Lecture 08

17 Torsors for groups

Suppose that G is a sheaf of groups. A *G-torsor* is traditionally defined to be a sheaf X with a free G-action such that $X/G \cong *$ in the sheaf category.

The requirement that the action $G \times X \to X$ is free means that the isotropy subgroups of G for the action are trivial in all sections, which is equivalent to requiring that all sheaves of fundamental groups for the Borel construction $EG \times_G X$ are trivial. There is an isomorphism of sheaves

$$\tilde{\pi}_0(EG \times_G X) \cong X/G.$$

Also the simplicial sheaf $EG \times_G X$ is the nerve of a sheaf of groupoids, which is given in each section by the translation category for the action of $G(U)$ on $X(U)$; this means, in particular, that all sheaves of higher homotopy groups for $EG \times_G X$ vanish.

It follows that a G-sheaf X is a *G-torsor* if and only if the map $EG \times_G X \to *$ is a local weak equivalence.

Example 17.1. The Borel construction

$$EH \times_H H = EH$$

for a group H is the nerve of the translation category for the action $H \times H \to H$ which is given by the multiplication of H. There is a unique map $e \xrightarrow{h} h$ for all $h \in H$, so that $EH \times_H H$ is a contractible simplicial set. If G is a sheaf of groups, then $EG \times_G G$ is contractible in each section, so that the map

$$EG \times_G G \to *$$

is a local weak equivalence, and G is a G-torsor. This object is often called the *trivial* G-torsor.

Example 17.2. Suppose that L/k is a finite Galois extension with Galois group G. Then the étale covering $\mathrm{Sp}(L) \to \mathrm{Sp}(k)$ has Čech resolution $C(L)$ and there is an isomorphism of simplicial schemes

$$C(L) \cong EG \times_G \mathrm{Sp}(L).$$

The simplicial presheaf map

$$C(L) \to *$$

on $Sch|_k$ is a local weak equivalence for the étale topology, so that $\mathrm{Sp}(L)$ represents a G-torsor for the étale topology on $\mathrm{Sp}(k)$, actually for all of the standard étale sites associated with k.

The category $G - \mathbf{tors}$ is the category whose objects are all G-torsors and whose maps are all G-equivariant maps between them.

Remark 17.3. If $f : X \to Y$ is a map of G-torsors, then f is induced as a map of fibres by the comparison of local fibrations

$$\begin{tikzcd} EG \times_G X \arrow[rr] \arrow[dr] & & EG \times_G Y \arrow[dl] \\ & BG & \end{tikzcd}$$

It follows that $f : X \to Y$ is a weak equivalence of constant simplicial sheaves, and is therefore an isomorphism. The category of G-torsors is therefore a groupoid.

Remark 17.4. Suppose that X is a G-torsor, and that the canonical map $X \to *$ has a (global) section $\sigma : * \to X$. Then σ extends, by multiplication, (also uniquely) to a G-equivariant map

$$\sigma_* : G \to X,$$

with $\sigma_*(g) = g \cdot \sigma_U$ for $g \in G(U)$. This map is an isomorphism of torsors, so that X is trivial with trivializing isomorphism σ_*. Conversely, if $\tau : G \to X$ is a map of torsors, then X has a global section $\tau(e)$. Thus a G-torsor X is trivial in the sense that it is isomorphic to G if and only if it has a global section.

Example 17.5. Suppose that X is a topological space. The category of sheaves on $op|_X$ can be identified up to equivalence with a category Top/X of spaces $Y \to X$ fibred over X. If G is a topological group, then G represents the sheaf $G \times X \to X$ given by projection. A sheaf with G-action consists of a map $Y \to X$ together with a G-action $G \times Y \to Y$ such that the map $Y \to X$ is G-equivariant for the trivial G-action on X. Such a thing is a G-torsor if the action $G \times Y \to Y$ is free and the map $Y/G \to X$ is an isomorphism. The latter implies that X has an open covering $i : U \subset X$ such that there are liftings

$$\begin{tikzcd} & Y \arrow[d] \\ U \arrow[ur, "\sigma"] \arrow[r, "i"'] & X \end{tikzcd}$$

Torsors are stable under pullback along continuous maps, and the map $U \times_X Y \to U$ is a G-torsor over U. The map σ induces a global section σ_* of this map, so that the pulled back torsor is trivial, and there is a commutative diagram

$$\begin{tikzcd} G \times U \arrow[r, "\cong"] \arrow[dr, "pr"'] & U \times_X Y \arrow[d] \\ & U \end{tikzcd}$$

where the displayed isomorphism is G-equivariant. It follows that a G-torsor over X is a principal G-bundle over X, and conversely.

Example 17.6. Suppose that U is an object of a small site $\mathcal{C}$. Composition with the canonical functor $\mathcal{C}/U \to \mathcal{C}$ induces a restriction functor

$$\mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C}/U),$$

written $F \mapsto F|_U$. The restriction functor is exact and preserves sheaf epimorphisms, and therefore takes G-torsors to $G|_U$-torsors. The global sections of $F|_U$ coincide with the elements of the set $F(U)$, so that a G-torsor X trivializes over U if and only if $X(U) \neq \emptyset$, or equivalently if and only if there is a diagram

$$\begin{tikzcd} & X \arrow[d] \\ U \arrow[ur] \arrow[r] & * \end{tikzcd}$$

The map $X \to *$ is a local epimorphism, so there is a covering family $U_\alpha \to *$ (ie. such that $\bigsqcup U_\alpha \to *$ is a local epimorphism) with $X(U_\alpha) \neq \emptyset$. In other words, every torsor trivializes over some covering family of the point $*$.

