[To do:
E_2 of the Serre spectral sequence
Some more basic Serre spectral sequence examples
potentially the Atiyah–Hirzebruch Leray–Serre spectral sequence
equivalence of cohomology of A_{PL} with singular cohomology
the Serre spectral sequence for A_{PL}
simplicial sets and product theorem
examples of symmetric spaces
Better interstitial wording
Expand treatment of Leray spectral sequence to give modern, full-strength version Pin down citations/attributions of theorems, expand theorem/author indices]

The rational cohomology of homogeneous spaces

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To Mom, Dad, and Drew,

who love me despite my failings,

and to Loring, who had faith I could write it

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Apology: How this book came to be

This monograph evolved from a dissertation whose purpose was to explore the satisfiability and consequences of a technical condition on Borel equivariant cohomology called *equivariant formality*, as applied to the *isotropy action* on a homogeneous space G/K, which the author eventually found required a detailed study of the singular cohomology of that space.

The standard way to compute $H^*(G/K;\mathbb{R})$ is to identify it with the Borel equivariant cohomology $H^*_K(G;\mathbb{R})$ and to determine it using the *Cartan model*. This model is already discused by the monograph of Guillemin and Sternberg [GS99] and by the forthcoming text [Tuar] of Tu, among other places, and a standard discussion involves a level of differential geometry and Lie theory the present author wanted to avoid in the present work. As applied to compute $H^*(G/K;\mathbb{R})$ it is also discussed in the tomes of Onishchik and of Greub–Halperin–Vanstone [Oni94, GHV76]. As it turns out, the Cartan model in the case of the author's thesis can be concisely constructed from mostly algebraic considerations, avoiding structure constants and indices, using the Serre spectral sequence and simple algebraic models, in a way which is much more economical and makes few presuppositions of the reader. The resulting theory is both simple and beautiful. Moreover, its historical development involved the discovery of spectral sequences, classifying spaces, and commutative models in rational homotopy theory, and thus an exposition of this historical question, surprisingly, gives a perfect opportunity to develop many fundamental notions of algebraic topology which fit together nicely into a second graduate course.

The author, having taught himself this material, initially put his own development in his dissertation operating on the spurious assumption that committee members would appreciate having all the background in one place. By the time he realized this was incorrect, he was already committed to producing a document that could serve as a reference.[1]

The existing literature

To explain our insistence on presenting yet another version of an old if insufficiently publicized story, some discussion of other expositions is in order.

The primary literature predominantly dates to a movement from 1949–53 clustered around Henri Cartan, and is presented rather telegraphically, littered with references to results whose proofs were never published, and reliant on an early version of sheaf theory which is now virtually forgotten. (These works will be cited in historical commentary throughout, especially in Chapter 8 and a version of this early account of sheaf theory and Borel's original derivation of the Cartan model are written up in Appendix C.) There are also long surveys by André and by

[1] It may also be that realizing his dissertation was the only published document he was ever likely to have complete creative control over, he went somewhat overboard. The present account is somewhat more streamlined than the thesis itself.

Rashevskii [And62, Ras69] summarizing the results of this school in greater detail, but still aimed at the professional, but the main secondary sources in English are the books of Onishchik and of Greub–Halperin–Vanstone [Oni94, GHV76].

Onishchik is relatively concise at under 300 pages, and surprisingly difficult to lay hands on. With a view toward classifying pairs (G, K) of compact, connected Lie groups with respect to the diffeomorphism, homeomorphism, or homotopy type of G/K, it develops Lie theory, a real version of Sullivan's rational homotopy theory, the Weil algebra, the theory of symmetric invariants, and the Cartan model. The Weil algebra appears *ex nihilo*, as it were, without reference to the connection and curvature forms associated to a principal bundle which were Weil's motivation. It is notable that through diligent use of filtration arguments, Onischik manages to completely avoid invoking spectral sequences. His end goal is classfication problems relating transitive actions

