Lecture 10

21 Localization for simplicial presheaves

Suppose that $\mathcal{C}$ is a small Grothendieck site, and that S is a set of cofibrations $A \to B$ in the category $s\operatorname{Pre}(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$.

I'm going to assume throughout this section that I is a simplicial presheaf on $\mathcal{C}$ with two disjoint global sections $0, 1 : * \to I$. The object I will be called an *interval*, whether it looks like one or not.

The examples of intervals that we are most likely to care about include the following:

1) the simplicial set Δ^1 with the two vertices $0, 1 : * \to \Delta^1$,
2) $B\pi(\Delta^1)$ with the two vertices $0, 1 : * \to \pi(\Delta^1)$ in the fundamental groupoid $\pi(\Delta^1)$ of Δ^1,
3) the affine line $\mathbb{A}^1$ over a scheme S with the rational points $0, 1 : S \to \mathbb{A}^1$.

The basic idea behind the flavour of localization theory which will be presented here, is that one wants to construct, in a minimal way, a homotopy theory on simplicial presheaves for which the cofibrations are the monomorphisms, all of the maps in the set S become weak equivalences, and the interval object I describes homotopies.

I sometimes write

$$\square^n = I^{\times n}.$$

There are face inclusions

$$d^{i,\epsilon} : \square^{n-1} \to \square^n, \ 1 \leq i \leq n, \ \epsilon = 0, 1,$$

with

$$d^{i,\epsilon}(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{i-1}, \epsilon, x_i, \dots, x_{n-1}).$$

Then there are subobjects $\partial\square^n$ and $\sqcap^n_{i,\epsilon}$ of $\square^n$ which are defined, respectively, by

$$\partial\square^n = \cup_{i,\epsilon}\, d^{i,\epsilon}(\square^{n-1}),$$

and

$$\sqcap^n_{i,\epsilon} = \cup_{(j,\gamma)\neq(i,\epsilon)}\, d^{j,\gamma}(\square^{n-1}).$$

The interval I is used to define homotopies. A *naive homotopy* between maps $f, g : X \to Y$ is a commutative diagram

$$\begin{tikzcd} X \arrow[d, "0"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "h"] & Y \\ X \arrow[u, "1"] \arrow[ur, "g"'] & \end{tikzcd}$$

Naive homotopies generate an equivalence relation: write

$$\pi(X, Y) = \pi_I(X, Y)$$

for the set of naive homotopy classes of maps $X \to Y$.

The class of *anodyne cofibrations* (or anodyne extensions) is the saturation of the set of inclusions $\Lambda(S)$ specified by

$$(C \times \square^n) \cup (D \times \sqcap^n_{(i,\epsilon)}) \subset D \times \square^n \tag{21.1}$$

where $C \to D$ is a member of the set of generating cofibrations for $s\,\mathrm{Pre}(\mathcal{C})$, and

$$(A \times \square^n) \cup (B \times \partial\square^n) \subset B \times \square^n \tag{21.2}$$

with $A \to B$ in the set S.

An *injective morphism* is a simplicial presheaf map $p : X \to Y$ which has the right lifting property with respect to all anodyne extensions, and a simplicial presheaf X is injective if the map $X \to *$ is an injective morphism.

A *weak equivalence* is a map $f : X \to Y$ which induces a bijection $\pi(Y, Z) \to \pi(X, Z)$ for all injective Z. A *cofibration* is just a monomorphism, and a *fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

It is an exercise to show that a map $f : Z \to W$ is a weak equivalence if and only if it is a naive homotopy equivalence. This means that there is a map $g : W \to Z$ and naive homotopies $f \cdot g \simeq 1_W$ and $g \cdot f \simeq 1_Z$.

Lemma 21.1. *1) Suppose that $C \to D$ is an anodyne cofibration. Then the induced map*

$$(C \times \square^1) \cup (D \times \partial\square^1) \subset D \times \square^1 \tag{21.3}$$

is anodyne.

2) All anodyne cofibrations are weak equivalences.

Proof. Show that that if $C \to D$ is in $\Lambda(S)$, then the induced map (21.3) is in $\Lambda(S)$. Then the proof of statement 1) is finished with a colimit argument.