Suppose that the picture

$$* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$$

is an object of the cocycle category $H(*, BG)$ in simplicial presheaves, and form the pullback

$$\begin{tikzcd} \mathrm{pb}(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\ EG \arrow[r, "\pi"'] & BG \end{tikzcd}$$

where $EG = B(G/*) = EG \times_G G$ and $\pi : EG \to BG$ is the canonical map. Then $\mathrm{pb}(Y)$ inherits a G-action from the G-action on EG, and the map

$$EG \times_G \mathrm{pb}(Y) \to Y \tag{17.1}$$

is a sectionwise weak equivalence (this is a consequence of Lemma 17.10 below). Also, the square is homotopy cartesian in sections where $Y(U) \neq \emptyset$, so there is a local weak equivalence

$$G|_U \to \mathrm{pb}(Y)|_U$$

over all such U. It follows that the canonical map $\mathrm{pb}(Y) \to \tilde{\pi}_0 \,\mathrm{pb}(Y)$ is a G-equivariant local weak equivalence, and hence that the maps

$$EG \times_G \tilde{\pi}_0 \,\mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are natural local weak equivalences. In particular, the G-sheaf $\tilde{\pi}_0 \operatorname{pb}(Y)$ is a G-torsor.

We therefore have a functor

$$H(*, BG) \to G - \mathbf{tors}$$

defined by sending $* \xleftarrow{\simeq} Y \to BG$ to the object $\tilde{\pi}_0 \operatorname{pb}(Y)$. The Borel construction defines a functor

$$G - \mathbf{tors} \to H(*, BG):$$

the G-torsor X is sent to the (canonical) cocycle

$$* \xleftarrow{\simeq} EG \times_G X \to BG.$$

One checks these functors are adjoint, and hence induce a bijection

$$\pi_0 H(*, BG) \cong \pi_0(G - \mathbf{tors}).$$

In view of the fact that $\pi_0(G - \mathbf{tors})$ is isomorphism classes of G-torsors, and we know that

$$\pi_0 H(*, BG) \cong [*, BG],$$

we have proved

Theorem 17.7. *Suppose that G is a sheaf of groups on a small Grothendieck site $\mathcal{C}$. Then there is a bijection*

$$[*, BG] \cong \{\textit{isomorphism classes of } G\textit{-torsors}\}$$

Remark 17.8. 1) Theorem 17.7 was first proved, by a different method, in [4].
2) The non-abelian invariant $H^1(\mathcal{C}, G)$ is traditionally defined to be the collection of isomorphism classes of G-torsors. The theorem therefore gives an identification

$$H^1(\mathcal{C}, G) \cong [*, BG].$$

Example 17.9. Suppose that k is a field. Let $\mathcal{C}$ be the étale site $et|_k$ for k, and identify the orthogonal group O_n with a sheaf of groups on this site. The non-abelian cohomology object $H^1_{et}(k, O_n)$ is well known to coincide with the set of isomorphism classes of non-degenerate symmetric bilinear forms over k of rank n. Thus, every such form q determines a morphism $* \to BO_n$ in the simplicial (pre)sheaf homotopy category, and this morphism determines the form q up to isomorphism.

Suppose that k is a field such that $char(k) \neq 2$. There are isomorphisms

$$\begin{aligned} H^*_{et}(BO_n, \mathbb{Z}/2) &\cong H^*(BO_n, \mathbb{Z}/2) \\ &\cong H^*_{et}(k, \mathbb{Z}/2)[HW_1, \dots, HW_n] \end{aligned}$$

where the polynomial generator HW_i has degree i. In fact HW_i is characterized by mapping to the i^{th}

elementary symmetric polynomial $\sigma_i(x_1, \ldots, x_n)$ under the isomorphism

$$\begin{aligned} H^*(BO_n, \mathbb{Z}/2) &\cong H^*(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2)^{\Sigma_n} \\ &\cong H^*_{et}(k, \mathbb{Z}/2)[x_1, \ldots, x_n]^{\Sigma_n}. \end{aligned}$$

where $(\ )^{\Sigma_n}$ denotes invariants for the symmetric group Σ_n

Every symmetric bilinear form α determines a map $\alpha : * \to BO_n$ in the simplicial presheaf homotopy category, and therefore induces a map

$$\alpha^* : H^*_{et}(BO_n, \mathbb{Z}/2) \to H^*_{et}(k, \mathbb{Z}/2),$$

and $HW_i(\alpha) = \alpha^*(HW_i)$ is the i^{th} Hasse-Witt class of α.

One can show that $HW_1(\alpha)$ is the pullback of the determinant $BO_n \to B\mathbb{Z}/2$, and $HW_2(\alpha)$ is the classical Hasse-Witt invariant of α.

The Steenrod algebra is used to calculate the relation between Hasse-Witt and Stiefel-Whitney classes for Galois representations. This calculation uses the Wu formulas for the action of the Steenrod algebra on elementary symmetric polynomials. See [4], [5].

