Lecture 12

24 T-spectra

Suppose that T is a pointed simplicial presheaf on a small site $\mathcal{C}$.

A *T-spectrum* X is a collection of pointed simplicial presheaves X^n, $n \geq 0$, with pointed maps $\sigma : T \wedge X^n \to X^{n+1}$. A map $f : X \to Y$ of T-spectra consists of pointed simplicial presheaf maps $f : X^n \to Y^n$ which respect structure in the sense that the diagrams

$$\begin{tikzcd} T \wedge X^n \arrow[r, "\sigma"] \arrow[d, "T\wedge f"'] & X^{n+1} \arrow[d, "f"] \\ T \wedge Y^n \arrow[r, "\sigma"'] & Y^{n+1} \end{tikzcd}$$

commute. Write $\mathrm{Spt}_T(\mathcal{C})$ for the category of T-spectra.

Say that a map $f : X \to Y$ of T-spectra is a *strict weak equivalence* (respectively *strict fibration*) if all maps $f : X^n \to Y^n$ are local weak equivalences (respectively injective fibrations) of pointed simplicial presheaves on $\mathcal{C}$.

A *cofibration* of T-spectra is a map $i : A \to B$ such that

- $i : A^0 \to B^0$ is a cofibration of simplicial presheaves, and
- all maps

  $$(T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1} \to B^{n+1}$$

  are cofibrations of simplicial presheaves.

If K is a pointed simplicial presheaf and X is a T-spectrum, then $X \wedge K$ has the obvious meaning:

$$(X \wedge K)^n = X^n \wedge K.$$

The *function complex* $\mathbf{hom}(X, Y)$ for T-spectra X and Y is the pointed simplicial set with

$$\mathbf{hom}(X, Y)_n = \{ \ X \wedge \Delta^n_+ \to Y \ \}.$$

Lemma 24.1. *With these definitions, the category of* $\mathrm{Spt}_T(\mathcal{C})$ *of* T*-spectra on* $\mathcal{C}$ *satisfies the definitions for a proper closed simplicial model category.*

The proof is the usual thing.

Suspensions and shifts work in $\mathrm{Spt}_T(\mathcal{C})$ just like for ordinary spectra:

- Given a pointed simplicial presheaf K, the *suspension spectrum* $\Sigma^\infty_T K$ is the T-spectrum

  $$K, T \wedge K, T^2 \wedge K, \dots$$

with $T^n = T \wedge \cdots \wedge T$ (n-fold smash power). The functor $K \mapsto \Sigma_T^\infty K$ is left adjoint to the 0-level functor $X \mapsto X^0$.

The suspension spectrum $\Sigma_T^\infty S^0$ is also denoted by S_T and is called the *T-sphere spectrum.*

- Given a T-spectrum X, $n \in \mathbb{Z}$,

$$X[n]^k = \begin{cases} X^{n+k} & n+k \geq 0 \\ * & n+k < 0 \end{cases}$$

Lemma 24.2. *Suppose given the diagram*

$$\begin{tikzcd} A \cap X \arrow[r] \arrow[d, "j_*"'] & X \arrow[d, "j"] \\ A \arrow[r, "i"'] & Y \end{tikzcd}$$

in spectra, where j is a cofibration and i is a levelwise cofibration. Then the induced map $j_* : A \cap X \to A$ is a cofibration.

The proof of Lemma 24.2 is set theoretic. The Lemma itself is a holdover from old approaches to constructing the stable category, and is not really needed now — see [1]

What now follows is a general set of tricks that applies to any set S of cofibrations $i : A \to B$ of $\mathrm{Spt}_T(\mathcal{C})$.

Suppose that α is a cardinal such that $\alpha > |\operatorname{Mor}(\mathcal{C})|$. Suppose also that $\alpha > |B|$ for all morphisms $i : A \to B$ appearing in the set S and that $\alpha > |S|$. Choose a cardinal λ such that $\lambda > 2^{\alpha}$.