Suppose that $i : C \to D$ is an anodyne cofibration and that Z is an injective argument. Then the lifting exists in any diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & Z \\ D \arrow[ur, dotted] & \end{tikzcd}$$

so that the map

$$i^* : \pi(D, Z) \to \pi(C, Z)$$

is surjective. If $f, g : D \to Z$ are morphisms such that there is a homotopy $h : C \times I \to Z$ between fi and gi, then the lifting exists in the diagram

$$\begin{tikzcd} (C \times \square^1) \cup (D \times \partial\square^1) \arrow[r, "{(h,(f,g))}"] \arrow[d] & Z \\ D \times \square^1 \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

(by part 1)) and the map H is a homotopy between f and g. It follows that the function

$$i^* : \pi(D, Z) \to \pi(C, Z)$$

is injective. $\square$

We shall sketch the proof of the following:

Theorem 21.2 (Cisinski)**.** *With the definitions given above, the simplicial presheaf category* $s\,\mathrm{Pre}(\mathcal{C})$ *has the structure of a left proper cubical model category.*

The cubical model structure involves the cubical set (the *cubical function complex*) $\mathbf{hom}(X, Y)$ whose n-cells are the maps $X \times \square^n \to Y$. This construction is supposed to satisfy a cubical version of Quillen's simplicial model axiom **SM7**. This is, however, an easy consequence of the proof of the rest of the Theorem.

There is a properness assertion as well:

Theorem 21.3. *Suppose that all cofibrations in the set S pull back to weak equivalences along all fibrations $p : X \to Y$ with Y fibrant. Then the model structure of Theorem 21.2 on $s\,\mathrm{Pre}(\mathcal{C})$ is proper.*

The condition in the statement of Theorem 21.3 means that, in every diagram

$$\begin{tikzcd} A \times_Y X \arrow[r, "i_*"] \arrow[d] & B \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "i"'] & B \arrow[r] & Y \end{tikzcd}$$

with p a fibration such that Y is fibrant, if i is a member of S then i_* is a weak equivalence.

Theorems 21.2 and 21.3 are special cases of more general results, which can be found in [7] (also [6]). In particular, [7] is where you should look for a proof of Theorem 21.3. Theorem 21.2 was originally proved by Cisinski [1], although he did not express it as it appears here. The main ideas of the proof are due to Cisinski.

1) Cardinality tricks

Suppose that T is some set of cofibrations of $\mathcal{A}$-sets, and choose a regular cardinal α such that $\alpha > |T|$ and that $\alpha > |D|$ for all $C \to D$ in T.

Suppose that $\lambda > 2^\alpha$ is regular.

Every $f : X \to Y$ has a functorial system of factorizations

$$
\begin{tikzcd}
X \arrow[r, "i_s"] \arrow[dr, "f"'] & E_s(f) \arrow[d, "f_s"] \\
& Y
\end{tikzcd}
$$

for $s < \lambda$ defined by the lifting property for maps in T, and which form the stages of a transfinite small object argument.

Specifically, given the factorization $f = f_s i_s$ form the pushout diagram

$$
\begin{tikzcd}
\bigsqcup_{\mathcal{D}} C \arrow[r] \arrow[d] & E_s(f) \arrow[d] \\
\bigsqcup_{\mathcal{D}} D \arrow[r] & E_{s+1}(f)
\end{tikzcd}
$$

where $\mathcal{D}$ runs through all diagrams

$$
\begin{tikzcd}
C \arrow[r] \arrow[d, "i"'] & E_s(f) \arrow[d] \\
D \arrow[r] & Y
\end{tikzcd}
$$

with i in T. Then $f_{s+1} : E_{s+1}(f) \to Y$ is the obvious induced map. Set $E_t(f) = \varinjlim_{s<t} E_s(f)$ at limit ordinals $t < \lambda$.

Then there is a functorial factorization

$$\begin{tikzcd} X \arrow[r, "i_\lambda"] \arrow[dr, "f"'] & E_\lambda(f) \arrow[d, "f_\lambda"] \\ & Y \end{tikzcd}$$

with $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$. Also f_λ has the right lifting property with respect to all $C \to D$ in T, and i_λ is in the saturation of T.