Here's the missing lemma:

Lemma 17.10. *Suppose that I is a small category and that $p : X \to BI$ is a simplicial set map. Let the pullback diagrams*

$$\begin{tikzcd} \mathrm{pb}(X)(i) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B(I/i) \arrow[r] & BI \end{tikzcd}$$

define the I-diagram $i \mapsto \mathrm{pb}(X)(i)$. Then the resulting map

$$\omega : \underrightarrow{\mathrm{holim}}_{\,i\in I}\ \mathrm{pb}(X)(i) \to X$$

is a weak equivalence.

Proof. The simplicial set

$$\underrightarrow{\mathrm{holim}}_{\,i\in I}\ \mathrm{pb}(X)(i)$$

is the diagonal of a bisimplicial set whose (n,m)-bisimplices are pairs

$$(x, i_0 \to \cdots \to i_n \to j_0 \to \cdots \to j_m)$$

where $x \in X_n$, the morphisms are in I, and $p(x)$ is the string

$$i_0 \to \cdots \to i_n.$$

The map

$$\omega : \underrightarrow{\mathrm{holim}}_{\,i\in I}\ \mathrm{pb}(X)(i) \to X$$

takes such an (n,m)-bisimplex to $x \in X_n$. The fibre over x can be identified with the simplicial set $B(i_n/I)$, which is contractible. $\square$

18 Torsors for groupoids

What's a set-valued functor $X : I \to \mathbf{Set}$?

The functor X consists of sets $X(i)$, $i \in \mathrm{Ob}(I)$ and functions $\alpha_* : X(i) \to X(j)$ for $\alpha : i \to j$ in $\mathrm{Mor}(I)$ such that $\alpha_* \beta_* = (\alpha \cdot \beta)_*$ for all composable pairs of morphisms in I and $(1_i)_* = 1_{X(i)}$ for all objects i of I.

The sets $X(i)$ can be collected together to give a set

$$\pi : X = \bigsqcup_{i \in \mathrm{Ob}(I)} X(i) \to \bigsqcup_{i \in \mathrm{Ob}(I)} = \mathrm{Ob}(I)$$

and the assignments $\alpha \mapsto \alpha_*$ can be collectively rewritten as a commutative diagram

$$\begin{tikzcd} X \times_{\pi,s} \mathrm{Mor}(I) \arrow[r, "m"] \arrow[d, "pr"'] & X \arrow[d, "\pi"] \\ \mathrm{Mor}(I) \arrow[r, "t"'] & \mathrm{Ob}(I) \end{tikzcd} \tag{18.1}$$

where $s, t : \mathrm{Mor}(I) \to \mathrm{Ob}(I)$ are the source and target maps, respectively, and

$$\begin{tikzcd}
X \times_{\pi,s} \mathrm{Mor}(I) \arrow[r, "pr"] \arrow[d] & \mathrm{Mor}(I) \arrow[d, "s"] \\
X \arrow[r, "\pi"'] & \mathrm{Ob}(I)
\end{tikzcd}$$

is a pullback. Then the notation is awkward, but the composition laws for the functor X translate into the commutativity of the diagrams

$$\begin{tikzcd}
X \times_{\pi,s} \mathrm{Mor}(I) \times_{t,s} \mathrm{Mor}(I) \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] & X \times_{\pi,s} \mathrm{Mor}(I) \arrow[d, "m"] \\
X \times_{\pi,s} \mathrm{Mor}(I) \arrow[r, "m"'] & X
\end{tikzcd} \tag{18.2}$$

and

$$\begin{tikzcd}
X \arrow[r, "e_*"] \arrow[dr, "1"'] & X \times_{\pi,s} \mathrm{Mor}(I) \arrow[d, "m"] \\
& X
\end{tikzcd} \tag{18.3}$$

Here, m_I is the composition law of the category I, and the map e_* is uniquely determined by the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[d, "1"'] & \mathrm{Ob}(I) \arrow[r, "e"] & \mathrm{Mor}(I) \arrow[d, "s"] \\
X \arrow[rr, "\pi"'] & & \mathrm{Ob}(I)
\end{tikzcd}$$

where the map e picks out the identity morphisms of I.

Thus, a functor $X : I \to \mathbf{Set}$ consists of a function $\pi : X \to \mathrm{Ob}(I)$ together with an action $m : X \times_{\pi,s} \mathrm{Mor}(I) \to X$ making the diagram (18.1) commute, such that the diagrams (18.2) and (18.3) also commute. This is the internal description, which can be used to define functors on category objects within specific categories.

Specifically, suppose that G is a sheaf of groupoids on a site $\mathcal{C}$. then a *sheaf-valued functor* X on G consists of a sheaf map $\pi : X \to \mathrm{Ob}(G)$, together with an action morphism $m : X \times_{\pi,s} \mathrm{Mor}(G) \to X$ in sheaves such that the diagrams corresponding to (18.1), (18.2) and (18.3) commute in the sheaf category.

Alternatively, X consists of set-valued functors

$$X(U) : G(U) \to \mathbf{Sets}$$

with $x \mapsto X(U)_x$ for $x \in \mathrm{Ob}(G(U))$, together with functions

$$\phi^* : X(U)_x \to X(V)_{\phi^*(x)}$$

for each $\phi : V \to U$ in $\mathcal{C}$, such that the assignment

$$U \mapsto X(U) = \bigsqcup_{x \in \mathrm{Ob}(G(U))} X(U)_x,\ U \in \mathcal{C},$$

defines a sheaf and the diagrams

$$
\begin{tikzcd}
X(U)_x \arrow[r, "\alpha_*"] \arrow[d, "\phi^*"'] & X(U)_y \arrow[d, "\phi^*"] \\
X(V)_{\phi^*(x)} \arrow[r, "(\phi^*(\alpha))_*"'] & X(V)_{\phi^*(y)}
\end{tikzcd}
$$

commute for each $\alpha : x \to y$ of $\mathrm{Mor}(G)$ and all $\phi : V \to U$ of $\mathcal{C}$.

