Lecture 11

22 Presheaves of spectra

Let $\mathcal{C}$ be a small site, as usual, and write $s\,\mathrm{Pre}(\mathcal{C})_*$ for the category of pointed simplicial presheaves $* \to X$, with base point preserving maps

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Recall that this category has a proper closed simplicial model structure, for which a map f as above is a local weak equivalence (respectively injective fibration, cofibration) if and only if the underlying map $f : X \to Y$ of simplicial presheaves is a local weak equivalence (respectively injective fibration, cofibration). The function complex $\mathbf{hom}(X, Y)$ is defined in simplicial degree n by

$$\mathbf{hom}(X, Y)_n = \{X \wedge \Delta^n_+ \to Y\}.$$

Here $\Delta^n_+ = \Delta^n \sqcup \{*\}$, with the disjoint point as base point.

Warning: A map f of pointed simplicial presheaves is a local weak equivalence if and only if it induces an isomorphism on sheaves of path components, and induces isomorphisms

$$\tilde{\pi}_n(X|_U, x) \to \tilde{\pi}_n(Y|_U, f(x))$$

for all $U \in \mathcal{C}$ and all (local) choices of base points $x \in X(U)$. It is *not* sufficient to check that the obvious maps $\tilde{\pi}_n(X, *) \to \tilde{\pi}_n(Y, *)$ (based at the canonical point $* \to X$) are isomorphisms.

A *presheaf of spectra* X consists of pointed simplicial presheaves X^n, $n \geq 0$ together with bonding maps

$$\sigma : S^1 \wedge X^n \to X^{n+1},\ n \geq 0.$$

Here, S^1 is identified with the constant pointed simplicial presheaf $U \mapsto S^1$. A map $f : X \to Y$ of presheaves of spectra consists of pointed simplicial presheaf maps $f : X^n \to Y^n$, $n \geq 0$, which respect structure in the obvious sense. Write $\mathrm{Spt}(\mathcal{C})$ for the category of presheaves of spectra on $\mathcal{C}$.

Generally speaking, any functorial construction for spectra also applies to presheaves of spectra, and we will see that, in general outline, the stable homotopy theory of presheaves of spectra is an analogue of ordinary stable homotopy theory.

Note that the ordinary category of spectra Spt (in pointed simplicial sets, following [2]) is the category of presheaves of spectra on the one-object, one-morphism category, so that general results about presheaves of spectra apply to spectra. Also, if I is a small category, then the category of I-diagrams $X : I \to \mathrm{Spt}$ is a category of presheaves of spectra on I^{op}, where I^{op} has the trivial topology: this means that results about presheaves of spectra apply to all categories of small diagrams of spectra, and to the category of spectra in particular.

Some examples:

1) Any spectrum A determines an associated constant presheaf of spectra $\Gamma^* A$ on $\mathcal{C}$, where

$$\Gamma^* A(U) = A,$$

and every morphism $\phi : V \to U$ induces the identity morphism $A \to A$. We shall often write $A = \Gamma^* A$ when there is no possibility of confusion. The *sphere spectrum* S in $\mathrm{Spt}(\mathcal{C})$ is the constant object $\Gamma^* S$ associated to the ordinary sphere spectrum.

Recall that the sphere spectrum S consists of the pointed simplicial sets

$$S^0, S^1, S^2, \dots,$$

where S^0 is the two-element set $\partial\Delta^1 = \{0, 1\}$ pointed by 0, $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle, and

$$S^n = S^1 \wedge \cdots \wedge S^1$$

is the n-fold smash power of copies of S^1 for $n \geq 1$.