Write $\mathcal{L}(X) = E_\lambda(X \to *)$.

Lemma 21.4. *1) Suppose that $t \mapsto X_t$ is a diagram of simplicial presheaves, indexed by $\omega > 2^\alpha$. Then the map*

$$\varinjlim_{t<\omega} \mathcal{L}(X_t) \to \mathcal{L}(\varinjlim_{t<\omega} X_t)$$

is an isomorphism.

2) The functor $X \mapsto \mathcal{L}(X)$ preserves cofibrations.

3) Suppose that γ is a cardinal with $\gamma > \alpha$, and let $\mathcal{F}_\gamma(X) =$ the subobjects of X having cardinality less than γ. Then the map

$$\varinjlim_{Y \in \mathcal{F}_\gamma(X)} \mathcal{L}(Y) \to \mathcal{L}(X)$$

is an isomorphism.

4) If $|X| < 2^\mu$ where $\mu \geq \lambda$ then $|\mathcal{L}(X)| < 2^\mu$.

5) Suppose that U, V are subobjects of X. Then the natural map

$$\mathcal{L}(U \cap V) \to \mathcal{L}(U) \cap \mathcal{L}(V)$$

is an isomorphism.

Proof. It suffices to prove all statements with $\mathcal{L}(X)$ replaced by $E_1(X)$. There is a pushout diagram

$$\begin{tikzcd} \bigsqcup_T (C \times \hom(C,X)) \arrow[r] \arrow[d] & X \arrow[d] \\ \bigsqcup_T (D \times \hom(C,X)) \arrow[r] & E_1 X \end{tikzcd}$$

Then, in sections,

$$E_1 X = \bigsqcup_T ((D(a) - C(a)) \times \hom(C,X)) \sqcup X(a).$$

so 5) follows. The remaining statements are exercises. □

Corollary 21.5. *Every simplicial presheaf map $f : X \to Y$ has a functorial factorization*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where j is anodyne and p is injective.

Suppose that $\alpha > |\Lambda(S)|$ and that $\alpha > |D|$ for all $C \to D$ in $\Lambda(S)$. Suppose that $\lambda > 2^{\alpha}$

Here is the bounded cofibration condition:

Lemma 21.6. *Suppose given a diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

of cofibrations such that i is a weak equivalence and $|A| < 2^{\lambda}$. Then there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < 2^{\lambda}$ and $B \cap X \to B$ is an equivalence.

Proof. The proof is due to Cisinski. It is innovative in the sense that it uses nothing but naive homotopy.

The map $i_* : \mathcal{L}X \to \mathcal{L}Y$ is a cofibration (by the previous lemma) and is a naive homotopy equivalence of injective objects. There is a map $\sigma : \mathcal{L}Y \to \mathcal{L}X$ such that $\sigma \cdot i_* \simeq 1$ via a naive homotopy $h : \mathcal{L}X \times \square^1 \to \mathcal{L}X$. Form the diagram

$$\begin{tikzcd} (\mathcal{L}Y \times \square^0) \cup (\mathcal{L}X \times \square^1) \arrow[r, "{(\sigma,h)}"] \arrow[d] & \mathcal{L}X \\ \mathcal{L}Y \times \square^1 \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

The other end of the homotopy H gives a map σ'

such that $\sigma' \cdot i_* = 1$, and $i_*\sigma' \simeq i_*\sigma \simeq 1$. We can therefore assume that $\sigma \cdot i_* = 1$.

Suppose that $A_s \subset Y$ and $|A_s| < 2^\lambda$. Then $|\mathcal{L}A_s \times \square^1| < 2^\lambda$. Also, there is a 2^λ-bounded subobject A_{s+1} such that $A_s \subset A_{s+1}$ and there is a diagram

$$\begin{array}{ccc} \mathcal{L}A_s \times \square^1 & \rightarrow & \mathcal{L}A_{s+1} \\ \downarrow & & \downarrow \\ \mathcal{L}Y \times \square^1 & \xrightarrow[K]{} & \mathcal{L}Y \end{array}$$

where $K : i_*\sigma \simeq 1$.