Suppose that $f : X \to Y$ is a morphism of $\mathrm{Spt}_T(\mathcal{C})$. Define a functorial system of factorizations

$$\begin{tikzcd} X \arrow[r, "i_s"] \arrow[dr, "f"'] & E_s(f) \arrow[d, "f_s"] \\ & Y \end{tikzcd}$$

of the map f indexed on all ordinal numbers $s < \lambda$ as follows:

1) Given the factorization (f_s, i_s) define the factorization (f_{s+1}, i_{s+1}) by requiring that the diagram

$$\begin{tikzcd} \bigvee_{\mathbf{D}} A \arrow[r, "(\alpha_{\mathbf{D}})"] \arrow[d, "\vee i"'] & E_s(f) \arrow[d] \\ \bigvee_{\mathbf{D}} B \arrow[r] & E_{s+1}(f) \end{tikzcd}$$

is a pushout, where the wedge is indexed over all diagrams $\mathbf{D}$ of the form

$$\begin{tikzcd} A \arrow[r, "\alpha_{\mathbf{D}}"] \arrow[d, "i"'] & E_s(f) \arrow[d, "f_s"] \\ B \arrow[r, "\beta_{\mathbf{D}}"'] & Y \end{tikzcd}$$

with $i : A \to B$ in the set S. Then the map

i_{s+1} is the composite

$$X \xrightarrow{i_s} E_s(f) \xrightarrow{g_*} E_{s+1}(f)$$

2) If s is a limit ordinal, set $E_s(f) = \varinjlim_{t<s} E_s(f)$.

Set $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$. Then there is an induced factorization

$$\begin{tikzcd} X \arrow[r, "i_\lambda"] \arrow[dr, "f"'] & E_\lambda(f) \arrow[d, "f_\lambda"] \\ & Y \end{tikzcd}$$

of the map f. Then i_λ is a cofibration. The map f_λ has the right lifting property with respect to the cofibrations $i : A \to B$ in S by a standard argument, since any map $\alpha : A \to E_\lambda(f)$ must factor through some $E_s(f)$ by the choice of cardinal λ.

Write $L(X) = E_\lambda(c)$ for the result of this construction when applied to the canonical map $c : X \to *$. Then we have the following:

Lemma 24.3. *1) Suppose that $t \mapsto X_t$ is a diagram of level cofibrations indexed by any cardinal $\gamma > 2^\alpha$. Then the natural map*

$$\varinjlim_{t<\gamma} L(X_t) \to L(\varinjlim_{t<\gamma} X_t)$$

is an isomorphism.

2) The functor $X \mapsto L(X)$ preserves level cofibrations.

3) Suppose that ζ is a cardinal with $\zeta > \alpha$, and let $\mathcal{F}_\zeta(X)$ denote the filtered system of subobjects of X having cardinality less than ζ. Then the natural map

$$\varinjlim_{Y \in \mathcal{F}_\zeta(X)} L(Y) \to L(X)$$

is an isomorphism.

4) If $|X| < 2^\omega$ where $\omega \geq \alpha$ then $|L(X)| < 2^\omega$.

5) Suppose that U, V are subobjects of a presheaf of T-spectra X. Then the natural map

$$L(U \cap V) \to L(U) \cap L(V)$$

is an isomorphism.

Proof. The argument is the same as for Lemma 22.4. □

Basic Assumptions: Suppose that S is a set of cofibrations such that

1) A is cofibrant for all $i : A \to B$ in S,

2) S includes the set I of generating maps

$$\Sigma_T^\infty C[-n] \to \Sigma_T^\infty D[-n], \ n \geq 0,$$

for the strict trivial cofibrations of $\mathrm{Spt}_T(\mathcal{C})$, which are induced by the α-bounded trivial cofibrations $C \to D$ of pointed simplicial presheaves, and

3) S includes all cofibrations

$$(A \wedge D) \cup (B \wedge C) \to B \wedge D, \ m \geq 0,$$

for $A \to B$ in S and all α-bounded pointed cofibrations $C \to D$ of simplicial presheaves.

A map $p : X \to Y$ is said to be *injective* if it has the right lifting property with respect to all maps of S. An object X is injective if the map $X \to *$ is injective. By construction, LX is injective for every object X. Every injective object is strictly fibrant.

Say that a map $f : X \to Y$ of $\mathrm{Spt}(\mathcal{C})$ is an L-*equivalence* if it induces a bijection

$$f^* : [Y, Z] \xrightarrow{\cong} [X, Z]$$

in morphisms in the strict homotopy category for every injective object Z.

Every strict equivalence $X \to Y$ is an L-equivalence.