2) The functor $A \mapsto \Gamma^* A$ is left adjoint to the global sections functor $\Gamma_* : \mathrm{Spt}(\mathcal{C}) \to \mathrm{Spt}$, where

$$\Gamma_* X = \varprojlim_{U \in \mathcal{C}} X(U).$$

3) If A is a sheaf (or presheaf) of abelian groups, the Eilenberg-Mac Lane presheaf of spectra $H(A)$ is the presheaf of spectra underlying the suspension object

$$A, S^1 \otimes A, S^2 \otimes A, \ldots$$

in the category of presheaves of spectra in simplicial abelian groups. As a simplicial abelian presheaf,

$$S^n \otimes A := \mathbb{Z}(S^n)/\mathbb{Z}(*) \otimes A$$

is an Eilenberg-Mac Lane object $K(A,n)$. Recall from Proposition 16.1 that if $j : K(A,n) \to GK(A,n)$ is an injective fibrant model of $K(A,n)$ then there are natural isomorphisms

$$\pi_j \Gamma_* GK(A,n) = \begin{cases} H^{n-j}(\mathcal{C}, A) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

We'll see later that these isomorphisms assemble to give an identification of the stable homotopy groups of global sections of a (stably) fibrant model for $H(A)$ with the cohomology of $\mathcal{C}$ with coefficients in A. In other words all sheaf cohomology groups are stable homotopy groups.

4) Every chain complex (bounded or unbounded) D determines a presheaf of spectra $H(D)$, which computes the hypercohomology of $\mathcal{C}$ with coefficients in D, via computing stable homotopy groups of global sections of a stably fibrant model. The spectrum objects in presheaves of simplicial R-modules give a model for the full derived category [4].

5) There is a presheaf of spectra K on $Sch|_S$, called the algebraic K-theory spectrum, such that $\pi_j K(U)$ is the j^{th} algebraic K-group $K_j(U)$ of the S-scheme U [5].

6) If X is a presheaf of spectra and $n \in \mathbb{Z}$ there is a presheaf of spectra $X[n]$ with

$$X[n]^k = \begin{cases} X^{n+k} & \text{if } n+k \geq 0, \\ * & \text{if } n+k < 0. \end{cases}$$

In ordinary stable homotopy theory, the shift $X \mapsto X[1]$ is equivalent to the suspension functor $X \mapsto$

$X \wedge S^1$, while the shift $X \mapsto X[-1]$ is equivalent to the (derived) loop functor $X \mapsto \Omega X$.

Say that a map $f : X \to Y$ of presheaves of spectra is a *strict weak equivalence* (respectively *strict fibration*) if all maps $f : X^n \to Y^n$ are local equivalences (respectively injective fibrations).

A *cofibration* $i : A \to B$ of $\mathrm{Spt}(\mathcal{C})$ is a map for which

- $i : A^0 \to B^0$ is a cofibration, and
- all maps

$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \to B^{n+1}$$

  are cofibrations.

The *function complex* $\mathbf{hom}(X, Y)$ for presheaves of spectra X, Y is defined in simplicial degree n in the usual way:

$$\mathbf{hom}(X, Y)_n = \{X \wedge \Delta^n_+ \to Y\}.$$

Proposition 22.1. *With these definitions, the category $s\,\mathrm{Pre}(\mathcal{C})$ satisfies the axioms for a proper closed simplicial model category.*

The proof of this result is an exercise. Of course, it's just an opening act.

A presheaf of spectra X has presheaves $\pi_n^s X$ of stable homotopy groups, defined by

$$U \mapsto \pi_n^s X(U).$$

As a presheaf

$$\pi_n^s X = \varinjlim_k \ \pi_{k+n} X^k$$

if X is strictly fibrant (otherwise make it so), where the map

$$\pi_{k+n} X^k \to \pi_{k+1+n} X^{k+1}$$

in the colimit diagram is the composite map (the "suspension map")

$$\pi_{k+n} X^k \xrightarrow{\sigma_*} \pi_{k+n} \Omega X^{k+1} \cong \pi_{k+1+n} X^{k+1}$$

which is induced by the adjoint bonding map $\sigma_* : X^k \to \Omega X^{k+1}$.

