Lecture 13

25 Descent theorems

1) **The Brown-Gersten descent theorem**

Suppose that S is a Noetherian scheme of finite dimension. Let $Zar|_S$ be the Zariski site of S.

Theorem 25.1. *Suppose that X is a simplicial presheaf on $Zar|_S$ such that*

1) the space $X(\emptyset)$ is contractible,

2) all stalks of X are contractible in the sense that the map $X_x \to *$ is a weak equivalences for each $x \in S$, and

3) the diagram

$$\begin{tikzcd} X(U \cup V) \arrow[r] \arrow[d] & X(U) \arrow[d] \\ X(V) \arrow[r] & X(U \cap V) \end{tikzcd}$$

associated to each pair of open subsets U, V of S is homotopy cartesian.

Then the map $X(U) \to *$ is a weak equivalence for each open subset U of S.

Proof. We show that $\pi_q X(U)$ is trivial for each $q \geq 0$ and each choice of base point $x \in X(U)$ under the assumption that $X(U) \neq \emptyset$ and $U \neq \emptyset$.

Suppose that $\alpha \in \pi_q X(U)$. Pick a maximal open subset $V \subset U$ such that $\alpha \mapsto 0$ in $\pi_q X(V)$. There are such subsets since $\pi_q X_x = 0$ for all $x \in U$, and S is Noetherian.

Say that a closed irreducible subset $C \subset S$ is *bad* if there is such an α, U, V such that $C \cap U \neq \emptyset$ and $C \subset S - V$. If some $V \neq U$ there are bad subsets C: this would be a closure in S of an irreducible component of $U - V$.

Pick a maximal bad subset C, with associated data $\alpha \in \pi_q X(U)$, maximal open $V \subset U$ such that $\alpha \mapsto 0$ in $\pi_q X(V)$, and such that C intersects U but misses V.

Take $y \in C \cap U$. There is an open subset $W \subset U$ such that $y \in W$ and $\alpha \mapsto 0$ in $\pi_q X(W)$. A long exact sequence argument says that there is an element $z \in \pi_{q+1} X(V \cap W)$ such that

$$\partial(z) = \alpha|_{V \cup W} \in \pi_q X(V \cup W).$$

Pick a maximal open subset $V' \subset V \cap W$ such that $z \mapsto 0$ in $\pi_{q+1} X(V')$.

Then C is a component of $S - V'$. In effect, C is contained in some component D of $S - V'$. If $D \cap V = \emptyset$ then $y \in C \cap U \subset D \cap U$ so that D is bad (for α) and $C = D$ by the maximality of C. If $D \cap V$ is non-empty then $D \cap U \neq \emptyset$ so that $D \cap (U \cap V) \neq \emptyset$ since D is irreducible, while $D \cap V' = \emptyset$ and D is bad (for z), and $C = D$ by maximality of C.

Suppose that $C, C_1, \dots, C_k$ is a list of the irreducible components of $X - V'$, and let F be the closed subset of $X - V'$ defined by the union

$$F = C_1 \cup \cdots \cup C_k.$$

Then $C - F$ is a non-trivial open subset of C as is $W \cap C$, and it follows that the intersection

$$(W - F) \cap C = (W \cap C) \cap (C - F) = W \cap (C - F)$$

is a non-trivial open subset of C (which is outside V) since C is irreducible. At the same time,

$$X - F = V' \cup (C - F),$$

so that

$$W - F = V' \cup (W \cap (C - F))$$

and $V \cap (W - F) = V'$.

It follows that, in the diagram

$$\begin{array}{ccc} \pi_{q+1}X(V\cap W) & \xrightarrow{\partial} & \pi_q X(V\cup W) \\ \downarrow & & \downarrow \\ \pi_{q+1}X(V\cap(W-F)) & \xrightarrow[\partial]{} & \pi_q X(V\cup(W-F)) \end{array}$$

the element $z \in \pi_{q+1}X(V\cap W)$ maps to zero in $\pi_{q+1}X(V\cap(W-F))$, so that $\alpha \in \pi_q X(U)$ restricts to 0 in $\pi_q X(V\cup(W-F))$. This contradicts the maximality of V, and it follows that there are no bad closed irreducible subsets in X.

We have therefore shown that there is a weak equivalence $X(U) \to *$ if $X(U) \neq \emptyset$. I claim that $X(S) \neq \emptyset$, and it follows that all $X(U)$ are not empty.