Lemma 24.4. *Suppose that $i : A \to B$ is a cofibration with A cofibrant. Then i is an L-equivalence if*

1) i induces a trivial fibration

$$i^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$$

for all injective Z, or

2) all injective Z have the right lifting property with respect to i and with respect to the cofibration

$$(A \wedge \Delta^1_+) \cup (B \wedge \partial\Delta^1_+) \to B \wedge \Delta^1_+.$$

Proof. The first claim is trivial.

The second claim is almost as easy: we must show that the induced function

$$i^* : \pi(B, Z) \to \pi(A, Z)$$

in naive homotopy classes is a bijection for all injective Z. This suffices, because A and B are cofibrant and Z is strictly fibrant.

Every morphism $A \to Z$ extends to a morphism $B \to Z$ because $Z \to *$ has the right lifting property with respect to i. It follows that i^* is surjective.

Given $f, g : B \to Z$, if there is a homotopy $h : A \wedge \Delta^1_+ \to Z$ from $f|_A$ to $g|_A$, then there is a diagram

$$\begin{tikzcd}
(B \wedge \partial\Delta^1_+) \cup (A \wedge \Delta^1_+) \arrow[r, "((f,g),h)"] \arrow[d] & Z \\
B \wedge \Delta^1_+ \arrow[ur, dotted] &
\end{tikzcd}$$

where the indicated lifting exists because Z is injective and the vertical map is a member of S. But then f and g are homotopic, so that i^* is injective. $\square$

Corollary 24.5. *All cofibrations appearing in the set S are L-equivalences.*

Proof. Every cofibration $i : A \to B$ appearing in the set S induces a trivial fibration

$$i^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$$

by construction. $\square$

A map $f : Z \to W$ between injective objects is an L-equivalence if and only if it is a strict equivalence. To see this, use cofibrant replacement and the fact that an L-equivalence between cofibrant injective objects is a homotopy equivalence.

A *cofibrant replacement* for a map $f : X \to Y$ is a commutative diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, "j"] \arrow[d, "\pi_X"'] & \tilde{Y} \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

in which the maps π_X and π_Y are trivial strict fibrations, $\tilde{X}$ is cofibrant and j is a cofibration. Any two cofibrant replacements for a fixed map f are strictly equivalent, by a standard argument. The map f is an L-equivalence if and only if it has a cofibrant replacement j which is an L-equivalence.

Note that if some cofibrant replacement j for f induces a trivial fibration

$$j^* : \mathbf{hom}(\tilde{Y}, Z) \to \mathbf{hom}(\tilde{X}, Z)$$

for all injective objects Z, then all cofibrant replacements for f have this property.

Lemma 24.6. *All cofibrations in the saturation of the set S are L-equivalences.*

Proof. The saturation of the set S is the family of cofibrations which has the left lifting property with respect to all injective maps $X \to Y$.

If the cofibration $j : C \to D$ is coproduct of members of S (hence with C and D cofibrant), then

$$j^* : \mathbf{hom}(D, Z) \to \mathbf{hom}(C, Z)$$

is a product of trivial fibrations and is therefore a trivial fibration.

Suppose given a pushout diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & C' \arrow[d, "j'"] \\ D \arrow[r] & D' \end{tikzcd}$$

where j is a coproduct of members of S and C' is cofibrant. Then from the pullback diagram

$$\begin{tikzcd} \mathbf{hom}(D', Z) \arrow[r] \arrow[d, "j'^*"'] & \mathbf{hom}(D, Z) \arrow[d, "j^*"] \\ \mathbf{hom}(C', Z) \arrow[r] & \mathbf{hom}(C, Z) \end{tikzcd}$$

we see that j'^* is a trivial fibration for all injective Z.

