Lecture 09

20 The Verdier hypercovering theorem

Suppose that $\mathcal{C}$ is a small Grothendieck site. As before, write $s\,\mathrm{Pre}(\mathcal{C})$ for the category of simplicial presheaves on the site $\mathcal{C}$. The discussion that follows will be confined to simplicial presheaves. It has an exact analog for simplicial sheaves.

Let A be a fixed choice of simplicial presheaf. The slice category $A/s\,\mathrm{Pre}(\mathcal{C})$ has all morphisms $x : A \to X$ as objects, and all diagrams

$$\begin{array}{ccc} & A & \\ {}^{x}\swarrow & & \searrow^{y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

as morphisms.

The intuition, in applications, is that $x : A \to X$ is a "base point" of X (geometric points for the étale topology are good examples to keep in mind) even though A could be non-trivial homotopically.

Racall that the category $A/s\,\mathrm{Pre}(\mathcal{C})$ inherits a local model structure from $s\,\mathrm{Pre}(\mathcal{C})$, in that a morphism $f : x \to y$ as above is a local weak equivalence (respectively cofibration, fibration) if and only if the underlying map $f : X \to Y$ is a local weak equivalence (respectively cofibration, fibration) of simplicial presheaves.

Remark 20.1. 1) Not all objects of the slice category are cofibrant: the identity morphsm $1 : A \to A$ is initial, and so an object $x : A \to X$ is cofibrant if an only if the map x is a cofibration of simplicial presheaves.

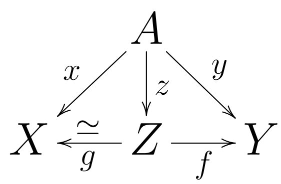
2) The unique map $A \to *$ taking values in the terminal simplicial presheaf $*$ is the terminal object of $A/s\operatorname{Pre}(\mathcal{C})$, and it follows that an object $x : A \to X$ is fibrant if and only if X is an injective fibrant simplicial presheaf.

Say that a map $f : x \to y$ in the slice category $A/s\operatorname{Pre}(\mathcal{C})$ is a *hypercover* if the underlying simplicial presheaf map $f : X \to Y$ is a hypercover (or a local trivial fibration). More generally, $f : x \to y$ is a *local fibration* if the map $f : X \to Y$ is a local fibration of simplicial presheaves. In particular, $x : A \to X$ is locally fibrant if X is locally fibrant.

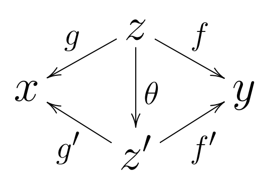
The theory of cocycle categories of [3] applies without change to the model category $A/s\,\mathrm{Pre}(\mathcal{C})$. Explicitly, a cocycle (g, f) from x to y is a diagram

$$x \xleftarrow[\simeq]{g} z \xrightarrow{f} y$$

in the slice category, or equivalently a commutative diagram of simplicial presheaf maps



for which the map g is a weak equivalence. These cocycles are the objects of a category $H(x, y)$ which has morphisms $\theta : (g, f) \to (g, f')$ given by the commutative diagrams



The category $H(x, y)$ is the category of cocycles from x to y.

The model structure on $A/s\,\mathrm{Pre}(\mathcal{C})$ is right proper, and weak equivalences in this structure are closed under finite products, because these properties both hold for the category of simplicial presheaves. Thus, Theorem 1 of [3] implies the following:

Lemma 20.2. *The function*

$$\phi : \pi_0 H(x,y) \to [x,y],$$

which is defined by $(g,f) \mapsto f \cdot g^{-1}$ *for a cocycle* (g,f) *in the slice category* $A/s\operatorname{Pre}(\mathcal{C})$*, is a bijection.*

Suppose that $f,g : x \to y$ are morphisms of the slice category $A/s\operatorname{Pre}(\mathcal{C})$. A (naive) pointed homotopy from f to g is a commutative diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ {\scriptstyle x\times\Delta^1}\downarrow & & \downarrow{\scriptstyle y} \\ X \times \Delta^1 & \xrightarrow[h]{} & Y \end{array}$$

such that h is a simplicial homotopy from f to g in the usual sense. Here, the projection map $pr : A \times \Delta^1 \to A$ onto A defines the constant homotopy on A.