Write $\tilde{\pi}_n^s X$ for the sheaf associated to the presheaf $\pi_n^s X$. The sheaves $\tilde{\pi}_n^s X$, $n \in \mathbb{Z}$, are the *sheaves of stable homotopy groups* of X.

Say that a map $f : X \to Y$ of presheaves of spectra is a *stable equivalence* if it induces isomorphisms

$$\tilde{\pi}_n^s X \xrightarrow{\cong} \tilde{\pi}_n^s Y$$

for all $n \in \mathbb{Z}$.

Observe that every strict equivalence is a stable equivalence.

Say that $p : Z \to W$ is a *stable fibration* if it has the right lifting property with respect to all maps which are cofibrations and stable equivalences.

While we're at it, suppose that $p : X \to Y$ is a strict fibration with fibre F. Then there are fibre sequences

$$F^k \xrightarrow{i} X^k \xrightarrow{p} Y^k$$

and corresponding long exact sequences

$$\dots \to \pi_{k+n}F^k \to \pi_{k+n}X^k \to \pi_{k+n}Y^k \xrightarrow{\partial} \pi_{k+n-1}F^k \to \dots$$

in presheaves of homotopy groups. The comparisons

$$\begin{tikzcd} F^k \arrow[r, "i"] \arrow[d, "\sigma_*"'] & X^k \arrow[r, "p"] \arrow[d, "\sigma_*"] & Y^k \arrow[d, "\sigma_*"] \\ \Omega F^{k+1} \arrow[r, "\Omega i"'] & \Omega X^{k+1} \arrow[r, "\Omega p"'] & \Omega Y^{k+1} \end{tikzcd}$$

induce comparisons of long exact sequences, and the (sheafified) filtered colimit is the long exact sequence

$$\dots \to \tilde{\pi}_n^s F \to \tilde{\pi}_n^s X \to \tilde{\pi}_n^s Y \xrightarrow{\partial} \tilde{\pi}_{n-1}^s F \to \dots \tag{22.1}$$

in sheaves of stable homotopy groups for a strict fibre sequence.

Suppose that $i : A \to B$ is a level cofibration (ie. all maps $A^n \to B^n$ are cofibrations) with cofibre B/A. From ordinary stable homotopy theory, all sequences

$$\pi_n^s A(U) \xrightarrow{i_*} \pi_n^s B(U) \xrightarrow{p_*} \pi_n^s B/A(U)$$

are exact, and in general the canonical map $S^1 \wedge X \to X[1]$ ($S^1 \wedge X$ is "fake suspension", but naturally stably equivalent to $X \wedge S^1$)) induces an isomorphism in stable homotopy groups. The Puppe sequence

$$A \xrightarrow{i} B \xrightarrow{p} B/A \to A \wedge S^1 \xrightarrow{i_*} B \wedge S^1$$

therefore induces natural exact sequences

$$\tilde{\pi}_n^s A \xrightarrow{i_*} \tilde{\pi}_n^s B \xrightarrow{p_*} \tilde{\pi}_n^s (B/A) \to \tilde{\pi}_{n-1}^s A \xrightarrow{i_*} \tilde{\pi}_{n-1}^s B \tag{22.2}$$

which patch together to give the long exact sequence in sheaves of stable homotopy groups for a level cofibre sequence.

Theorem 22.2. *With the definitions of cofibration, stable equivalence and stable cofibration given above, the category* $\mathrm{Spt}(\mathcal{C})$ *satisfies the axioms for a proper closed simplicial model category.*

Lemma 22.3. *A map $p : X \to Y$ is a stable fibration and a stable equivalence if and only if all maps $p : X^n \to Y^n$ are trivial injective fibrations of simplicial presheaves.*

Proof. If all $p : X^n \to Y^n$ are trivial injective fibrations, then p has the right lifting property with respect to all cofibrations, and is therefore a stable fibration. The map p is also a stable equivalence because it is a strict equivalence.