Suppose given a pushout diagram

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "j"'] & E \arrow[d] \\ D \arrow[r] & D \cup_C E \end{tikzcd}$$

with j as above and E arbitrary. Then there is a factorization

$$
\begin{tikzcd}
C \arrow[r, "i"] \arrow[dr, "\alpha"'] & \tilde{E} \arrow[d, "\pi"] \\
& E
\end{tikzcd}
$$

of α with π a strictly trivial fibration and i a cofibration, and there is an induced commutative diagram

$$
\begin{tikzcd}
\tilde{E} \arrow[r, "\tilde{j}_*"] \arrow[d, "\pi"'] & D \cup_C \tilde{E} \arrow[d, "\pi_*"] \\
E \arrow[r, "j_*"'] & D \cup_C E
\end{tikzcd}
$$

The map π is a strict equivalence, so that π_* is a strict equivalence by properness. The map $\tilde{j}_*$ induces a trivial fibration

$$(\tilde{j}_*)^* : \mathbf{hom}(D \cup_C \tilde{E}, Z) \to \mathbf{hom}(\tilde{E}, Z)$$

for all injective Z, by the previous paragraph. It follows that some cofibrant replacement of the map

$$j_* : E \to D \cup_C E$$

induces a corresponding function complex weak equivalence.

Suppose given a string of morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \dots$$

such that each f_i is an L-equivalence. Take a "cofibrant replacement"

$$\begin{tikzcd} A_0 \arrow[r, "i_1"] \arrow[d, "\pi_0"'] & A_1 \arrow[r, "i_2"] \arrow[d, "\pi_1"'] & A_2 \arrow[r] \arrow[d, "\pi_2"] & \dots \\ X_0 \arrow[r, "f_1"'] & X_1 \arrow[r, "f_2"'] & X_2 \arrow[r] & \dots \end{tikzcd}$$

in which A_0 is cofibrant, all i_k are cofibrations and all π_j are trivial strict fibrations. Then all maps i_k induce trivial fibrations

$$i_k^* : \mathbf{hom}(A_k, Z) \to \mathbf{hom}(A_{k-1}, Z)$$

for all injective Z, so the cofibration $A_0 \to \varinjlim_i A_i$ induces a trivial fibration

$$\mathbf{hom}(\varinjlim_i A_i, Z) \to \mathbf{hom}(A_0, Z).$$

for all injective Z. The map

$$\varinjlim_i A_i \to \varinjlim_i X_i$$

is a (sectionwise) weak equivalence, and it follows that some cofibrant replacement for the map $X_0 \to \varinjlim_i X_i$ induces a trivial fibration in all function complexes taking values in injective objects Z.

It follows that every member $i : A \to B$ of the saturation of S has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & Z \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

such that π is injective and j is a member of the saturation of S which is also an L-equivalence. The map i has the left lifting property with respect to all injective maps such as π, so that i is a retract of j. $\square$

Corollary 24.7. *1) The natural map $j : X \to LX$ is an L-equivalence.*

2) A map $f : X \to Y$ is an L-equivalence if and only if the induced map $Lf : LX \to LY$ is a strict equivalence.

Lemma 24.8. *Suppose that $\gamma \geq \alpha$. Suppose further that $i : X \to Y$ is a level cofibration and a strict equivalence and that $A \subset Y$ is an γ-bounded subobject. Then there is a γ-bounded subobject $B \subset Y$ with $A \subset B$ such that the level cofibration $B \cap X \to B$ is a strict equivalence.*

Proof. First of all, consider the diagram of cofibrations

$$\begin{tikzcd} & X^0 \arrow[d, "i"] \\ A^0 \arrow[r] & Y^0 \end{tikzcd}$$

Then by Lemma 10.2 (the bounded cofibration condition for simplicial presheaves) there is a subobject $B^0 \subset Y^0$ such that B^0 is γ-bounded, $A^0 \subset B^0$ and $B^0 \cap X^0 \to B^0$ is a local weak equivalence.

Form the diagram

$$\begin{tikzcd} T \wedge A^0 \arrow[r] \arrow[d, "\sigma"'] & T \wedge B^0 \arrow[r] & T \wedge Y^0 \arrow[d, "\sigma"] \\ A^1 \arrow[rr] & & Y^1 \end{tikzcd}$$

Then the induced map

$$A^1 \cup_{T \wedge A^0} T \wedge B^0 \to Y^1$$

factors through a γ-bounded subobject $C^1 \subset Y^1$. There is a γ-bounded subobject $B^1 \subset Y^1$ such that $C^1 \subset B^1$ and $B^1 \cap X^1 \to B^1$ is a local weak equivalence. The composite

$$T \wedge B^0 \to A^1 \cup_{T \wedge A^0} T \wedge B^0 \to C^1 \subset B^1$$

is the bonding map up to level 1 for the object B.