Suppose that $X(S) = \emptyset$. Pick a maximal non-empty open subset $U \subset S$ such that $X(U) \neq \emptyset$. Take $x \in S - U$ and pick an open subset $V \subset S$ with $y \in V$ and $X(V) \neq \emptyset$. The open subsets U and V exist because all stalks of X are non-empty. Then there is a homotopy cartesian diagram

$$\begin{array}{ccc} X(U\cup V) & \longrightarrow & X(U) \\ \downarrow & & \downarrow \\ X(V) & \longrightarrow & X(U\cap V) \end{array}$$

in which $X(U)$, $X(V)$ and $X(U\cup V)$ are non-empty contractible spaces. Then a homotopy lifting argument shows that $X(U \cup V)$ is non-empty. This contradicts the maximality of U if $U \neq S$. $\square$

The following result is the Brown-Gersten descent theorem:

Theorem 25.2. *Suppose that X is a simplicial presheaf on $Zar|_S$ such that*

1) *the map $X(\emptyset) \to *$ is a weak equivalence, and*

2) *the diagram*

$$\begin{tikzcd} X(U \cup V) \arrow[r] \arrow[d] & X(U) \arrow[d] \\ X(V) \arrow[r] & X(U \cap V) \end{tikzcd}$$

associated to each pair of open subsets U, V of S is homotopy cartesian.

Let $j : X \to Z$ be an injective fibrant model. Then j is a sectionwise equivalence.

Proof. It suffices to show that the induced map $j : X(S) \to Y(S)$ is a weak equivalence. The map $X(U) \to Y(U)$ is global sections of the restriction of $j|_U$ to the Zariski site $Zar|_U$, for all open subschemes $U \subset S$, and the restricted map $j|_U$ is an injective fibrant model by Lemma 16.3.

Find a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"'] & Y \arrow[d, "p"] \\ & Z \end{tikzcd}$$

such that i is a sectionwise equivalence and p is a sectionwise Kan fibration. Then the simplicial presheaf Y satisfies conditions 1) and 2) of the statement of the Theorem, and the local weak equivalence $p : Y \to Z$ is an injective fibrant model for Y.

Suppose that $x \in Z(S)$ is a vertex of $Z(S)$, and form the pullback diagram

$$\begin{tikzcd} F_x \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ * \arrow[r, "x"'] & Z \end{tikzcd}$$

in simplicial presheaves. Then the simplicial presheaf F_x satisfies the conditions of Theorem 25.1, and is therefore sectionwise contractible.

In particular, the map $F_x(S) \to *$ is a weak equivalence, so that $F_x(S)$ is non-empty, and the vertex x lifts to $Y(S)$. This is true for all vertices of $Z(S)$, so the induced map $\pi_0 Y(S) \to \pi_0 Z(S)$ is surjective.

All fibres $F_{p(y)}$ associated to all vertices $y \in Y(S)$ are sectionwise contractible. It follows that the map $\pi_0 Y(S) \to \pi_0 Z(S)$ is injective, and that all homomorphisms

$$\pi_n(Y(S), y) \to \pi_n(Z(S), p(y))$$

are isomorphisms. □

2) **The Nisnevich descent theorem**

Following [4], we use the notation $(Sm|_S)_{Nis}$ to denote the category of smooth S-schemes with the Nisnevich topology.

An *elementary distinguished square* is a pullback diagram in $(Sm|_S)_{Nis}$

$$\begin{tikzcd} \phi^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "\phi"] \\ U \arrow[r, "j"'] & T \end{tikzcd} \tag{25.1}$$

such that j is an open immersion, ϕ is étale, and such that the induced morphism

$$\phi^{-1}(T - U) \to T - U$$

of closed subschemes (with reduced structure) is an isomorphism.

Remark 25.3. An elementary distinguished square is completely specified by a diagram

$$\begin{tikzcd} Z \times_T V \arrow[r] \arrow[d, "\phi_*", "\cong"'] & V \arrow[d, "\phi"] \\ Z \arrow[r, "i"'] & T \end{tikzcd}$$

such that ϕ is étale and i is a closed immersion. In effect, if Z is reduced, then $Z \times_T V$ is reduced since ϕ_* is étale [3], and is therefore the reduced closed subscheme of V on the closed subset $\phi^{-1}(Z)$.