Suppose that p is a stable fibration and a stable equivalence. Then p has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "p"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where j is a cofibration and q is a trivial strict fibration. But then j is stable equivalence as well as a cofibration, so that the lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ W \arrow[r, "q"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

so that p is a retract of q and is therefore a trivial strict fibration. $\square$

Choose an infinite cardinal α such that $|\operatorname{Mor}(\mathcal{C})| <$

α. Say that a presheaf of spectra A is α-bounded if all pointed simplicial sets $A^n(U)$, $n \geq 0$, $U \in \mathcal{C}$ are α-bounded. Observe that every presheaf of spectra X is a union of its α-bounded subobjects.

Lemma 22.4. *Suppose given a cofibration $i : X \to Y$ which is a stable equivalence, and suppose that $A \subset Y$ is an α-bounded subobject. Then there is an α-bounded subobject $B \subset Y$ such that $A \subset B$ and the cofibration $B \cap X \to B$ is a stable equivalence.*

Proof. Note that $\tilde{\pi}_n^s Z = 0$ if and only if for all $x \in \pi_n^s Z(U)$ there is a covering sieve $\phi : V \to U$ such that $\phi^*(x) = 0$ for all ϕ in the covering.

The sheaves $\tilde{\pi}_n^s(Y/X)$ are trivial (sheafify the natural long exact sequence for a cofibration), and

$$\tilde{\pi}_n^s(Y/X) = \varinjlim_{C} \tilde{\pi}_n^s(C/C \cap X)$$

where C varies over all α-bounded subobjects of Y. The list of elements of all $x \in \pi_n^s(A/A \cap X)(U)$ is α-bounded. For each such x there is an α-bounded subobject $B_x \subset X$ such that

$$x \mapsto 0 \in \tilde{\pi}_n(B_x/B_x \cap X).$$

It follows that there is an α-bounded subobject

$$B_1 = A \cup (\cup_x B_x)$$

such that all $x \mapsto 0 \in \tilde{\pi}_n(B_1/B_1 \cap X)$.

Write $A = B_0$. Then inductively, we can produce an ascending sequence

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of α-bounded subobjects of Y such that all presheaf homomorphisms

$$\pi_n^s(B_i/B_i \cap X) \to \tilde{\pi}_n^s(B_{i+1}/B_{i+1} \cap X)$$

are trivial. Set $B = \cup_i B_i$. Then B is α-bounded and all sheaves $\tilde{\pi}_n^s(B/B \cap X)$ are trivial. $\square$

Lemma 22.5. *The class of stably trivial cofibrations has a generating set, namely the set I of all α-bounded stably trivial cofibrations.*

Proof. The class of cofibrations of $\mathrm{Spt}(\mathcal{C})$ is generated by the set J of cofibrations

$$\Sigma^\infty A[-n] \to \Sigma^\infty B[-n]$$

which are induced by α-bounded cofibrations $A \to B$ of pointed simplicial presheaves.

Suppose given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where j is a cofibration, B is α-bounded, and f is a stable equivalence. Then f has a factorization $f = q \cdot i$ where i is a cofibration and q is a strictly trivial fibration, hence a stable equivalence, and the lifting exists in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "j"'] & X \arrow[d, "i"] \\ & Z \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

The cofibration $i : X \to Z$ is a stable equivalence, and the image $\theta(B) \subset Z$ is α-bounded, so there is an α-bounded subobject $D \subset Z$ with $\theta(B) \subset D$ such that $D \cap X \to D$ is a stable equivalence. It follows that there is a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & D \cap X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

of the original diagram through an α-bounded stably trivial cofibration.

Now suppose that $i : C \to D$ is a stably trivial cofibration. Then i has a factorization

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[dr, "i"'] & E \arrow[d, "p"] \\ & D \end{tikzcd}$$

where j is a cofibration in the saturation of the set of α-bounded stably trivial cofibrations and p has the right lifting property with respect to all α-bounded stably trivial cofibrations. The map j is a stable equivalence since the class of stably trivial cofibrations is closed under pushout (by a long exact sequence argument) and composition. It follows that p is a stable equivalence, and therefore has the right lifting property with respect to all α-bounded cofibrations (by the trick of the last paragraph) and hence with respect to all cofibrations. It follows that i is a retract of the map j. □

Remark 22.6. The proof of Lemma 22.5 is a concrete implementation of Jeff Smith's "solution set condition" argument. See also [1].