Equivalently, such a pointed homotopy is a map

$$h : (X \times \Delta^1) \cup_{A \times \Delta^1} A \to Y.$$

In the pushout diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ {\scriptstyle x\times\Delta^1}\downarrow & & \downarrow \\ X \times \Delta^1 & \xrightarrow[pr_*]{} & (X \times \Delta^1) \cup_{A \times \Delta^1} A \end{array}$$

the map pr_* is a weak equivalence if the map $x : A \to X$ is a cofibration, or if x is a cofibrant object of $A/s\,\mathrm{Pre}(\mathcal{C})$. In that case, the pushout object is a cylinder for x in the slice category.

Every object $x : A \to X$ has a cofibrant model, meaning a diagram

$$\begin{array}{ccc} A & \xrightarrow{v} & Z \\ & {\scriptstyle x}\searrow & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

such that v is a cofibration and p is a weak equivalence. If the maps $f, g : x \to y$ are pointed homotopic and $p : v \to x$ is a cofibrant model of x, then the composites fp and gp are pointed homotopic and therefore represent the same map in the homotopy category since v is cofibrant. But then p is an isomorphism in that category, so that $f = g$ in the homotopy category.

The objects of the category $Triv/x$ are the pointed homotopy classes of maps $[p] : z \to x$ which are represented by hypercovers $p : z \to x$. The morphisms of this category are commutative triangles of pointed homotopy classes of maps in the obvious sense.

There is a contravariant set-valued functor which takes an object $[p] : z \to x$ of $Triv/x$ to the set $\pi(z,y)$ of pointed homotopy classes of maps between z and y. There is a function

$$\phi_h : \varinjlim_{[p]:z\to x} \pi(z,y) \to [x,y]$$

which is defined by sending the diagram of pointed homotopy classes

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y$$

to the morphism $f \cdot p^{-1}$ in the homotopy category.

The colimit

$$\varinjlim_{[p]:z\to x} \pi(z,y)$$

is the set of path components of a category $H_h(x,y)$ whose objects are the pictures of pointed homotopy classes

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y,$$

such that $p : z \to x$ is a hypercover, and whose morphisms are the commutative diagrams

$$\begin{array}{ccccc} & & z & & \\ & {\scriptstyle [p]} \swarrow & & \searrow {\scriptstyle [f]} & \\ x & & \downarrow {\scriptstyle [\theta]} & & y \\ & {\scriptstyle [p']} \nwarrow & & \nearrow {\scriptstyle [f']} & \\ & & z' & & \end{array} \tag{20.1}$$

in pointed homotopy classes of maps. The map ϕ_h therefore has the form

$$\phi_h : \pi_0 H_h(x,y) \to [x,y]$$

The following result is a generalized Verdier hypercovering theorem:

Theorem 20.3. *The function*

$$\phi_h : \pi_0 H_h(x,y) \to [x,y]$$

is a bijection if y is locally fibrant.

Remark 20.4. Theorem 20.3 specializes to a generalization of the standard form of the Verdier hypercovering theorem [1, p.425], [2] if $A = \emptyset$, for the unique map $x : \emptyset \to X$. The object X is not required to be locally fibrant.

There are multiple variations of the category $H_h(x,y)$:

1) Write $H'_h(x,y)$ for the category whose objects are pictures

$$x \xleftarrow{p} z \xrightarrow{[f]} y$$

where p is a hypercover and $[f]$ is a pointed homotopy class of maps. The morphisms of $H'_h(x,y)$

are diagrams

$$\begin{array}{ccccc} & {\scriptstyle p} \swarrow & z & \searrow {\scriptstyle [f]} & \\ x & & \downarrow {\scriptstyle [\theta]} & & y \\ & {\scriptstyle p'} \nwarrow & z' & \nearrow {\scriptstyle [f']} & \end{array} \qquad (20.2)$$

such that $[\theta]$ is a fibrewise pointed homotopy class of maps over x, and $[f'][\theta] = [f]$ as pointed homotopy classes. There is a functor

$$\omega : H'_h(x, y) \to H_h(x, y)$$

which is defined by the assignment $(p, [f]) \mapsto ([p], [f])$, and which sends the morphism (20.2) to the morphism (20.1).

2) Write $H''_h(x, y)$ for the category whose objects are the pictures

$$x \xleftarrow{p} z \xrightarrow{[f]} y$$

where p is a hypercover and $[f]$ is a pointed simplicial homotopy class of maps (as before). The morphisms of $H''_h(X, Z)$ are commutative diagrams

$$\begin{array}{ccc} & {\scriptstyle p} \swarrow & z \\ x & & \downarrow {\scriptstyle \theta} \\ & {\scriptstyle p'} \nwarrow & z' \end{array}$$

such that $[f' \cdot \theta] = [f]$. There is a canonical functor

$$H''_h(x, y) \xrightarrow{\omega'} H'_h(x, y)$$

which is the identity on objects, and takes morphisms θ to their associated fibrewise pointed homotopy classes.