Example 25.4. If U and V are open subschemes of a smooth S-scheme T, then the diagram of inclusions

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \cup V \end{tikzcd}$$

is an elementary distinguished square in $Sm|_S$.

Example 25.5. Suppose that $x \in S$ is a closed point of S, and suppose that $\phi : U \to S$ is an étale morphism such that there is a section

$$\begin{tikzcd} & U \arrow[d, "\phi"] \\ \mathrm{Sp}(k(x)) \arrow[ur, "y"] \arrow[r, "x"'] & S \end{tikzcd}$$

over the residue field $k(x)$ of x. If $\phi(z) = x$, then z and x have the same (maximal) dimension [3, I.3.16], so that z is closed in U. The set-theoretic fibre $\phi^{-1}(x)$ is therefore a finite set of closed points, of the form

$$\phi^{-1}(x) = \{y, y_1, \ldots, y_k\}.$$

Let V be the open subset $U - \{y_1, \ldots, y_k\}$ of U, and let $\phi|_V$ be the restiction of ϕ to V. Then there is a diagram

$$\begin{tikzcd} & V \arrow[d, "\phi|_V"] \\ \mathrm{Sp}(k(x)) \arrow[ur, "y"] \arrow[r, "x"'] & S \end{tikzcd}$$

Then $\phi|_V$ induces an isomorphism

$$\mathrm{Sp}(k(y)) \cong \mathrm{Sp}(k(x)),$$

and $\mathrm{Sp}(k(y))$ is the reduced closed fibre of $\phi|_V$ over the closed subscheme $\mathrm{Sp}(k(x))$ of S. Let U be the open subscheme $S - \{x\}$ of S, with inclusion $j : U \subset S$. Then the pullback diagram

$$\begin{tikzcd} \phi|_V^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "\phi_V"] \\ U \arrow[r, "j"'] & S \end{tikzcd}$$

is an elementary distinguished square.

Every elementary distinguished square defines a Nisnevich cover $\{j : U \subset T, \phi : V \to T\}$ of

X, because every map $\mathrm{Sp}(k) \to X$ with k a field factors through one of the two maps.

Following [4], say that a simplicial presheaf X *has the BG-property* if

1) the space $X(\emptyset)$ is contractible, and
2) X takes elementary distinguished squares (25.1) to homotopy cartesian diagrams

$$\begin{tikzcd} X(T) \arrow[r, "j^*"] \arrow[d, "\phi^*"'] & X(V) \arrow[d] \\ X(U) \arrow[r] & X(\phi^{-1}(U)) \end{tikzcd} \tag{25.2}$$

If X has the BG-property and U, V are open subschemes of a smooth S-scheme T, then the diagram

$$\begin{tikzcd} X(U \cup V) \arrow[r] \arrow[d] & X(V) \arrow[d] \\ X(U) \arrow[r] & X(U \cap V) \end{tikzcd}$$

is homotopy cartesian, so that the restriction of X to the Zariski site $Zar|_T$ satisfies the conditions of Theorem 25.2. It follows in particular that the canonical map

$$X(U \sqcup V) \to X(U) \times X(V)$$

is a weak equivalence for all smooth S-schemes U, V, which means precisely that the simplicial presheaf X is *additive* — see [1].

Lemma 25.6. *Suppose that Z is an injective fibrant simplicial presheaf on the smooth Nisnevich site $(Sm|_S)_{Nis}$. Then Z has the BG-property.*

Proof. Every open immersion $j : U \to T$ is a cofibration of simplicial presheaves, and all induced inclusions

$$(U \times \Delta^n) \cup (T \times \Lambda^n_k) \subset T \times \Delta^n$$

are trivial cofibrations. It follows that the map $j^* : Z(T) \to Z(U)$ is a Kan fibration.

The square (25.1) is a pushout in the category of sheaves (and simplicial sheaves) for the Nisnevich topology on the smooth site $Sm|_S$. Thus, if Z' is an injective fibrant simplicial sheaf, then the diagram of simplicial set maps

$$\begin{tikzcd} Z'(T) \arrow[r, "j^*"] \arrow[d, "\phi^*"'] & Z'(V) \arrow[d] \\ Z'(U) \arrow[r] & Z'(\phi^{-1}(U)) \end{tikzcd}$$

is a pullback in which both vertical maps are Kan fibrations, and is therefore homotopy cartesian.