Proof of Theorem 22.2. According to Lemma 22.5, a map is a stable fibration if and only if it has the right lifting property with respect to all α-bounded stably trivial cofibrations. A small object argument therefore implies that every map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where j is a stably trivial cofibration and p is a stable fibration.

Similarly, Lemma 22.3 implies that a map is a stable fibration and a stable weak equivalence if and only if it is a strict fibration and a strict weak equivalence. There is a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

for any map $f : X \to Y$ where i is a cofibration and p is a strict fibration and a strict equivalence — this gives the corresponding factorization for the stable structure.

We have therefore proved **CM5**. The potentially non-trivial part of **CM4** is a consequence of Lemma 22.3. The remaining closed model axioms are trivial.

The closed simplicial model axiom **SM7** is proved by showing (by induction on n) that if $i : A \to B$ is a stably trivial cofibration then all maps

$$i \wedge \partial\Delta^n_+ : A \wedge \partial\Delta^n_+ \to B \wedge \partial\Delta^n_+$$

are stable equivalences.

Left and right properness are consequences of long exact sequences in stable homotopy groups. $\square$

23 The stable category: basic properties

We begin with a discusion of stable fibrations and stably fibrant objects.

The stable model structure on $\mathrm{Spt}(\mathcal{C})$ is cofibrantly generated, so there is a functorial stably fibrant model construction

$$j : X \to LX$$

Note that if X and Y are stably fibrant, any stable equivalence $f : X \to Y$ must be a strict equivalence. This is a consequence of Lemma 22.3. It follows that a map $f : X \to Y$ of arbitrary presheaves of spectra is a stable equivalence if and only if the induced map $LX \to LY$ is a strict equivalence. Thus a map $f : X \to Y$ is an L-equivalence (in the sense that $LX \to LY$ is a strict equivalence) if and only if it is a stable equivalence.

We also have the following:

A4 The functor L preserves strict equivalence.

A5 The maps $j_{LX}, Lj_X : LX \to LLX$ are strict weak equivalences.

A6′ Stable equivalences are preserved by pullback along stable fibrations.

The following characterization of stable fibrations is a formal consequence of these axioms. See [2], but a proof will appear later (Theorem 27.6) in these notes in a different context.

Theorem 23.1. *A map $p : X \to Y$ of* $\mathrm{Spt}(\mathcal{C})$ *is a stable fibration if and only it is a strict fibration and the diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & LX \arrow[d, "Lp"] \\ Y \arrow[r, "j"'] & LY \end{tikzcd}$$

is strictly homotopy cartesian.

There is a different recognition procedure for stably fibrant objects, given by Proposition 23.4 below. The ideas in the proof of that result begin with the following:

Lemma 23.2. *Suppose that $p : X \to Y$ is a stable fibration. Then the diagrams*

$$\begin{tikzcd} X^n \arrow[r, "\sigma_*"] \arrow[d, "p"'] & \Omega X^{n+1} \arrow[d, "\Omega p"] \\ Y^n \arrow[r, "\sigma_*"'] & \Omega Y^{n+1} \end{tikzcd}$$

are strictly homotopy cartesian.

Proof. Since p is a stable fibration, any stably trivial cofibration $\theta : A \to B$ induces a homotopy cartesian diagram

$$\begin{tikzcd} \mathbf{hom}(B, X) \arrow[r, "p_*"] \arrow[d, "\theta^*"'] & \mathbf{hom}(B, Y) \arrow[d, "\theta^*"] \\ \mathbf{hom}(A, X) \arrow[r, "p_*"'] & \mathbf{hom}(A, Y) \end{tikzcd}$$

If $\theta : A \to B$ is a stable equivalence between cofibrant objects, then the diagram above is still homotopy cartesian. In effect, θ has a factorization $\theta = \pi \cdot j$ where j is a trivial cofibration and $\pi \cdot i = 1$ for some level trivial cofibration i. It follows that the diagram above is a composite of homotopy cartesian diagrams, and is therefore homotopy cartesian.