3) Let $H_{hyp}(x,y)$ be the full subcategory of $H(x,y)$ whose objects are the cocycles

$$x \xleftarrow{p} z \xrightarrow{f} y$$

with p a hypercover. There is a functor

$$\omega'' : H_{hyp}(x,y) \to H''_h(x,y)$$

which takes a cocycle (p,f) to the object $(p,[f])$.

Lemma 20.5. *Suppose that y is locally fibrant. Then the inclusion functor $i : H_{hyp}(x,y) \subset H(x,y)$ is a homotopy equivalence.*

Proof. Objects of the cocycle category $H(x,y)$ can be identified with maps $(g,f) : z \to x \times y$ such that the morphism g is a weak equivalence, and morphisms of $H(x,y)$ are commutative triangles in the obvious way. Maps of the form (g,f) have functorial factorizations

$$\begin{array}{ccc} z & \xrightarrow{j} & v \\ & {\scriptstyle (g,f)} \searrow & \downarrow {\scriptstyle (p,g')} \\ & & x \times y \end{array} \tag{20.3}$$

such that j is a pointwise trivial cofibration and (p,g') is a pointwise Kan fibration. It follows that

(p, g') is a local fibration and the map p, or rather the composite

$$z \xrightarrow{(p,g')} x \times y \xrightarrow{pr} x,$$

is a local weak equivalence. The projection map pr is a local fibration since y is locally fibrant, so the map p is also a local fibration, and hence a hypercover.

It follows that the assignment $(u, g) \mapsto (p, g')$ defines a functor

$$\psi' : H(x, y) \to H_h(x, y).$$

The weak equivalences j of the diagram (20.3) define homotopies $p' \cdot i \simeq 1$ and $i \cdot \psi' \simeq 1$. □

Proof of Theorem 20.3. The composite

$$H(x, y) \xrightarrow{\psi'} H_{hyp}(x, y) \xrightarrow{\omega''} H''_h(x, y) \xrightarrow{\omega'} H'_h(x, y) \xrightarrow{\omega} H_h(x, y)$$

is the functor ψ, and the composite

$$\begin{aligned} \pi_0 H(x, y) \xrightarrow{\psi'_*} \pi_0 H_{hyp}(x, y) \xrightarrow{\omega''_*} \pi_0 H''_h(x, y) \xrightarrow{\omega'_*} \pi_0 H'_h(x, y) \\ \xrightarrow{\omega_*} \pi_0 H_h(x, y) \xrightarrow{\phi_h} [x, y] \end{aligned} \tag{20.4}$$

is the bijection ϕ of Lemma 20.2. The function ψ'_* is a bijection by Lemma 20.5, and the functions ω''_*, ω'_* and ω_* are surjective, as is the function ϕ_h.

The functions which make up the string (20.4) are therefore all bijections. □

The following is a corollary of the proof of Theorem 20.3 which deserves independent mention:

Corollary 20.6. *Suppose that the object* $y : A \to Y$ *of* $A/s\,\mathrm{Pre}(\mathcal{C})$ *is locally fibrant. Then the induced functions*

$$\pi_0 H_{hyp}(x,y) \xrightarrow{\omega''_*} \pi_0 H''_h(x,y) \xrightarrow{\omega'_*} \pi_0 H'_h(x,y) \xrightarrow{\omega_*} \pi_0 H_h(x,y)$$

are bijections, and all of these sets are isomorphic to the set $[x,y]$ *of morphisms* $x \to y$ *in the homotopy category* $\mathrm{Ho}(s/\,\mathrm{Pre}(\mathcal{C}))$*.*

The bijections of the path component objects in the statement of Corollary 20.6 with the set $[x,y]$ all represent specific variants of the Verdier hypercovering theorem.

References

[1] Kenneth S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1974.

[2] J. F. Jardine. Simplicial objects in a Grothendieck topos. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, pages 193–239. Amer. Math. Soc., Providence, RI, 1986.

[3] J. F. Jardine. Cocycle categories. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 185–218. Springer, Berlin, 2009.

[4] J.F. Jardine. *Local Homotopy Theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.