If Z is an injective fibrant simplicial presheaf, there is a local weak equivalence $\eta : Z \to Z'$ such that Z' is an injective fibrant simplicial sheaf. The map η is a sectionwise weak equivalence, and the property of taking elementary distinguished squares to homotopy cartesian diagrams is an invariant of sectionwise equivalence.

The map η induces a weak equivalence

$$Z(\emptyset) \to Z'(\emptyset) \cong *$$

of simplicial sets. □

It makes perfect sense to talk about simplicial presheaves X on the small Nisnevich site $(et|_S)_{Nis}$ which have the BG-property: one restricts the discussion to S-schemes $T \to S$ which are étale over S. Then a simplicial presheaf Y on the smooth site $(Sm|_S)_{Nis}$ has the BG-property if and only if the restrictions to the small sites $(et|_T)_{Nis}$ have the BG-property, for all smooth S-schemes $T \to S$.

Now here is the analogue of Theorem 25.1 for the Nisnevich topology:

Theorem 25.7. *Suppose that X is a simplicial presheaf on the small Nisnevich site $(et|_S)_{Nis}$ such that*

1) X has the BG-property, and

2) the map $X \to *$ is a local weak equivalence for the Nisnevich topology.

Then X is sectionwise contractible in the sense that the map $X(U) \to *$ is a weak equivalence of simplicial sets for each étale S-scheme $U \to S$.

Proof. It suffices to show that the global sections map $X(S) \to *$ is a weak equivalence. The restriction of X to the site $(et|_T)$ for each étale S-scheme $T \to S$ also satisfies conditions 1) and 2), and the map $Z(T) \to *$ would be a weak equivalence for each T.

Write $\mathcal{O}_x$ for the local ring $\mathcal{O}_{x,S}$ of $x \in S$, and let $x : \mathrm{Sp}(\mathcal{O}_x) \to S$ be the canonical map.

Suppose that $\phi : T \to S$ is an S-scheme, and write

$$T_x = \mathrm{Sp}(\mathcal{O}_x) \times_S T.$$

Let x^p be the left adjoint of the direct image functor

$$x_* : \mathrm{Pre}(et|_{\mathrm{Sp}(\mathcal{O}_x)})_{Nis} \to \mathrm{Pre}(et|_S)_{Nis},$$

where

$$x_*(Y)(T) = Y(T_x).$$

The global sections simplicial set $x^p X(\mathcal{O}_x)$ is the Zariski stalk of X at the point x. The functor x^p preserves local weak equivalences for the Nisnevich topology, since it is defined by a site morphism.

It is a consequence of Lemma 25.8 below that $x^p X$ satisfies the BG-property on $Sm|_{\mathrm{Sp}(\mathcal{O}_x)}$.

Suppose that $\mathcal{O}_x$ has dimension 0, so that $\mathcal{O}_x$ is an Artinian local ring. It well known that the functor

$$U \mapsto U \times_{\mathrm{Sp}(\mathcal{O}_x)} \mathrm{Sp}(k(x))$$

defines an equivalence of categories

$$et|_{\mathrm{Sp}(\mathcal{O}_x)} \to et|_{\mathrm{Sp}(k)}.$$

Every diagram

$$\begin{tikzcd} & U \arrow[d] \\ \mathrm{Sp}(k(x)) \arrow[r] \arrow[ur] & \mathrm{Sp}(\mathcal{O}_x) \end{tikzcd}$$

with ϕ étale therefore determines a section $\sigma : \mathrm{Sp}(\mathcal{O}_x) \to U$ of the map ϕ. It follows that the global sections functor takes sheaf epimorphisms on the Nisnevich site $(Sm|_{\mathrm{Sp}(\mathcal{O}_x)})_{Nis}$ to surjections. In effect, if $p : F \to F'$ is a sheaf epi and $\alpha \in F'(\mathcal{O}_x)$ there is an étale map $\phi : U \to \mathrm{Sp}(\mathcal{O}_x)$ having a section σ such that $\phi^*(\alpha)$ lifts to $F(U)$, and then $\alpha = \sigma^*\phi^*(\alpha)$ lifts to $F(\mathcal{O}_x)$. It follows that the global sections functor $X \mapsto X(\mathcal{O}_x)$ takes local weak equivalences to weak equivalences of simplicial sets.