From this alternative point of view, it's easy to see that a sheaf-valued functor X on G defines a natural simplicial (pre)sheaf homomorphism

$$p : \underrightarrow{\mathrm{holim}}_G\, X \to BG.$$

One makes the construction sectionwise.

NB: This story is a direct generalization of what we saw for sheaves Y with actions by sheaves of groups H. The Borel construction $EH \times_H Y$ is the homotopy colimit $\underrightarrow{\mathrm{holim}}_H Y$.

I say that a sheaf-valued functor X on a sheaf of groupoids G is a *G-torsor* if the canonical map

$$\underrightarrow{\mathrm{holim}}_G\, X \to *$$

is a local weak equivalence.

A *morphism* $f : X \to Y$ of G-torsors is a natural transformation of G-functors, namely a sheaf morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{Ob}(G) & \end{tikzcd}$$

fibred over $\mathrm{Ob}(G)$ which respects the multiplication maps.

The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & \underrightarrow{\mathrm{holim}}_{G}\, X \arrow[d, "p"] \\ \mathrm{Ob}(G) \arrow[r] & BG \end{tikzcd}$$

is homotopy cartesian in each section by Quillen's Theorem B [2, IV.5.2] (more specifically, Lemma 5.7), since G is a (pre)sheaf of groupoids, and is therefore homotopy cartesian in simplicial sheaves. It follows that a morphism $f : X \to Y$ of G-torsors specializes to a weak equivalence $X \to Y$ of constant simplicial sheaves, which is therefore an isomorphism. It follows that the category

$$G - \mathbf{tors}$$

of G-torsors is a groupoid.

Clearly, every G-torsor X has an associated canonical cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_{G}\, X \xrightarrow{p} BG,$$

and this association defines a functor

$$\phi : G - \mathbf{tors} \to H(*, BG)$$

taking values in the simplicial sheaf cocycle category.

Now suppose given a cocycle

$$* \xleftarrow{\simeq} Y \xrightarrow{g} BG$$

in simplicial sheaves and form the pullback diagrams

$$\begin{tikzcd}
\mathrm{pb}(Y)(U)_x \arrow[r] \arrow[d] & Y(U) \arrow[d, "g"] \\
B(G(U)/x) \arrow[r] & BG(U)
\end{tikzcd}$$

of simplicial sets for each $x \in \mathrm{Ob}(G(U)), U \in \mathcal{C}$, and set

$$\mathrm{pb}(Y)(U) = \bigsqcup_{x \in \mathrm{Ob}(G(U))} \mathrm{pb}(Y)(U)_x.$$

Then the resulting simplicial presheaf map $\mathrm{pb}(Y) \to \mathrm{Ob}(G)$ defines a simplicial presheaf-valued functor on G. There is a sectionwise weak equivalence

$$\underrightarrow{\mathrm{holim}}_G \ \mathrm{pb}(Y) \to Y \simeq *$$

by Lemma 17.10, and the diagram

$$\begin{tikzcd}
\mathrm{pb}(Y) \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}_G \ \mathrm{pb}(Y) \arrow[d] \\
\mathrm{Ob}(G) \arrow[r] & BG
\end{tikzcd}$$

is sectionwise homotopy cartesian. It follows that the natural transformation

$$\mathrm{pb}(Y) \to \tilde{\pi}_0(\mathrm{pb}(Y))$$

of simplicial presheaf-valued functors on G is a local weak equivalence. Thus, there are local weak equivalences

$$\underrightarrow{\mathrm{holim}}_G\ \tilde{\pi}_0\,\mathrm{pb}(Y) \simeq \underrightarrow{\mathrm{holim}}_G\ \mathrm{pb}(Y) \simeq Y \simeq *,$$

and the sheaf-valued functor $\tilde{\pi}_0\,\mathrm{pb}(Y)$ on G is a G-torsor. These constructions are functorial on $H(*, BG)$ and so there is a functor

$$\psi : H(*, BG) \to G - \mathbf{tors}.$$

Theorem 18.1. *The functors ϕ and ψ induce a homotopy equivalence*

$$B(G - \mathbf{tors}) \simeq BH(*, BG).$$

Corollary 18.2. *The functors ϕ and ψ induce a bijection*

$$\pi_0(G - \mathbf{tors}) \cong [*, BG].$$

There are multiple possible proofs of Corollary 18.2 (see also [7]), but it is convenient here to use a trick for diagrams of simplicial sets which are indexed by groupoids.