Construct the remaining objects B^n, $n \geq 1$, inductively according to this recipe. $\square$

Lemma 24.9. *Suppose given a cofibration $i : X \to Y$ which is an L-equivalence, and suppose that $A \subset Y$ is a 2^λ-bounded subobject, where λ is chosen as above. Then there is a 2^λ-bounded subobject $B \subset Y$ with $A \subset B$ and such that the cofibration $B \cap X \to B$ is an L-equivalence.*

Proof. Write $B_0 = A$, and set $\kappa = 2^\lambda$.

Consider the diagram

$$\begin{tikzcd} & LX \arrow[d] \\ LB_0 \arrow[r] & LY \end{tikzcd}$$

Then the maps are level cofibrations (Lemma 24.3.2) and $LX \to LY$ is a strict equivalence by assumption. The object LB_0 is κ-bounded by Lemma 24.3.4, so there is a κ-bounded subobject $C_1 \subset LY$ with $LB_0 \subset C_1$ such that $C_1 \cap LX \to C_1$ is a strict equivalence, by Lemma 24.8. Since C_1 is κ-bounded there is a κ-bounded subobject $B_1 \subset Y$ with $B_0 \subset B_1$ such that $C_1 \subset LB_1$ (Lemma 24.3.3). Proceeding inductively we find κ-bounded subobjects

$$C_1 \subset C_2 \subset \dots$$

of LY and κ-bounded subobjects

$$B_0 \subset B_1 \subset B_2 \subset \dots$$

indexed by $i < \kappa$, such that C_s and B_s are defined at limit ordinals s by colimits, and

$$LB_i \subset C_{i+1} \subset LB_{i+1}$$

and $C_i \cap LX \to C_i$ is a level weak equivalence. Write $B = \varinjlim_{i<\kappa} B_i$. Then B is κ-bounded, and

$$L(B) = \varinjlim_{i<\kappa} L(B_i) = \varinjlim_{i<\kappa} C_i$$

by Lemma 24.3.1 and construction. Also

$$\begin{aligned} L(B \cap X) = L(B) \cap L(X) &= \varinjlim_{i<\kappa} L(B_i) \cap L(X) \\ &\cong \varinjlim_{i<\kappa} C_i \cap L(X) \end{aligned}$$

by Lemma 24.3.1 and 24.3.5 and construction. It follows that the map

$$B \cap X \to B$$

is an L-equivalence. □

Say that a cofibration is *L-trivial* if it is an L-equivalence.

Lemma 24.10. *The set of κ-bounded L-trivial cofibrations is a generating set for the class of L-trivial cofibrations.*

Proof. Run the solution set argument of Lemma 22.5 using Lemma 24.9 for the set of κ-bounded cofibrations. Recall that the κ-bounded cofibrations generate the class of cofibrations. $\square$

Say that a map $p : X \to Y$ is an *L-fibration* if it has the right lifting property with respect to all L-trivial cofibrations. Observe that every L-fibration is a strict fibration, since S contains a generating set for the class of strict trivial cofibrations.

Lemma 24.11. *A map $p : X \to Y$ is an L-fibration and an L-equivalence if and only if p is a trivial strict fibration.*

Proof. We need only show that p is a trivial strict fibration if it is an L-fibration and an L-equivalence, but this is the usual proof: find a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "p"'] & W \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where j is a cofibration and π is a trivial strict fibration. But then j is an L-equivalence so the lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ W \arrow[r, "\pi"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

so that p is a retract of π. $\square$

Theorem 24.12. *Suppose that S is a set of cofibrations which satisfies the list of basic assumptions above. Let the L-equivalences and L-fibrations be defined relative to the set S. Then with these definitions the category $\mathrm{Spt}_T(\mathcal{C})$ satisfies the axioms for a closed simplicial model category.*

Proof. Every map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & W \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p is an L-fibration and j is a cofibration and an L-equivalence, by Lemma 24.6 and Lemma 24.10.

Every map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

such that i is a cofibration and q is a strictly trivial fibration. But then q is an L-fibration and an L-equivalence.

The rest of the closed model axioms are trivial to verify.