This is the successor ordinal step in the construction of a system $s \mapsto A_s$ with $s < \lambda$ (recall that $\lambda > 2^\alpha$) and $A = A_0$. Let $B = \varinjlim_s A_s$. Then, by construction, B is 2^λ-bounded and the restriction of the homotopy K to $\mathcal{L}B \times \square^1$ factors through the inclusion $j_* : \mathcal{L}B \to \mathcal{L}Y$.
There is a pullback

$$\begin{array}{ccc} \mathcal{L}(B \cap X) & \xrightarrow{\tilde{j}} & \mathcal{L}X \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow{\scriptstyle i_*} \\ \mathcal{L}B & \xrightarrow[j_*]{} & \mathcal{L}Y \end{array}$$

and $i_*\sigma(\mathcal{L}B) \subset \mathcal{L}B$. It follows that there is a map $\sigma' : \mathcal{L}B \to \mathcal{L}(B \cap X)$ such that $\sigma' \cdot \tilde{i} = 1$. K restricts to a homotopy $\mathcal{L}B \times \square^1 \to \mathcal{L}B$ (by construction), and this is a homotopy $\tilde{i}\sigma' \simeq 1$. $\square$

2) Trivial cofibrations are preserved by pushout

Note first that anodyne extensions are closed under pushout.

Lemma 21.7. *Suppose given a diagram*

$$\begin{tikzcd} C \arrow[r, "f,g"] \arrow[d, "i"'] & E \\ D & \end{tikzcd}$$

where i is a cofibration, and suppose that there is a naive homotopy $h : C \times \square^1 \to E$ from f to g. Then $g_* : D \to D \cup_g E$ is a weak equivalence if and only if $f_* : D \to D \cup_f E$ is a weak equivalence.

Proof. There are pushout diagrams

$$\begin{tikzcd} C \arrow[r, "d_0"] \arrow[d, "i"'] & C \times \square^1 \arrow[rr, "h"] \arrow[d, "i_*"] & & E \arrow[d, "i_*"] \\ D \arrow[r, "d_{0*}"'] & D \cup_C (C \times \square^1) \arrow[rr, "h'"'] \arrow[d, "j"'] & & D \cup_f E \arrow[d, "j_*"] \\ & D \times \square^1 \arrow[rr, "h_*"'] & & (D \times \square^1) \cup_h E \end{tikzcd}$$

where the top composite is f. The maps d_{0*}, j and j_* are anodyne cofibrations. Thus $f_* = h' \cdot d_{0*}$ is a weak equivalence if and only if h' is a weak equivalence, and h' is a weak equivalence if and only if h_* is a weak equivalence. Thus, f_* is a weak equivalence if and only if h_* is a weak equivalence. Similarly, g_* is a weak equivalence if and only if h_* is a weak equivalence. $\square$

Lemma 21.8. *Suppose that $i : C \to D$ is a trivial cofibration. Then the cofibration*

$$(C \times \square^1) \cup (D \times \partial\square^1) \to D \times \square^1$$

is a weak equivalence.

Proof. The diagram

$$\begin{tikzcd} C \times \partial\square^1 \arrow[r] \arrow[d] & D \times \partial\square^1 \arrow[r] \arrow[d] & \mathcal{L}D \times \partial\square^1 \arrow[d] \\ C \times \square^1 \arrow[r] & D \times \square^1 \arrow[r] & \mathcal{L}D \times \square^1 \end{tikzcd}$$

induces a diagram

$$\begin{tikzcd} (C \times \square^1) \cup (D \times \partial\square^1) \arrow[r] \arrow[d] & (C \times \square^1) \cup (\mathcal{L}D \times \partial\square^1) \arrow[d] \\ D \times \square^1 \arrow[r] & \mathcal{L}D \times \square^1 \end{tikzcd}$$

in which the horizontal maps are anodyne extensions, and hence weak equivalences.