Suppose that Γ is a small groupoid, and let $s\mathbf{Set}^{\Gamma}$ be the category of Γ-diagrams in simplicial sets. Let $s\mathbf{Set}/B\Gamma$ be the category of simplicial set morphisms $Y \to B\Gamma$. The homotopy colimit defines a functor

$$\underrightarrow{\mathrm{holim}}_{\Gamma} : s\mathbf{Set}^{\Gamma} \to s\mathbf{Set}/B\Gamma.$$

This functor sends a diagram $X : \Gamma \to s\mathbf{Set}$ to the canonical map $\underrightarrow{\mathrm{holim}}_{\Gamma} X \to B\Gamma$. On the other hand, given a simplicial set map $Y \to B\Gamma$, the collection of pullback diagrams

$$\begin{tikzcd} \mathrm{pb}(Y)_x \arrow[r] \arrow[d] & Y \arrow[d] \\ B(\Gamma/x) \arrow[r] & B\Gamma \end{tikzcd}$$

defines an Γ-diagram $\mathrm{pb}(Y) : \Gamma \to s\mathbf{Set}$ which is functorial in $Y \to B\Gamma$.

Lemma 18.3. *Suppose that Γ is a groupoid. Then the functors*

$$\mathrm{pb} : s\mathbf{Set}/B\Gamma \leftrightarrows s\mathbf{Set}^{\Gamma} : \underrightarrow{\mathrm{holim}}_{\Gamma}$$

form an adjoint pair: pb *is left adjoint to* $\underrightarrow{\mathrm{holim}}_{\Gamma}$.

Proof. Suppose that X is a Γ-diagram and that $p : Y \to B\Gamma$ is a simplicial set over $B\Gamma$. Suppose given a natural transformation

$$f : \mathrm{pb}(Y)_n \to X_n.$$

and let x be an object of Γ. Then an element of $(\mathrm{pb}(Y)_x)_n$ can be identified with a pair

$$(x, a_0 \to \cdots \to a_n \xrightarrow{\alpha} x)$$

where the string of arrows is in Γ and $p(x)$ is the string $a_0 \to \ldots a_n$. Then f is uniquely determined by the images of the elements

$$f(x, a_0 \to \cdots \to a_n \xrightarrow{1} a_n)$$

in $X_n(a_n)$. Since Γ is a groupoid, an element $y \in X(a_n)$ uniquely determines an element

$$(y_0, a_0) \to (y_1, a_1) \to \ldots (y_n, a_n)$$

with $y_n = y$. It follows that there is a natural bijection

$$\hom_\Gamma(\mathrm{pb}(Y)_n, X_n) \cong \hom_{B\Gamma_n}(Y_n, (\underrightarrow{\mathrm{holim}}_\Gamma X)_n).$$

Extend simplicially to get the adjunction isomorphism

$$\hom_\Gamma(\mathrm{pb}(Y), X) \cong \hom_{B\Gamma}(Y, \underrightarrow{\mathrm{holim}}_\Gamma X).$$

$\square$

Proof of Theorem 18.1. It follows from Lemma 18.3 that the functor ψ is left adjoint to the functor ϕ. $\square$

Example: Suppose that H is a groupoid and that $x \in \mathrm{Ob}(H)$. The groupoid H/x has a terminal object and hence determines a cocycle

$$* \xleftarrow{\simeq} B(H/x) \to BH.$$

If $a \in \mathrm{Ob}(H)$ then in the pullback diagram

$$\begin{tikzcd} \mathrm{pb}(B(H/x))(a) \arrow[r] \arrow[d] & B(H/x) \arrow[d] \\ B(H/a) \arrow[r] & BH \end{tikzcd}$$

the object $\mathrm{pb}(B(H/x))(a)$ is the nerve of a groupoid whose objects are the diagrams

$$a \xleftarrow{\alpha} b \xrightarrow{\beta} x$$

in H, and whose morphisms are the diagrams

$$\begin{tikzcd} & b \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd] & \\ a & & x \\ & b' \arrow[ul, "\alpha'"] \arrow[ur, "\beta'"'] & \end{tikzcd}$$

In the presence of such a picture, $\beta \cdot \alpha^{-1} = \beta' \cdot (\alpha')^{-1}$. There are uniquely determined diagrams

$$\begin{tikzcd} & b \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd] & \\ a & & x \\ & a \arrow[ul, "1"] \arrow[ur, "\beta\cdot\alpha^{-1}"'] & \end{tikzcd}$$

for each object $a \xleftarrow{\alpha} b \xrightarrow{\beta} x$. It follows that there is a natural bijection

$$\pi_0 \operatorname{pb}(B(H/x)(a) \cong \hom_H(a,x)$$

and that

$$\operatorname{pb}(B(H/x))(a) \to \pi_0 \operatorname{pb}(B(H/x))(a)$$

is a natural weak equivalence.

It follows that there are weak equivalences

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}_{\,a\in H} \operatorname{pb}(B(H/x))(a) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & B(H/x) \simeq * \\
\underrightarrow{\mathrm{holim}}_{\,a\in H} \hom_H(a,x)
\end{tikzcd}$$

so that the functor $a \mapsto \hom_H(a,x)$ defines an H-torsor. Here, the function

$$\beta_* : \hom_H(a,x) \to \hom_H(b,x)$$

induced by $\beta : a \to b$ is precomposition with β^{-1}.

To put it a different way, each $x \in H$ determines a H-torsor $a \mapsto \hom_H(a,x)$, which we'll call $\hom_H(\ ,x)$ and there is a functor

$$H \to H - \mathbf{tors}$$

which is defined by $x \mapsto \hom_H(\ ,x)$.

Observe that the maps $\hom_H(\ ,x) \to Y$ classify elements of $Y(x)$ for all functors $Y : H \to \mathbf{Set}$.