Thus, if $x \in S$ has dimension 0, and the simplicial presheaf X satisfies the conditions of the Theorem, then $X(\mathcal{O}_x)$ is contractible. This is true for all schemes S which are Noetherian and of finite dimension.

We show by induction on the dimension of $x \in S$ that $X(\mathcal{O}_x)$ is contractible. Take an element $x \in S$ and assume that $X(\mathcal{O}_y)$ is contractible for all points y (in "all" schemes S) of smaller dimension.

Write x for the closed point of $\mathrm{Sp}(\mathcal{O}_x)$, and suppose given an element $\alpha \in \pi_k x^p X(\mathcal{O}_x)$. Then α is 0 locally for the Nisnevich topology, so that, following the prescription of Example 25.5, there is an étale morphism $\phi : V \to \mathrm{Sp}(\mathcal{O}_x)$ with a diagram

$$\begin{tikzcd} V \times_{\mathrm{Sp}(O_x)} \mathrm{Sp}(k(x)) \arrow[r] \arrow[d, "\cong"'] & V \arrow[d, "\phi"] \\ \mathrm{Sp}(k(x)) \arrow[r, "x"'] & \mathrm{Sp}(\mathcal{O}_x) \end{tikzcd}$$

such that $\phi^*(\alpha) = 0$ in $\pi_k x^p X(V)$. Write $U = \mathrm{Sp}(\mathcal{O}_x) - \{x\}$. Then all points of U and all points of $\phi^{-1}(U)$ have dimension smaller than that of x, and $x^p X$ satisfies the assumption of the Theorem. It follows from Theorem 25.1 that the spaces $x^p X(U)$ and $x^p X(\phi^{-1}(U))$ are contractible. Then $x^p X$ satisfies the BG-property, and it follows that the map

$$\phi^* : x^p X(\mathcal{O}_x) \to x^p X(V)$$

is a weak equivalence. But then $\alpha = 0$ in $\pi_k x^p X(\mathcal{O}_x)$.

All homotopy groups and the set of path components of $x^p X(\mathcal{O}_x)$ are therefore trivial if the space $x^p X(\mathcal{O}_x)$ is non-empty.

For this, we can find a diagram

$$\begin{tikzcd} & V \arrow[d, "\phi"] \\ \mathrm{Sp}(k(x)) \arrow[ur] \arrow[r, "x"'] & S \end{tikzcd}$$

with ϕ étale and such that $X(V) \neq \emptyset$, since all Nisnevich stalks of X are non-empty. Pull back the map ϕ over $\mathrm{Sp}(\mathcal{O}_x)$ to create a picture

$$\begin{tikzcd} & V_x \arrow[d, "\phi_x"] \\ \mathrm{Sp}(k(x)) \arrow[ur, "y"] \arrow[r, "x"'] & \mathrm{Sp}(\mathcal{O}_x) \end{tikzcd}$$

with $x^p X(V_x) \neq \emptyset$. Now cut out all closed points of V_x in the fibre over x except for y to construct a picture

$$\begin{tikzcd} V' \otimes_{\mathcal{O}_x} k(x) \arrow[r] \arrow[d, "\cong"'] & V' \arrow[d, "\phi'"] \\ \mathrm{Sp}(k(x)) \arrow[r, "x"'] & \mathrm{Sp}(\mathcal{O}_x) \end{tikzcd}$$

just as before, but with $x^p X(V') \neq \emptyset$. The induced map

$$\phi'^* : x^p X(\mathcal{O}_x) \to x^p X(V')$$

is a weak equivalence once again, so that $x^p X(\mathcal{O}_x)$ is non-empty. $\square$

Lemma 25.8. *Suppose that the simplicial presheaf X on $(Sm|_S)_{Nis}$ has the BG-property, and let $\mathcal{O}_x$ be the local ring of $x \in S$ with canonical map $x : \mathrm{Sp}(\mathcal{O}_x) \to S$. Then the inverse image $x^p X$ on $(Sm|_{\mathrm{Sp}(\mathcal{O}_x)})_{Nis}$ has the BG-property.*

Proof. Suppose that $f : T \to \mathrm{Sp}(\mathcal{O}_x)$ is a $\mathcal{O}_x$-scheme which is locally of finite type. Then there is an open affine neighbourhood U of x in S and a U-scheme $f' : T' \to U$ which is locally of finite type, with an isomorphism of $\mathcal{O}_x$-schemes

$$T \cong \mathrm{Sp}(\mathcal{O}_x) \times_U T'$$

If f is an open immersion, respectively closed immersion, or étale, then the "thickening" f' can be chosen to have the same property. In particular, if $\phi : V \to \mathrm{Sp}(\mathcal{O}_x)$ is étale and has étale thickening $\phi' : V' \to U$ over an open neighbourhood U, then there is an isomorphism

$$x^p X(V) = \varinjlim_{x \in W \subset U} X(W \times_U V'),$$

where W varies over the open neighbourhoods of x which are contained in U.