For the closed simplicial model structure, we need to show that if $i : A \to B$ is a cofibration and an L-equivalence, then all maps

$$i \wedge \partial\Delta^n_+ : A \wedge \partial\Delta^n_+ \to B \wedge \partial\Delta^n_+$$

are L-equivalences. By replacing by a cofibrant model if necessary, it is enough to assume that A is cofibrant. Then one uses the usual patching argument for the category of cofibrant objects in the L-model structure for $\mathrm{Spt}_T(\mathcal{C})$ to compare pushouts of the form

$$\begin{tikzcd} A \wedge \partial\Delta^{n-1}_+ \arrow[r] \arrow[d] & A \wedge \Lambda^n_{k+} \arrow[d] \\ A \wedge \Delta^{n-1}_+ \arrow[r] & A \wedge \partial\Delta^n_+ \end{tikzcd}$$

to show inductively that the question reduces to showing that the map

$$i \vee i : A \vee A \to B \vee B$$

is an L-equivalence. But $i \vee i$ has the left lifting property with respect to all L-fibrations, and must therefore be an L-trivial cofibration. $\square$

Lemma 24.13. *The L-structure on* $\mathrm{Spt}_T(\mathcal{C})$ *is* *left proper: given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d] \\ B \arrow[r, "f_*"'] & D \end{tikzcd}$$

in which i is a cofibration, if f is an L-equivalence then f_* is an L-equivalence.

Proof. The original diagram may be replaced up to strict weak equivalence by a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "i"'] & C' \arrow[d] \\ B \arrow[r, "f'_*"'] & D' \end{tikzcd}$$

in which f' is a cofibration and an L-equivalence. But then f'_* is also an L-trivial cofibration and is in particular an L-equivalence. □

Lemma 24.14. *Every injective object is L-fibrant, so that the L-fibrant T-spectra coincide with the injective T-spectra.*

Proof. Suppose that X is injective, and suppose given a diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \\ B & \end{tikzcd}$$

where the morphism i is a cofibration and an L-equivalence. Then $\alpha = \alpha' \cdot j$ for some map $\alpha' : LA \to X$ since X is injective, and so there is a diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & LA \arrow[r, "\alpha'"] \arrow[d, "Li"] & X \\ B \arrow[r, "j"'] & LB & \end{tikzcd}$$

which factorizes the original. The map Li is a strict equivalence by Corollary 24.7.

One finishes the argument in the usual way: Li has a factorization

$$\begin{tikzcd} LA \arrow[r, "i'"] \arrow[dr, "Li"'] & W \arrow[d, "p"] \\ & LB \end{tikzcd}$$

where i' is a cofibration, p is a strict fibration and both maps are strict weak equivalences. Then X is strictly fibrant so there is a map $\sigma : W \to X$ such that $\sigma \cdot i' = \alpha'$, and there is a map $\theta : B \to W$ such that $p \cdot \theta = j$ and $\theta \cdot i = i' \cdot j$. $\square$

Now we can go further, to give a general recognition principle for L-fibrations. The most complete statement (Theorem 24.17 below) depends on right properness for the L-structure, which will be addressed in a subsequent section.

Lemma 24.15. *Suppose that $p : X \to Y$ is a strict fibration between L-fibrant T-spectra. Then p is an L-fibration.*

Proof. Suppose given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd} \tag{24.1}$$

where i is a cofibration and an L-equivalence. Then the induced map $i_* : LA \to LB$ is a strict equivalence, as are the L-fibrant model maps $j : X \to LX$ and $j : Y \to LY$. The induced diagram

$$\begin{tikzcd} LA \arrow[r] \arrow[d, "i_*"'] & LX \arrow[d, "p_*"] \\ LB \arrow[r] & LY \end{tikzcd}$$

has a factorization

$$\begin{tikzcd} LA \arrow[r, "j_A"] \arrow[d, "i_*"'] & V_X \arrow[r, "p_X"] \arrow[d, "i'"] & LX \arrow[d, "p_*"] \\ LB \arrow[r, "j_B"'] & V_Y \arrow[r, "p_Y"'] & LY \end{tikzcd}$$

such that j_A and j_B are strict trivial cofibrations and p_X and p_Y are strict fibrations. In the pullback diagram

$$\begin{tikzcd} V_X \times_{LX} X \arrow[r] \arrow[d, "j_{X*}"'] & X \arrow[d, "j_X"] \\ V_X \arrow[r, "p_X"'] & LX \end{tikzcd}$$

the map j_{X*} is a strict equivalence. The corresponding map j_{Y*} in the diagram

