps labelled j are local weak equivalences, $GK(A,n)$ is injective fibrant and p is an injective fibration. Let $F = p^{-1}(0)$. Then F is injective fibrant and the induced map

$$K(A, n-1) \to F$$

is a local weak equivalence, by Exercise 16.2. Write $GK(A, n-1)$ for F.

We have sectionwise fibre sequences

$$\begin{aligned} GK(A, n-1)(U) \to & GWK(A, n-1)(U) \\ & \to GK(A,n)(U) \end{aligned}$$

for all $U \in \mathcal{C}$. The map

$$GWK(A, n-1) \to *$$

is a trivial injective fibration, and is therefore a sectionwise trivial fibration. It follows that

$$\pi_j GK(A,n)(U) \cong \pi_{j-1} GK(A, n-1)(U)$$

for $j \geq 1$, so that

$$\pi_j GK(A,n)(U) \cong H^{n-j}(\mathcal{C}/U, \tilde{A}|_U)$$

for $1 \leq j \leq n$ by induction on n. □

Example: Suppose $\mathcal{C}$ is the big site $(Sch|_S)_{et}$ for a scheme S with the étale topology and that U is an S-scheme in this site. Then $\mathcal{C}/U$ is isomorphic to the site $(Sch|_U)_{et}$. If A is a sheaf on the big étale site for S, and if $K(A,n) \to GK(A,n)$ is an injective fibrant model for $K(A,n)$, then the presheaves of homotopy groups for $GK(A,n)$ have the form

$$\pi_j GK(A,n)(U) \cong \begin{cases} H_{et}^{n-j}(U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

Similar statements obtain for all other geometric topologies on categories of S-schemes.

Suppose that X is a presheaf of locally connected pointed Kan complexes, and form the Postnikov tower

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
 & P_2X \arrow[r, "j"] \arrow[d] & GP_2X \arrow[d, "p"] \\
 & P_1X \arrow[r, "j"] \arrow[d] & GP_1X \arrow[d, "p"] \\
X \arrow[r] \arrow[ur] \arrow[uur] & P_0X \arrow[r, "j"'] & GP_0X
\end{tikzcd}$$

where all maps labelled j are injective fibrant models and the maps p are injective fibrations.

The fibre of $GP_nX \to GP_{n-1}X$ is sectionwise equivalent to $GK(\tilde{\pi}_nX, n)$, where

$$\tilde{\pi}_nX = \tilde{\pi}_n(X, *)$$

is the n^{th} homotopy group sheaf, based at the global base point.

Now take $U \in \mathcal{C}$ and consider the tower of fibrations

$$GP_0X(U) \leftarrow GP_1X(U) \leftarrow GP_2X(U) \leftarrow \dots$$

The fibre $GK(\tilde{\pi}_nX, n)(U)$ of the map

$$GP_nX(U) \to GP_{n-1}X(U)$$

has homotopy groups

$$\pi_j GK(\tilde{\pi}_n X, n)(U) \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{\pi}_n X|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

and so the tower of fibrations spectral sequence (with the Thomason re-indexing trick [10, 5.54]) determines a spectral sequence with

$$E_2^{s,t}(U) = H^s(\mathcal{C}/U, \tilde{\pi}_s X|_U)$$

This is the (unstable) *descent spectral sequence* — it is actually a presheaf of spectral sequences. One sometimes sees this spectral sequence referred to as either a cohomological or topological descent spectral sequence.

There are two issues:

1) the spectral sequence might or might not converge to

$$\pi_{t-s} \varprojlim GP_n X(U)$$

2) it can be a bit of work to show that the map $X \to \varprojlim_n GP_n X$ is a local weak equivalence.

Both issues can be resolved (ie. the spectral sequence converges and the map of 2) is a local weak equivalence) if X is locally connected in the sense that $\tilde{\pi}_0 X \cong *$ and there is a uniform bound on cohomological dimension for all sheaves $\tilde{\pi} X|_U$. See [4].

There are also "finite" descent spectral sequences, which are Bousfield-Kan spectral sequences arising from function complexes $\mathbf{hom}(V, Z)$, where $V \to *$ is a local weak equivalence and Z is injective fibrant. In particular, V could be the Čech resolution $C(U) \to *$ which is associated to a local epimorphism $U \to *$ of sheaves (or presheaves).

Example: Suppose that L/k is a finite Galois extension of a field k with Galois group G. Then, by Galois theory, there is an isomorphism

$$G \times \mathrm{Sp}(L) \xrightarrow{\cong} \mathrm{Sp}(L) \times \mathrm{Sp}(L)$$

of k-schemes which induces an isomorphism

$$EG \times_G \mathrm{Sp}(L) \cong C(\mathrm{Sp}(L))$$

on simplical sheaves (even simplicial schemes) on any of the étale sites for the field k. It follows that the canonical map

$$EG \times_G \mathrm{Sp}(L) \to *$$

is a local weak equivalence for the étale topology. Thus, if Z is an injective fibrant simplicial presheaf, then the induced map

$$Z(k) \cong \mathbf{hom}(*, Z) \to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Z)$$

is a weak equivalence of simplicial sets. At the same time, the Bousfield-Kan spectral sequence for the function complex on the right has the form

$$E_2^{s,t} = H^s(G, \pi_t Z(L)) \Rightarrow \pi_{t-s} Z(k).$$

This is a finite Galois descent spectral sequence for the homotopy groups of the global sections space $Z(k)$ of Z. It is also referred to as a homotopy fixed points spectral sequence, since the function complex

$$\mathbf{hom}(EG \times_G, \mathrm{Sp}(L), Z) = \underleftarrow{\mathrm{holim}}_G\, Z(L)$$

is the traditional homotopy fixed points complex for the action of G on the space $Z(L)$.

On the other hand, the full Galois (or étale) cohomological descent spectral sequence for Z has the form

$$E_2^{s,t} = H^s(\Omega, \tilde{\pi}_t Z) \Rightarrow \pi_{t-s} Z(k),$$

(provided that it converges to the right thing), where Ω is the absolute Galois group of k.

One often says that a simplicial presheaf X on an étale site for k satisfies *finite descent* if the map

$$X(k) \cong \mathbf{hom}(*, X) \to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X)$$

x is a weak equivalence for every finite Galois extension L/k. The question of whether a given simplicial presheaf X (like an algebraic K-theory presheaf) satisfies finite descent is also sometimes called the *homotopy fixed points problem.*

Warning: You might be tempted (many were) to say that finite descent for X implies that X satisfies descent for the étale topology on k, but you would be wrong. Such claims hold only in very special cases — see [7], [10], [8], [9].

References

[1] Lawrence Breen. Extensions du groupe additif. *Inst. Hautes Études Sci. Publ. Math.*, 48:39–125, 1978.

[2] D. Dugger. Classification spaces of maps in model categories. Preprint, `http://www.uoregon.edu/~ddugger/`, 2006.

[3] W. G. Dwyer and D. M. Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.

[4] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[5] J. F. Jardine. Universal Hasse-Witt classes. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, pages 83–100. Amer. Math. Soc., Providence, RI, 1989.

[6] J. F. Jardine. Higher spinor classes. *Mem. Amer. Math. Soc.*, 110(528):vi+88, 1994.

[7] J. F. Jardine. *Generalized Étale Cohomology Theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[8] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.

[9] J.F. Jardine. Galois descent criteria. Preprint, `http://uwo.ca/math/faculty/jardine/`, 2018.

[10] R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.

[11] Vladimir Voevodsky. Motivic cohomology with $\mathbf{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (98):59–104, 2003.

[12] Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.*, (98):1–57, 2003.