It follows that if $j : Z \to V$ is a closed immersion in an étale $\mathcal{O}_x$-scheme $\phi : V \to \mathrm{Sp}(\mathcal{O}_x)$, and $\psi : \tilde{V} \to V$ is an étale morphism with pullback diagram

$$\begin{tikzcd} Z \times_V \tilde{V} \arrow[r] \arrow[d, "\cong"'] & \tilde{V} \arrow[d, "\psi"] \\ Z \arrow[r, "j"'] & V \end{tikzcd}$$

then there is a thickened diagram

$$\begin{tikzcd} Z' \times_{V'} \tilde{V}' \arrow[r] \arrow[d, "\cong"'] & \tilde{V}' \arrow[d, "\psi'"] & \\ Z' \arrow[r, "j'"'] & V' \arrow[dr, "\phi'"] & \\ & & U \end{tikzcd}$$

over some open neighbourhood U of x. The corresponding elementary distinguished square

$$\begin{tikzcd} \psi^{-1}(V-Z) \arrow[r] \arrow[d] & \tilde{V} \arrow[d, "\psi"] \\ V-Z \arrow[r, "j"'] & V \end{tikzcd}$$

therefore has a thickening

$$\begin{tikzcd} \psi'^{-1}(V'-Z') \arrow[r] \arrow[d] & \tilde{V}' \arrow[d, "\psi'"] \\ V'-Z' \arrow[r, "j'"'] & V' \end{tikzcd}$$

over U, and the diagram

$$\begin{tikzcd} x^pX(V) \arrow[r] \arrow[d] & x^pX(V-Z) \arrow[d] \\ x^pX(\tilde{V}) \arrow[r] & x^pX(\psi^{-1}(V-Z)) \end{tikzcd} \tag{25.3}$$

is a filtered colimit of homotopy cartesian squares

$$\begin{tikzcd} X(W\times_U V') \arrow[r] \arrow[d] & X((W\times_U V')-(W\times_U Z')) \arrow[d] \\ X(W\times_U \tilde{V}) \arrow[r] & X(\psi^{-1}((W\times_U V')-(W\times_U Z'))) \end{tikzcd}$$

The diagram (25.3) is therefore homotopy cartesian. □

Here is the Morel-Voevodsky statement of the Nisnevich descent theorem:

Theorem 25.9. *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves on $(Sm|_S)_{Nis}$, and suppose that both X and Y satisfy the BG-property. Then all maps $X(T) \to Y(T)$ in sections are weak equivalences of simplicial sets.*

Proof. Suppose that $x \in Y(S)$ is a global section, and let F_x be the sectionwise homotopy fibre of the map f. Then the restriction of F_x to the small site $(et|_S)_{Nis}$ satisfies the hypotheses of Theorem 25.7, and so the map $F_x(T) \to *$ is a weak equivalence for all étale S-schemes T. It follows that the map

$$f : X(S) \to Y(S)$$

in global sections is a weak equivalence.

All restrictions

$$j|_T : X|_T \to Y|_T$$

to $(Sm|_T)_{Nis}$ for smooth S-schemes T satisfy the same assumptions, so that all maps $X(T) \to Y(T)$ are weak equivalences. $\square$

The following result is the analogue, for the Nisnevich topology, of Theorem 25.2. The statement is equivalent to Theorem 25.9.