In general, every global section x of a sheaf of groupoids G determines a G-torsor $\hom_G(\ ,x)$ which is constructed sectionwise according to the recipe above. In particular, this is the torsor associated by the pullback construction to the cocycle

$$* \xleftarrow{\simeq} B(G/x) \to BG.$$

The torsors $\hom_G(\ ,x)$ are the trivial torsors for the sheaf of groupoids G. There is a functor

$$j : \Gamma_* G \to G - \mathbf{tors}$$

which is defined by $j(x) = \hom_G(\ ,x)$.

Observe that torsor (iso)morphisms

$$\begin{tikzcd} \hom_G(\ ,x) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Ob}(G) & \end{tikzcd}$$

are in bijective correspondence with global sections of X which map to $x \in \mathrm{Ob}(G)$ under the structure map $X \to \mathrm{Ob}(G)$. Such maps are trivializations of the torsor X.

These constructions restrict nicely. If $\phi : V \to U$ is a morphism of the underlying site $\mathcal{C}$ then composition with ϕ defines a functor

$$\phi_* : \mathcal{C}/V \to \mathcal{C}/U,$$

and composition with ϕ_* determines a restriction functor

$$\phi^* : \mathrm{Pre}(\mathcal{C}/U) \to \mathrm{Pre}(\mathcal{C}/V)$$

which takes $F|_U$ to $F|_V$ for any presheaf F on $\mathcal{C}$. All restriction functors take sheaves to sheaves and are exact. Thus, ϕ^* takes a $G|_U$-torsor to a $G|_V$ torsor. In particular,

$$\phi^* \hom_{G|_U}(\ ,x) = \hom_{G|_V}(\ ,x_V)$$

for all $x \in G(U)$. The functor ϕ^* also preserves cocycles.

The upshot is that there is a presheaf of groupoids $G - \mathbf{Tors}$ on the site $\mathcal{C}$ with

$$G - \mathbf{Tors}(U) = G|_U - \mathbf{tors}$$

and a presheaf of categories $\mathbb{H}(*, BG)$ with

$$\mathbb{H}(*, BG)(U) = H(*, BG|_U).$$

and there are functors

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr] & G - \mathbf{Tors} \arrow[d, "\phi"] \\ & \mathbb{H}(*, BG) \end{tikzcd}$$

where ϕ induces a sectionwise weak equivalence

$$\phi_* : B(G - \mathbf{Tors}) \xrightarrow{\simeq} B\mathbb{H}(*, BG)$$

by Theorem 18.1, and the displayed map is defined by sending an object $x \in G(U)$ to the cocycle $B(G|_U/x) \to BG|_U$.

The images $\hom(\ ,x)$ of the functor $j : G \to G-\mathbf{Tors}$ are the *trivial torsors*, and maps (isomorphisms) $\hom(\ ,x) \to X$ of G-torsors are global sections of X. Every G-torsor X has sections along some cover, since $\underrightarrow{\mathrm{holim}}_G X \to *$ is a local weak equivalence, so every G-torsor is locally trivial.

Proposition 18.4. *Suppose that G is a sheaf of groupoids on a small site $\mathcal{C}$. Then the induced maps*

$$\begin{tikzcd} BG \arrow[r, "j_*"] \arrow[dr] & B(G-\mathbf{Tors}) \arrow[d, "\phi_*"] \\ & B\mathbb{H}(*, BG) \end{tikzcd}$$

are local weak equivalences of simplicial sheaves.

Proof. The functor j is fully faithful in all sections (exercise), and the map

$$j_* : \tilde{\pi}_0 BG \to \tilde{\pi}_0 B(G-\mathbf{Tors})$$

is a sheaf epimorphism. But the fact that j is fully faithful in all sections means that the presheaf map

$$j_* : \pi_0 BG \to \pi_0 B(G-\mathbf{Tors})$$

is a monomorphism in all sections. $\square$

19 Stacks and homotopy theory

Write Pre($\mathbf{Gpd}(\mathcal{C})$) for the category of presheaves of groupoids on a small site $\mathcal{C}$.

Say that a morphism $f : G \to H$ of presheaves of groupoids is a weak equivalence (respectively fibration) if and only if the induced map $f_* : BG \to BH$ is a local weak equivalence (respectively injective fibration). A morphism $i : A \to B$ is a cofibration if it has the left lifting property with respect to all trivial fibrations.

The fundamental groupoid functor $X \mapsto \pi(X)$ is left adjoint to the nerve functor. It follows that every cofibration $A \to B$ of simplicial presheaves induces a cofibration $\pi(A) \to \pi(B)$ of presheaves of groupoids. The class of cofibrations $A \to B$ is closed under pushout along arbitrary morphisms $A \to G$, because cofibrations are defined by a left lifting property.

There is a function complex construction for presheaves of groupoids: the simplicial set $\mathbf{hom}(G, H)$ has for n-simplices all morphisms

$$\phi : G \times \pi(\Delta^n) \to H.$$

There is a natural isomorphism

$$\mathbf{hom}(G,H) \cong \mathbf{hom}(BG,BH),$$

which sends the simplex ϕ to the composite

$$BG\times\Delta^n \xrightarrow{1\times\eta} BG\times B\pi(\Delta^n) \cong B(G\times\pi(\Delta^n)) \xrightarrow{\phi_*} BH.$$

The following result appears in [3]:

Proposition 19.1. *With these definitions, the category* $\mathrm{Pre}(\mathbf{Gpd}(\mathcal{C}))$ *satisfies the axioms for a right proper closed simplicial model category.*

Proof. The inductive model structure for the category $s\,\mathrm{Pre}(\mathcal{C})$ is cofibrantly generated. It follows easily that every morphism $f : G \to H$ has a factorization

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & H \end{tikzcd}$$

such that j is a cofibration and p is a trivial fibration.