The diagrams of the statement of the Lemma arise from the stable equivalences

$$\Sigma^\infty S^1[-1-n] \to S[-n].$$

□

Corollary 23.3. *If X is stably fibrant, then all X^n are injective fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences.*

Proposition 23.4. *A presheaf of spectra X is stably fibrant if and only if all X^n are injective fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences.*

Proof. Suppose that all X^n are injective fibrant and all $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences. Then the simplicial presheaves X^n and ΩX^{n+1} are injective fibrant and cofibrant, so that all σ_* are homotopy equivalences. It follows that all spaces $X^n(U)$ are fibrant and that all maps $\sigma_* : X^n(U) \to \Omega X^{n+1}(U)$ are weak equivalences of pointed simplicial sets. It follows that all maps

$$\pi_k X^n(U) \to \pi^s_{k-n} X(U)$$

are isomorphisms.

Suppose that $j : X \to LX$ is a stably fibrant model for X. Then all spaces $LX^n(U)$ are fibrant and all maps $LX^n(U) \to \Omega LX^{n+1}(U)$ are weak equivalences, and so all maps

$$\pi_k LX^n(U) \to \pi^s_{k-n} LX(U)$$

are isomorphisms. The map j induces an isomorphism in all sheaves of stable homotopy groups, and so the maps $j : X^n \to LX^n$ induce isomorphisms

$$\tilde{\pi}_k X^n \to \tilde{\pi}_k LX^n$$

of sheaves of homotopy groups for $k \geq 0$. X and LX are presheaves of infinite loop spaces, and so the maps $X^n \to LX^n$ are local weak equivalences of simplicial presheaves (exercise). In particular, $j : X \to LX$ is a strict weak equivalence.

Now consider the lifting problem

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \\ B \arrow[ur, dotted] & \end{tikzcd}$$

where i is a stably trivial cofibration. Then the induced map $i_* : LA \to LB$ is a strict equivalence of stably fibrant objects, by assumption.

Take a factorization

$$\begin{tikzcd} LA \arrow[rr, "L\alpha"] \arrow[dr, "j'"'] & & LX \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

where j' is a cofibration and a strict weak equivalence and p is a strict fibration. Then LB is strictly fibrant, so there is a map $\zeta : Z \to LB$ such that the diagram

$$\begin{tikzcd} LA \arrow[r, "j'"] \arrow[d, "Li"'] & Z \arrow[dl, "\zeta"] \\ LB & \end{tikzcd}$$

commutes. The map ζ is therefore a strict equivalence. Form the pullback

$$\begin{tikzcd} Z \times_{LX} X \arrow[r, "p_*"] \arrow[d, "j_*"'] & X \arrow[d, "j"] \\ Z \arrow[r, "p"'] & LX \end{tikzcd}$$

and observe that the map j_* is a strict weak equivalence since j is a strict weak equivalence and p is a strict fibration. Then there is a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Z \times_{LX} X \arrow[r, "p_*"] \arrow[d, "\zeta j_*"] & X \\ B \arrow[r, "j"'] & LB & \end{tikzcd}$$

in which the top composite is the map $\alpha : A \to X$. The vertical map ζj_* is a strict weak equivalence and X is strictly fibrant, so the lifting problem can be solved. $\square$

The bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$ of a presheaf of spectra X have adjoints $\sigma_* : X^n \to$

ΩX^{n+1}. Write $\Omega^{\infty} X^n$ for the colimit of the system

$$X^n \xrightarrow{\sigma_*} \Omega X^{n+1} \xrightarrow{\Omega\sigma_*} \Omega^2 X^{n+2} \xrightarrow{\Omega^2\sigma_*} \dots.$$