Theorem 25.10. *Suppose that X is a simplicial presheaf on $(Sm|_S)_{Nis}$ which satisfies the BG-property, and let $j : X \to Z$ be an injective fibrant model for the Nisnevich topology. Then all maps $X(T) \to Z(T)$ in sections are weak equivalences of simplicial sets.*

3) Motivic descent

In all that follows, given simplicial presheaves X, Y, the *internal function complex* $\mathbf{Hom}(X,Y)$ is the simplicial presheaf with

$$\mathbf{Hom}(X,Y)(U) = \mathbf{hom}(X|_U, Y|_U)$$

for U in the underlying site $\mathcal{C}$. The natural isomorphism

$$\hom(X \times A, Y) \cong \hom(A, \mathbf{Hom}(X,Y))$$

is the exponential law for simplicial presheaves A, X and Y. Given an injective fibration $p : X \to Y$ and a cofibration $i : A \to B$, then an adjointness argument implies that the induced map

$$\mathbf{Hom}(B,X) \xrightarrow{(i^*,p_*)} \mathbf{Hom}(A,X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B,Y)$$

is an injective fibration which is trivial if either i or p is trivial.

Recall from the examples in Section 22 (Lecture 10) that the motivic model structure on the simplicial presheaf category

$$s\,\mathrm{Pre}(Sm|_S)_{Nis}$$

can be constructed by specializing Theorem 22.2 to the case where S is the generating set of trivial cofibrations for the injective model structure on

$$s\,\mathrm{Pre}(Sm|_S)_{Nis}$$

and the interval I is the affine line $\mathbb{A}^1$.

In particular, injective (equivalently fibrant) objects for the theory are defined by having the right lifting property with respect to the maps

$$(C \times \square^n) \cup (D \times \sqcap^n_{(i,\epsilon)}) \subset D \times \square^n \qquad (25.4)$$

where $C \to D$ is a member of the set of generating cofibrations for $s\,\mathrm{Pre}(\mathcal{C})$, and the maps

$$(A \times \square^n) \cup (B \times \partial\square^n) \subset B \times \square^n \qquad (25.5)$$

with $A \to B$ in the generating set S of trivial cofibrations.

Recall the notation: $\square^n = I^{\times n}$, (which in the present case is the affine plane $\mathbb{A}^n$), and there are face inclusions

$$d^{i,\epsilon} : \square^{n-1} \to \square^n,\ 1 \leq i \leq n,\ \epsilon = 0, 1,$$

with

$$d^{i,\epsilon}(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{i-1}, \epsilon, x_i, \dots, x_{n-1}).$$

Then there are subobjects $\partial\square^n$ and $\sqcap^n_{i,\epsilon}$ of $\square^n$ which are defined, respectively, by

$$\partial\square^n = \cup_{i,\epsilon}\ d^{i,\epsilon}(\square^{n-1}),$$

and

$$\sqcap^n_{i,\epsilon} = \cup_{(j,\gamma)\neq(i,\epsilon)}\ d^{j,\gamma}(\square^{n-1}).$$

Observe that

$$\square^m \times \square^n = \square^{m+n},$$

and that there are induced relations

$$\begin{aligned}
(\partial\square^m \times \square^n) \cup (\square^m \times \partial\square^n) &= \partial\square^{m+n} \\
(\sqcap^m_{i,\epsilon} \times \square^n) \cup (\square^m \times \partial\square^n) &= \sqcap^{m+n}_{i,\epsilon} \\
(\partial\square^m \times \square^n) \cup (\square^m \times \sqcap^n_{j,\epsilon}) &= \sqcap^{m+n}_{m+j,\epsilon}
\end{aligned} \tag{25.6}$$

Lemma 25.11. *A simplicial presheaf X is injective for the motivic model structure if and only if X is an injective fibrant simplicial presheaf (for the Nisnevich topology) and the injective fibration*

$$0^* : \mathbf{Hom}(\mathbb{A}^1, X) \to \mathbf{Hom}(*, X)$$

is trivial.

Proof. If X is injective, then X has the right lifting property with respect to all generating trivial cofibrations ((25.5), $n = 0$), and is therefore injective fibrant.

The object X also has the right lifting property with respect to the maps

$$(C \times \mathbb{A}^1) \cup (D \times *) \to D \times \mathbb{A}^1$$

defined by the set of generating cofibrations $C \to D$ ((25.4), $n = 1$). It follows that the map

$$0^* : \mathbf{Hom}(\mathbb{A}^1, X) \to \mathbf{Hom}(*, X)$$

has the right lifting property with respect to all $C \to D$, and is therefore a trivial injective fibration.