There is a factorization

$$\begin{tikzcd} C \arrow[r, "i'"] \arrow[dr, "i"'] & D' \arrow[d, "p"] \\ & D \end{tikzcd}$$

where i' is anodyne and p is both injective and a weak equivalence. In the induced diagram

$$\begin{tikzcd} (C \times \square^1) \cup (\mathcal{L}D' \times \partial\square^1) \arrow[r] \arrow[d] & (C \times \square^1) \cup (\mathcal{L}D \times \partial\square^1) \arrow[d] \\ \mathcal{L}D' \times \square^1 \arrow[r] & \mathcal{L}D \times \square^1 \end{tikzcd}$$

the top horizontal map is induced by the homotopy equivalence

$$\mathcal{L}D' \times \partial\square^1 \to \mathcal{L}D \times \partial\square^1,$$

and is therefore an equivalence by Lemma 21.7. The bottom horizontal map is also a homotopy equivalence. The left hand vertical map is an equivalence by comparison with the map

$$(C \times \square^1) \cup (D' \times \partial\square^1) \to D' \times \square^1$$

which is an anodyne extension. $\square$

Lemma 21.9. *The class of trivial cofibrations is closed under pushout.*

Proof. If $j : C \to D$ is a cofibration and a weak equivalence, then every map $\alpha : C \to Z$ with Z injective extends to a map $D \to Z$.

In effect, there is a homotopy $h : C \times \square^1 \to Z$ from α to a map $\beta \cdot j$ for some map $\beta : D \to Z$, and then the homotopy extends:

$$\begin{tikzcd} (C \times \square^1) \cup (D \times \{1\}) \arrow[r, "(h,\beta)"] \arrow[d] & Z \\ D \times \square^1 \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

Note that the vertical map is an anodyne extension.

Now suppose given a pushout diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & C' \arrow[d, "j'"] \\ D \arrow[r] & D' \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} (C \times \square^1) \cup (D \times \partial\square^1) \arrow[r] \arrow[d] & (C' \times \square^1) \cup (D' \times \partial\square^1) \arrow[d] \\ D \times \square^1 \arrow[r] & D' \times \square^1 \end{tikzcd}$$

is a pushout. The left vertical map is a trivial cofibration by Lemma 21.8, and therefore has the left lifting property with respect to the map $Z \to *$. Thus, if two maps $f, g : D' \to Z$ restrict to homotopic maps on C', then $f \simeq g$. $\square$

3) Many injective maps are fibrations

Lemma 21.10. *Suppose that the map $p : X \to Y$ is injective and that Y is injective. Then p is a fibration.*

Proof. Suppose given a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array} \tag{21.4}$$

where i is a trivial cofibration. Then there is a map $\theta : B \to X$ such that $\theta \cdot i = \alpha$ since X is injective.

The constant homotopy $A \times \square^1 \xrightarrow{pr} A \xrightarrow{\alpha} X$ extends to a homotopy $h : B \times \square^1 \to Y$ as in the diagram

$$\begin{tikzcd} (A \times \square^1) \cup (B \times \partial\square^1) \arrow[r, "{(p\alpha pr_A,(\beta,p\theta))}"] \arrow[d] & Y \\ B \times \square^1 \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

since the vertical map is a trivial cofibration (Lemma 21.8) and Y is injective. It follows that there is a homotopy

$$\begin{array}{ccc} A \times \square^1 & \xrightarrow{\alpha pr_A} & X \\ {\scriptstyle i\times i}\downarrow & & \downarrow{\scriptstyle p} \\ B \times \square^1 & \xrightarrow[h]{} & Y \end{array}$$

from the original diagram to a diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "p\theta"'] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

Find the indicated lifting in the diagram

$$\begin{tikzcd}
(A \times \square^1) \cup B \arrow[r, "(\alpha pr_A, \theta)"] \arrow[d] & X \arrow[d, "p"] \\
B \times \square^1 \arrow[r, "h"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$

to show that the required lifting exists for the original diagram (21.4). $\square$

Corollary 21.11. *Every injective object is fibrant.*

4) Final approach

Lemma 21.12 (CM4). *Suppose that $p : X \to Y$ is a fibration and a weak equivalence. Then p has the right lifting property with respect to all cofibrations.*

Proof. Suppose first that Y is injective. Then p is a naive homotopy equivalence, and has a section $\sigma : Y \to X$ (exercise).