The other factorization axiom can be proved the same way, provided one knows that if $i : A \to B$ is a trivial cofibration of simplicial presheaves and the diagram

$$\begin{tikzcd} \pi(A) \arrow[r] \arrow[d, "i_*"'] & G \arrow[d, "i'"] \\ \pi(B) \arrow[r] & H \end{tikzcd}$$

is a pushout, then the map i' is a local weak equivalence. But one can prove the corresponding statement for ordinary groupoids, and the general case follows by a Boolean localization argument (exercise).

The claim is proved for ordinary groupoids by observing that in all pushout diagrams

$$\begin{tikzcd} \pi(\Lambda^n_k) \arrow[r] \arrow[d, "i_*"'] & G \arrow[d, "i'"] \\ \pi(\Delta^n) \arrow[r] & H \end{tikzcd}$$

the map i_* is an isomorphism for $n \geq 2$ and is the inclusion of a strong deformation retraction if $n = 1$. The classes of isomorphisms and strong deformation retractions are both closed under pushout in the category of groupoids.

All other closed model axioms are trivial to verify, as is right properness. The simplicial model axiom **SM7** has an elementary argument, which ultimately follows from the fact that the fundamental groupoid functor preserves products. $\square$

One can make the same definitions for sheaves of groupoids: say that a map $f : G \to H$ of sheaves of groupoids is a weak equivalence (respectively fibration) if the associated simplicial sheaf map $f_* : BG \to BH$ is a local weak equivalence (respectively injective fibration). Cofibrations are defined by a left lifting property, as before.

Write $\mathrm{Shv}(\mathbf{Gpd}(\mathcal{C}))$ and observe that the forgetful functor i and associated sheaf functor L^2 induce an adjoint pair

$$L^2 : \mathrm{Pre}(\mathbf{Gpd}(\mathcal{C})) \leftrightarrows \mathrm{Shv}(\mathbf{Gpd}(\mathcal{C})) : i$$

According to the definitions, the forgetful functor i preserves fibrations and trivial fibrations. Moreover, the canonical map $\eta : BG \to iL^2BG$ is always a local weak equivalence. The method of proof of Proposition 19.1 and formal nonsense now combine to prove the following

Proposition 19.2. *1) With these definitions, the category* $\mathrm{Shv}(\mathbf{Gpd}(\mathcal{C}))$ *of sheaves of groupoids satisfies the axioms for a right proper closed simplicial model category.*

2) The adjoint pair

$$L^2 : \mathrm{Pre}(\mathbf{Gpd}(\mathcal{C})) \leftrightarrows \mathrm{Shv}(\mathbf{Gpd}(\mathcal{C})) : i$$

forms a Quillen equivalence.

One could say that the model structures of Proposition 19.1 and 19.2 are the injective model structures for presheaves and sheaves of groupoids on a site $\mathcal{C}$, respectively. Of course, part 2) of Proposition 19.2 says that these model structures are Quillen equivalent.

Part 1) of Proposition 19.2 was first proved in [10]. This was a breakthrough result, in that it enabled the following definition:

Definition: A sheaf of groupoids H is said to be a *stack* if it satisfies descent for the injective model structure on $\mathrm{Shv}(\mathbf{Gpd}(\mathcal{C}))$. This means that every injective fibrant model $j : H \to H'$ should be a sectionwise weak equivalence.

Observe that if $j : H \to H'$ is a fibrant model in sheaves (or presheaves) of groupoids, then the induced map $j_* : BH \to BH'$ is a fibrant model in simplicial presheaves. Thus, H is a stack if and only if the simplicial presheaf BH satisfies descent.

Every fibrant object is a stack, because fibrant objects satisfy descent. This means that every fibrant model $j : G \to H$ of a sheaf of groupoids

G is a *stack completion*. This model j can be constructed functorially, since the injective model structure on $\mathrm{Shv}(\mathbf{Gpd}(\mathcal{C}))$ is cofibrantly generated. We can therefore speak unambiguously about "the" stack completion of a sheaf of groupoids G — the stack completion is also called the associated stack.

Similar definitions can also be made for presheaves of groupoids. This means, effectively, that stacks can be identified with homotopy types of presheaves or sheaves of groupoids, within the respective injective model structures.

Example: Suppose that $G \times X \to X$ is an action of a sheaf of groups G on a sheaf X. Then the Borel construction $EG \times_G X$ is the nerve of a sheaf of groupoids $E_G X$. The stack completion

$$j : E_G X \to [X/G]$$

is called the *quotient stack*. Many stacks which arise in nature are quotient stacks. In particular, $G \cong E_G *$, so that $[*/G]$ is sectionwise equivalent to the stack associated to the group G.

A G-torsor over X is a G-equivariant map $P \to X$ where P is a G-torsor. A morphism of G-torsors over X is a commutative diagram

$$\begin{tikzcd} P \arrow[rr, "\theta"] \arrow[dr] & & P' \arrow[dl] \\ & X & \end{tikzcd}$$

of G-equivariant morphisms, where P and P' are G-torsors. Write $G - \mathbf{tors}/X$ for the corresponding groupoid.