The the induced isomorphisms

$$\Omega^{\infty} X^n \xrightarrow[\cong]{\sigma_*} \Omega(\Omega^{\infty} X^{n+1})$$

and the objects $\Omega^{\infty} X^n$ define a presheaf of spectra $\Omega^{\infty} X$, and the maps $X^n \to \Omega^{\infty} X^n$ assemble to define a natural map

$$\tilde{\eta} : X \to \Omega^{\infty} X.$$

The map $\tilde{\eta}$ is a stable equivalence if X is strictly fibrant (exercise).

Corollary 23.5. *1) The presheaf of spectra*

$$QX = F\Omega^{\infty} FX$$

is stably fibrant, for any presheaf of spectra X.

2) Let $\eta : X \to QX$ be the natural map which is defined by the composite

$$X \xrightarrow{j} FX \xrightarrow{\tilde{\eta}} \Omega^{\infty} FX \xrightarrow{j} F\Omega^{\infty} FX,$$

where the morphisms j are natural strictly fibrant models. Then $\eta : X \to QX$ is a natural stably fibrant model.

The additivity property for the stable category $\mathrm{Ho}(\mathrm{Spt}(\mathcal{C}))$ is a basic consequence of existence of the long exact sequences in stable homotopy groups for strict fibrations and level cofibrations.

Lemma 23.6. *Suppose that X and Y are spectra. Then the canonical inclusion*

$$c : X \vee Y \to X \times Y$$

is a natural stable equivalence.

Proof. The sequence

$$0 \to \tilde{\pi}_k^s X \to \tilde{\pi}_k^s (X \vee Y) \to \tilde{\pi}_k^s Y \to 0$$

arising from the level cofibration $X \subset X \vee Y$ is split exact, as is the sequence

$$0 \to \tilde{\pi}_k^s X \to \tilde{\pi}_k^s (X \times Y) \to \tilde{\pi}_k^s Y \to 0$$

arising from the fibre sequence $X \to X \times Y \to Y$. It follows that the map $X \vee Y \to X \times Y$ induces an isomorphism in all sheaves of stable homotopy groups. □

Remark 23.7. The stable homotopy category

$$\mathrm{Ho}(\mathrm{Spt}(\mathcal{C}))$$

is *additive*: the sum of two maps $f, g : X \to Y$ is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \underset{\simeq}{\xleftarrow{c}} Y \vee Y \xrightarrow{\nabla} Y.$$

Here also is a basic consequence of the existence of the long exact sequence in sheaves of stable homotopy groups for a levelwise cofibration:

Lemma 23.8. *Suppose that*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "j"'] & D \end{tikzcd} \tag{23.1}$$

is a pushout in **Spt** *where i is a levelwise cofibration. Then there is a long exact sequence in sheaves of stable homotopy groups*

$$\dots \xrightarrow{\partial} \tilde{\pi}_k^s A \xrightarrow{(i,\alpha)} \tilde{\pi}_k^s C \oplus \tilde{\pi}_k^s B \xrightarrow{j-\beta} \tilde{\pi}_k^s D \xrightarrow{\partial} \tilde{\pi}_{k-1}^s A \to \dots \tag{23.2}$$

The sequence (23.2) is the *Mayer-Vietoris sequence* for the cofibre square (23.1). The boundary map $\partial : \tilde{\pi}_k^s D \to \tilde{\pi}_{k-1}^s A$ is the composite

$$\tilde{\pi}_k^s D \to \tilde{\pi}_k^s D/C = \tilde{\pi}_k^s B/A \xrightarrow{\partial} \tilde{\pi}_{k-1}^s A.$$

We finish this section with the basic relation between fibre and cofibre sequences in the stable category: they are the same thing, up to stable equivalence.