The map σ is a trivial cofibration so the lift exists in the diagram

$$\begin{tikzcd}
(Y \times \square^1) \cup (X \times \partial\square^1) \arrow[rr, "(\sigma \cdot pr, (1_X, \sigma \cdot p))"] \arrow[d] & & X \arrow[d, "p"] \\
X \times \square^1 \arrow[r, "p \times 1"'] \arrow[urr, dotted, "H"] & Y \times \square^1 \arrow[r, "pr"'] & Y
\end{tikzcd}$$

since the left vertical map is a weak equivalence by Lemma 21.8. It follows that the identity diagram on $p : X \to Y$ is naively homotopic to the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma\cdot p"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ Y \arrow[r, "1"'] \arrow[ur, dotted, "\sigma"] & Y \end{tikzcd}$$

Thus, any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

is naively homotopic to a diagram which admits a lifting. It follows that p has the right lifting property with respect to all cofibrations.

If Y is not injective, form the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & Z \arrow[d, "q"] \\ Y \arrow[r, "j_Y"'] & \mathcal{L}(Y) \end{tikzcd}$$

where j is an anodyne cofibration, q is injective, and j_Y is an injective model for Y. Then q is a fibration by Lemma 21.9 and is a weak equivalence, so that q has the right lifting property with respect to all cofibrations, by the previous paragraph.

Factorize the map $X \to Y \times_{\mathcal{L}(Y)} Z$ as

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & W \arrow[d, "\pi"] \\ & Y \times_{\mathcal{L}(Y)} Z \end{tikzcd}$$

where π has the right lifting property with respect to all cofibrations and i is a cofibration. Write q_* for the induced map $Y \times_{\mathcal{L}(Y)} Z \to Y$. Then the composite $q_*\pi$ has the right lifting property with respect to all cofibrations and is therefore a homotopy equivalence. The cofibration i is also a weak equivalence, and it follows that the lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q_*\pi"'] \arrow[ur] & Y \end{tikzcd}$$

so that p is a retract of a map which has the right lifting property with respect to all cofibrations. $\square$

Corollary 21.13. *A map $p : X \to Y$ is a fibration and a weak equivalence if and only if it has the right lifting property with respect to all cofibrations.*

Proof of Theorem 21.2. The cofibration/trivial fibration factorization statement of **CM5** is also a consequence of Corollary 21.13: every map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & W \arrow[ur, "p"'] & \end{tikzcd}$$

where i is a cofibration and p is a fibration and a weak equivalence.

The trivial cofibration/fibration factorization statement follows from the bounded cofibration condition: every $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & Z \arrow[ur, "q"'] & \end{tikzcd}$$

where j is a cofibration and a weak equivalence and q is a fibration. In order to conclude that j is a weak equivalence, we need to know that trivial cofibrations are closed under pushout, but this is Lemma 21.9.

All simplicial presheaves are cofibrant for the present model structure. Left properness therefore follows from general nonsense about categories of cofibrant objects — see [3, II.8.5]. $\square$

Examples:

1) Homotopy theory of simplicial presheaves

Suppose that S is a generating set of trivial cofibrations $A \to B$ for the inductive model structure on $s\,\mathrm{Pre}(\mathcal{C})$, and that $I = \Delta^1$ is the standard interval.

An injective model $j : X \to \mathcal{L}(X)$ is an injective fibrant model since all anodyne extensions are trivial cofibrations for the injective structure and all injective objects are injective fibrant. Thus, every weak equivalence (for the "new" model structure) is a local weak equivalence. If $f : X \to Y$ is a local weak equivalence, then $\mathcal{L}(X) \to \mathcal{L}(Y)$ is a local weak equivalence between injective fibrant models, and is therefore a (standard) homotopy equivalence; it follows that f is a weak equivalence in the "new" sense.