If $P \to X$ is a G-torsor over X, then the induced map of Borel constructions

$$* \xleftarrow{\simeq} EG \times_G P \to EG \times_G X$$

is an object of the cocycle category

$$H(*, EG \times_G X),$$

and the assignment is functorial. Conversely, if the diagram

$$* \xleftarrow{\simeq} U \to EG \times_G X$$

is a cocycle, then the induced map

$$\tilde{\pi}_0 \operatorname{pb}(U) \to \tilde{\pi}_0 \operatorname{pb}(EG \times_G X) \xrightarrow[\cong]{\epsilon} X$$

is a G-torsor over X. The two functors are adjoint, and we have proved

Lemma 19.3. *There is a weak equivalence*

$$B(G - \mathbf{tors}/X) \simeq BH(*, EG \times_G X).$$

In particular, there is an induced bijection

$$\pi_0(G-\mathbf{tors}/X) \cong [*, EG \times_G X].$$

Lemma 19.3 was proved by a different method in [6]. There is a generalization of this result, having essentially the same proof, for the homotopy colimit $\underrightarrow{\mathrm{holim}}_G\, X$ of a diagram X on a sheaf of groupoids G. See [8].

Remark 19.4. A diagram

$$G \xleftarrow{p} H \xrightarrow{q} G'$$

of morphisms of sheaves of groupoids such that the induced maps

$$BG \xleftarrow{p_*} BH \xrightarrow{q_*} BG'$$

are local trivial fibrations is called a Morita morphism, and sheaves of groupoids G, K are said to be *Morita equivalent* if there is a string of Morita morphisms connecting them.

Clearly if G and K are Morita equivalent then they are weakly equivalent. Conversely, if $f : G \to H$ is a weak equivalence, take the cocycle

$$G \xrightarrow{(1,f)} G \times H$$

and find a factorization

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr, "{(1,f)}"'] & K \arrow[d, "{(p_1,p_2)}"] \\ & G \times H \end{tikzcd}$$

such that j is a weak equivalence and (p_1, p_2) is a fibration. Then the induced map

$$BK \xrightarrow{(p_{1*},p_{2*})} BG \times BH$$

is an injective hence local fibration, and the projection maps $BG \times BH \to BG$ and $BG \times BH \to BH$ are local fibrations since BG and BH are locally fibrant. It follows that the maps

$$G \xleftarrow{p_1} K \xrightarrow{p_2} H$$

define a Morita morphism.

It also follows that sheaves of groupoids G and H are weakly equivalent if and only if they are Morita equivalent. The same holds for presheaves of groupoids with the obvious expanded definition of Morita equivalence.

Example: A *gerbe* is traditionally defined to be a locally connected stack. Alternatively, a gerbe is a presheaf of groupoids G such that $\tilde{\pi}_0 BG = *$. Weak equivalence classes of gerbes are classified by path components of a cocycle category taking values in presheaves of 2-groupoids — see [8], [9]. Sets of such weak equivalence classes form the various flavours of Giraud's non-abelian H^2 functors [1].

Lemma 19.5. *Suppose that G is a fibrant sheaf of groupoids. Then the morphisms*

$$\begin{tikzcd} BG \arrow[r, "j_*"] \arrow[dr] & B(G - \mathbf{Tors}) \arrow[d, "\phi_*"] \\ & B\mathbb{H}(*, BG) \end{tikzcd}$$

are sectionwise weak equivalences of simplicial sheaves.

Proof. The morphism j is already fully faithful in all sections. Thus, it suffices to show that all maps

$$j_* : \pi_0 BG(U) \to \pi_0 B(G - \mathbf{Tors})(U)$$

is surjective for all $U \in \mathcal{C}$. For this, it suffices to assume that the site $\mathcal{C}$ has a terminal object t and show that the map

$$\pi_0 BG(t) \to \pi_0 B\mathbb{H}(*, BG)(t) = \pi_0 BH(*, BG)$$

is surjective.

In every cocycle

$$* \xleftarrow{s} U \xrightarrow{f} BG$$

the map s is a local weak equivalence, so there is a homotopy commutative diagram

$$\begin{tikzcd} U \arrow[d, "s"'] \arrow[dr, "f"] & \\ * \arrow[r, "x"'] & BG \end{tikzcd}$$

since BG is injective fibrant. This means that the cocycles (s, f), (s, xs) and $(1, x)$ are all in the same path component of $H(*, BG)$. $\square$

Lemma 19.6. *Suppose that G is a sheaf of groupoids. Then the maps $j : G \to G-$* **Tors** *and $\phi j : G \to \mathbb{H}(*, BG)$ are models for the stack completion, up to sectionwise weak equivalence.*

Proof. Suppose that $i : G \to H$ is a fibrant model for G. Then $i_* : BG \to BH$ is a local weak equivalence, so that the induced map

$$i_* : B\mathbb{H}(*, BG) \to B\mathbb{H}(*, BH)$$

is a sectionwise equivalence. Thus, it follows from Lemma 19.5 that $B\mathbb{H}(*, BG)$ is sectionwise equivalent to an injective fibrant object, namely BH, and therefore satisfies descent. $\square$

Remark 19.7. The presheaf of categories $\mathbb{H}(*, BG)$ is a fine example of what should be meant by a stack in categories: such an object should be a presheaf of categories D such that the nerve BD satisfies descent.

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