Lemma 23.9. *Suppose that $A \xrightarrow{i} X \xrightarrow{\pi} X/A$ is a levelwise cofibre sequence in* $\mathrm{Spt}(\mathcal{C})$, *and let*

F be the strict homotopy fibre of the map $\pi : X \to X/A$. Then the induced map $i_* : A \to F$ is a stable equivalence.

Proof. Choose a strict fibration $p : Z \to X/A$ such that $Z \to *$ is a strict weak equivalence. Form the pullback

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi_*"] \arrow[d, "p_*"'] & Z \arrow[d, "p"] \\ X \arrow[r, "\pi"'] & X/A \end{tikzcd}$$

Then $\tilde{X}$ is the strict homotopy fibre of π and the maps $i : A \to X$ and $* : A \to Z$ together determine a map $i_* : A \to \tilde{X}$. We show that i_* is a stable equivalence.

Pull back the cofibre square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & * \arrow[d] \\ X \arrow[r, "\pi"'] & X/A \end{tikzcd}$$

along the strict fibration p to find a levelwise cofibre square

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d, "\tilde{i}"'] & U \arrow[d] \\ \tilde{X} \arrow[r] & Z \end{tikzcd}$$

The spectrum object Z is contractible, so a Mayer-Vietoris sequence argument (Lemma 23.8) implies that the map $\tilde{A} \to \tilde{X} \times U$ is a stable equivalence.

Also, from the fibre square

$$\begin{tikzcd}
\tilde{A} \arrow[r] \arrow[d] & U \arrow[d] \\
A \arrow[r] & *
\end{tikzcd}$$

we see that the map $\tilde{A} \to A \times U$ is an isomorphism. The map $i_* : A \to \tilde{X}$ induces a section $\theta : A \to \tilde{A}$ of the map $\tilde{A} \to A$ which composes with the projection $\tilde{A} \to U$ to give the trivial map $* : A \to U$. It follows that there is a commutative diagram

$$\begin{tikzcd}
& A \arrow[r, "i_*"] \arrow[dl, "(1_A,*)"'] \arrow[d, "\theta"] & \tilde{X} \arrow[d, "(1_{\tilde{X}},*)"] \\
A \times U \arrow[dr, "pr"'] & \tilde{A} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & \tilde{X} \times U \arrow[dl] \\
& U &
\end{tikzcd}$$

It follows that A is the stable homotopy fibre of the map $\tilde{A} \to U$, and so i_* is a stable equivalence. $\square$

Lemma 23.10. *Suppose that F is the fibre of a strict fibration $p : E \to B$, and let $i : F \to E$ be the levelwise inclusion of F in E. Then the* *induced map $\gamma : E/F \to B$ is a stable equivalence.*

Proof. There is a diagram

$$\begin{tikzcd}
F \arrow[rr, "i"] \arrow[dd, "="'] \arrow[dr, "j'_*"] & & E \arrow[rr, "\pi"] \arrow[dd, "="'] \arrow[dr, "j'"] & & E/F \arrow[dd, "\gamma"] \arrow[dr, "="] & \\
& F' \arrow[rr, "i'"] \arrow[dl, "\theta_*"] & & U \arrow[rr, "p'"] \arrow[dl, "\theta"] & & E/F \arrow[dl, "\gamma"] \\
F \arrow[rr, "i"'] & & E \arrow[rr, "p"'] & & B &
\end{tikzcd}$$

where p' is a strict fibration, j' is a cofibration and a strict equivalence, and θ exists by a lifting property:

$$\begin{tikzcd}
E \arrow[r, "="] \arrow[d, "j'"'] & E \arrow[d, "p"] \\
U \arrow[r, "\gamma p'"'] \arrow[ur, dotted, "\theta"] & B
\end{tikzcd}$$

Then the map j'_* is a stable equivalence by Lemma 23.9, so that θ_* is a stable equivalence. The map θ is a strict equivalence, so it follows from a comparison of long exact sequences in stable homotopy groups (22.1) that γ is a stable equivalence. $\square$

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