2) Motivic homotopy theory

Suppose that S is a scheme of finite dimension (typically a field), and let $(Sm|_S)_{Nis}$ be the category of smooth schemes of finite type over S, equipped with the Nisnevich topology. A covering family for the Nisnevich topology is an étale covering family $\phi_i : V_i \to U$ in the category of

S-schemes such that every map $\mathrm{Sp}(K) \to U$ lifts to some V_i, for all fields K. Nisnevich originally called this topology the "completely decomposed topology" or "cd-topology" [9], because of the way it behaves over fields — see [4].

The motivic model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ can be constructed in two ways:

a) Let S consist of the generating set of the trivial cofibrations for the injective model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$, plus the 0-section $* \to \mathbb{A}^1$, and let $I = \Delta^1$.

b) Let S be the generating set of trivial cofibrations for the injective model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ and let $I = \mathbb{A}^1$ with the global sections $0, 1 : * \to \mathbb{A}^1$.

It's an exercise to show that the two model structures coincide: show that every anodyne cofibration of one structure is a trivial cofibration of the other, and so the two structures have same injective objects. It follows that the two classes of weak equivalences coincide.

The motivic model structure is called the $\mathbb{A}^1$-model structure in [8]. Strictly speaking, the Morel-Voevodsky model structure is on the category of simplicial sheaves on the smooth Nisnevich site, but the model structures for simplicial sheaves and simplicial presheaves are Quillen equivalent by the usual argument [5]. There are many other models for motivic homotopy theory, including model structures on presheaves and sheaves (not simplicial!) on the smooth Nisnevich site [5], and all the models arising from test categories [6].

3) Localized model structures

Suppose that $f : A \to B$ is a cofibration of simplicial presheaves on a site $\mathcal{C}$. Let S consist of the generating set of trivial cofibrations for the injective model structure on $s\,\mathrm{Pre}(\mathcal{C})$, plus the cofibration f. Let $I = \Delta^1$. The resulting model structure is the f-local model structure on $s\,\mathrm{Pre}(\mathcal{C})$. The motivic model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ is a special case of this construction, as are all of the standard f-local theories for simplicial sets.

4) Quasi-categories

The quasi-category model structure on the category $s\mathbf{Set}$ of simplicial sets is the model structure given by the theorem for the set S of *inner anodyne extensions*

$$\Lambda^n_k \subset \Delta^n,\ 0 < k < n,$$

and the interval $I = B\pi(\Delta^1)$.

Theorem 21.14. *Suppose that $f : * \to A$ is a global section of a simplicial presheaf A on a small site $\mathcal{C}$. Then the f-local model structure on $s\operatorname{Pre}(\mathcal{C})$ is proper.*

Proof. We only need to verify right properness.

According to Theorem 21.3, and since the standard injective model structure on $s\operatorname{Pre}(\mathcal{C})$ is proper, it is enough to show that the map f_* is a weak equivalence in all pullback diagrams

$$\begin{tikzcd} F \arrow[r, "f_*"] \arrow[d] & A \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ * \arrow[r, "f"'] & A \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

such that p is a fibration and Y is fibrant.

The map $t : A \to *$ is a weak equivalence and Y is fibrant, so there is a map $v : * \to Y$ and a homotopy h making the diagram

$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d, "0"'] & A \arrow[d, "0"] \arrow[drr, "\alpha"] & & \\ \Delta^1 \arrow[r] & A \times \Delta^1 \arrow[rr, "h"] & & Y \\ * \arrow[r, "f"'] \arrow[u, "1"] & A \arrow[r, "t"'] \arrow[u, "1"'] & * \arrow[ur, "v"'] & \end{tikzcd}$$

commute. All instances of the maps 0 and 1 pull back to weak equivalences along p since the standard injective model structure is proper. It therefore suffices to show that the map f_* in the pullback diagram

$$\begin{tikzcd} F_v \arrow[r, "f_*"] \arrow[d] & A \times F_v \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ * \arrow[r, "f"'] & A \arrow[r, "vt"'] & Y \end{tikzcd}$$

is a weak equivalence, where F_v is the fibre of p over v, but this is obvious since f_* is anodyne. $\square$

Corollary 21.15. *The motivic model structure on the category* $s\operatorname{Pre}(Sm|_S)_{Nis}$ *of simplicial presheaves on the smooth Nisnevich site is proper.*

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