$$\begin{tikzcd} LA \arrow[r, "j_A"] \arrow[d] & V_X \arrow[d] & V_X \times_{LX} X \arrow[l, "j_{X*}"'] \arrow[d] \\ LB \arrow[r, "j_B"'] & V_Y & V_Y \times_{LY} Y \arrow[l, "j_{Y*}"] \end{tikzcd}$$

is also a strict equivalence. It follows that the induced map

$$V_X \times_{LX} X \to V_Y \times_{LY} Y$$

is a strict equivalence, and that the diagram (24.1) has a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & V_X \times_{LX} X \arrow[r] \arrow[d, "\simeq"] & X \arrow[d, "p"] \\ B \arrow[r] & V_Y \times_{LY} Y \arrow[r] & Y \end{tikzcd}$$

in which the middle vertical map is a strict equivalence. The result follows by a standard argument: one factorizes the middle vertical map as a trivial strict cofibration followed by a trivial strict fibration. □

Proposition 24.16. *Suppose that $p : X \to Y$ is a strict fibration. Then p is an L-fibration if* *the diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] & LX \arrow[d, "Lp"] \\ Y \arrow[r, "i"'] & LY \end{tikzcd} \tag{24.2}$$

is strictly homotopy cartesian.

Proof. Suppose that the diagram (24.2) is strictly homotopy cartesian. There is a factorization

$$\begin{tikzcd} LX \arrow[r, "j"] \arrow[dr, "Lp"'] & Z \arrow[d, "q"] \\ & LY \end{tikzcd}$$

of LP such that j is an L-equivalence and q is an injective fibration. But then Z is injective, hence L-fibrant, so that j is a strict equivalence. It also follows from Lemma 24.15 that q is an L-fibration. By pulling back q along i, we see from the hypothesis that the induced map

$$X \to Y \times_{LY} Z$$

is a strict equivalence. Every trivial strict fibration is an L-fibration, and it follows that p is a retract of an L-fibration, and hence is itself an L-fibration. $\square$

Theorem 24.17. *Suppose that the L-structure of Theorem 24.12 is right proper. Suppose that*

$p : X \to Y$ is a strict fibration. Then p is an L-fibration if and only if the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Lp} \\ Y & \xrightarrow[i]{} & LY \end{array} \tag{24.3}$$

is strictly homotopy cartesian.

Proof. We already have Proposition 24.16.

Suppose that the map $p : X \to Y$ is an L-fibration, and take a factorization

$$\begin{tikzcd} LX \arrow[r, "j"] \arrow[dr, "Lp"'] & Z \arrow[d, "q"] \\ & LY \end{tikzcd}$$

of the map Lp such that q is an L-fibration and j is an L-trivial cofibration. Then j is an L-equivalence between L-fibrant T-spectra, so that j is a strict equivalence on account of Lemma 24.11.

The induced map $i_* : Y \times_{LY} Z \to Z$ is an L-equivalence by the right properness assumption, so that the canonical map $\theta : X \to Y \times_{LY} Z$ is an L-equivalence, and the map

$$\begin{tikzcd} X \arrow[rr, "\theta"] \arrow[dr, "p"'] & & Y \times_{LY} Z \arrow[dl, "q_*"] \\ & Y & \end{tikzcd}$$

is an equivalence of fibrant objects for the model structure on $\mathrm{Spt}_T(\mathcal{C})/Y$ which is induced by the L-structure on $\mathrm{Spt}_T(\mathcal{C})$. Form the diagram

$$\begin{tikzcd}
V_1 \arrow[rr, "\tilde{\theta}"] \arrow[d, "\pi_1"'] & & V_2 \arrow[d, "\pi_2"] \\
X \arrow[r, "\theta"] \arrow[dr, "p"'] & Y \times_{LY} Z \arrow[d, "q_*"] & \\
& Y &
\end{tikzcd}$$

where π_1 and π_2 are trivial strict fibrations and V_1 and V_2 are cofibrant. Then $\tilde{\theta}$ is a weak equivalence between objects of $\mathrm{Spt}_T(\mathcal{C})/T$ which are both fibrant and cofibrant, and is therefore a (fibrewise) homotopy equivalence, and hence a strict weak equivalence. □

References

[1] J.F. Jardine. *Local Homotopy Theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.