For the converse, the map 0^* has the right lifting property with respect to all cofibrations

$$(C \times \square^k) \cup (D \times \partial\square^k) \subset D \times \square^k,$$

and so the relations (25.6) can be used to show that X has the right lifting property with respect to all inclusions (25.4). The simplicial presheaf X also has the right lifting property with respect to all trivial cofibrations

$$(A \times \square^n) \cup (B \times \partial\square^n) \subset B \times \square^n$$

which are induced by trivial cofibrations $A \to B$. It follows that X is an injective object. $\square$

Remark 25.12. Suppose that S consists of all generating trivial cofibrations $A \to B$ for the injective structure plus the map $0 : * \to \mathbb{A}^1$ and the interval I is Δ^1.

If Z is injective (ie. fibrant) for this structure, then Z is injective fibrant, and $* \to \mathbb{A}^1$ is a weak equivalence, so that all maps

$$(C \times \mathbb{A}^1) \cup (D \times *) \subset D \times \mathbb{A}^1$$

induced by cofibrations $C \to D$ are weak equivalences. It follows that the map

$$0^* : \mathbf{Hom}(\mathbb{A}^1, Z) \to \mathbf{Hom}(*, Z)$$

is a trivial injective fibration.

Conversely, if Z is injective fibrant and 0^* is trivial, then Z has the right lifting property with respect to all (local) trivial cofibrations (25.5), and the map 0^* has the right lifting property with respect to all cofibrations

$$(C \times \square^k) \cup (D \times \partial\square^k) \subset D \times \square^k.$$

It follows that Z has the right lifting property with respect to the cofibrations (25.5) (use the relations (25.6)).

Suppose that the simplicial presheaf X is injective for the Nisnevich topology. The injective fibration

$$0^* : \mathbf{Hom}(\mathbb{A}^1, X) \to \mathbf{Hom}(*, X)$$

is given in sections corresponding to smooth S-schemes $T \to S$ by the map

$$X(\mathbb{A}^1 \times T) \to X(T)$$

associated to the 0-sections map $T \to \mathbb{A}^1 \times T$. The injective fibration 0^* is a local weak equivalence if and only if it is a sectionwise weak equivalence. The latter is equivalent to the assertion that all projections $\mathbb{A}^1 \times T \to T$ induce weak equivalences

$$X(T) \to X(\mathbb{A}^1 \times T). \tag{25.7}$$

Remark 25.13. In general, if the map (25.7) is a weak equivalence for all smooth S-schemes T, we say that X has or satisfies the *homotopy property*. The term comes from algebraic K-theory: it is a central result of the subject (and a theorem of Quillen [5]) that the algebraic K-theory functor satisfies the homotopy property for all regular Noetherian schemes T. Explicitly, this means that the projection $\mathbb{A}^1 \times T \to T$ induces a weak equivalence

$$K(T) \xrightarrow{\simeq} K(\mathbb{A}^1 \times T)$$

of spaces or spectra for all such T.

The homotopy property is also a central concept for other geometric cohomology theories: the assertion that étale cohomology with torsion coefficients satisfies the homotopy property is a consequence of the smooth base change theorem [3].

The following "motivic descent theorem" is a corollary of the Nisnevich descent theorem (Theorem 25.10):

Theorem 25.14. *Suppose that X is a simplicial presheaf on $(Sm|_S)_{Nis}$ such that*

1) *X satisfies the BG-property, and*
2) *every projection $\mathbb{A}^1 \times T \to T$ induces a weak equivalence*

$$X(T) \to X(\mathbb{A}^1 \times T).$$

Let $j : X \to Z$ be a motivic fibrant model. Then j is a sectionwise weak equivalence. Conversely, if a motivic fibrant model $j : X \to Z$ is a sectionwise weak equivalence, then X satisfies conditions 1) and 2).

Proof. Suppose that X satisfies conditions 1) and 2), and let $j : X \to Z$ be an injective fibrant model for the Nisnevich topology. Then j is a sectionwise equivalence by Theorem 25.10. All 0-section maps $T \to \mathbb{A}^1 \times T$ (these are sections of projections) induce weak equivalences

$$Z(\mathbb{A}^1 \times T) \to Z(T).$$

It follows that the injective fibration

$$0^* : \mathbf{Hom}(\mathbb{A}^1, Z) \to \mathbf{Hom}(*, Z)$$

is trivial, so that Z is motivic fibrant.

The converse is a consequence of Lemma 25.11. $\square$

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