The book of Greub, Halperin, and Vanstone, on the other hand, comprises nearly 600 pages. It develops the necessary background in great generality, finally arriving at the target results on the cohomology of homogeneous spaces in Ch. XI [GHV76, p. 457]. The development is an earlier language than that now current[2] and the notation, which is highly nonstandard, is, as Samelson notes in his otherwise favorable review [Sam77], not indexed. The book's thoroughness and the generality of the formulations result perforce in an ouroboric format where the topological results at the end are notational permutations of algebraic results obtained toward the beginning. Hence the most possible is said about any topic touched, and reading a proof involving a topological space is a recursive process with three to four iterations.[3] The list of notations used by Onishchik is unfortunately incomplete.

Our approach

The present book cannot hope and would not presume to compete with the existing secondary literature in terms of scope or depth. What it can do, by way of contrast, is present the material as briefly and directly as possible, through a purely topological lens, assuming minimal prerequisites, and with a complete index of notation. Thus the goal of this monograph is to arrive along a quasigeodesic path at Chevalley's and Cartan's respective theorems on the cohomology of principal bundles $G \to P \to B$ and homogeneous spaces G/K, showing both how one computes this in general and in many specific examples. There are many other paths one could go down along the way, and throughout these detours are clearly marked. The Serre spectral sequence is developed from scratch, Lie theory is quoted only when necessary, which is not often, and the results are seen to follow for essentially algebraic reasons from the presence of the multiplication on a Lie group and the existence of commutative models. Some language of rational homotopy theory is thus used, particularly in Chapter 4 where we introduce the algebra of polynomial differential forms, which allows one to circumvent an approximation of BG by manifolds, the bulk of Lie theory, and the development of sheaf theory. The hope is that this will inspire a reader to learn

[2] Many results are phrased in a sort of first draft of a version of the language of rational homotopy, which was just coming into being at the time, and for which the later work [FHT01] of Halperin has become a standard reference.

[3] The major pattern is that results on homogeneous space in Ch. XI are a rewriting of those on Lie algebra cohomology in Ch. X, which specialize analogous results on "P-differential algebras" in Ch. III, bearing the same relation to results about "P-spaces" in Ch. II. These last are bilinear maps $P \otimes S \longrightarrow S$ of degree 1 of graded vector spaces, where P is odd-dimensional, which the authors note are exactly the same as modules over the polynomial algebra $\bigvee \boldsymbol{P}$; here $\bigvee$ denotes the symmetric algebra and $\boldsymbol{P}$ the evenly-graded suspension of P. The results of these early chapters are mostly translatable into results about Sullivan algebras.

more about rational homotopy theory without requiring her to learn it immediately.

This exposition presents a direct, historically honest account, demonstrates the essential simplicity of the determination of $H^*(G/K;\mathbb{Q})$, and offers motivation for the study of rational homotopy theory without building up the entire edifice of this general theory, already well-developed in Felix *et al.* [FHT01]. We require throughout only basic Lie theory and differential and algebraic topology, much less than that contained in the respective books of Bröcker–tom Dieck, Tu, and Hatcher [BtD85, Tu11, Hat02], and all of which is summarized in the appendices, in the hopes that the whole be legible to a second-year graduate student interested in topology. The author believes the resulting account to be the most accessible available account of some essential material which should be better known.

The rational cohomology of homogeneous spaces

Introduction

> *The following definition of the cohomology of a compact space is an extension of de Rham's definition of the cohomology of the algebra of their exterior differential forms (E. Cartan suggested that definition in 1928 after he succeeded in understanding a sentence written by H. Poincaré in 1899).*
>
> —Jean Leray [Ler72]

> *Let $\widetilde{M}$ be a fibre bundle over M with projection P and fibre F. Using cohomology groups with rational coefficients, the author defines, for each dimension p, a characteristic isomorphism of a factor group of a subgroup of the cohomology group $H^p(F)$ onto a group similarly related to $H^{p+1}(M)$. It is stated that one of these, suitably interpreted, is the characteristic cocycle of the bundle. [This could only be so if the coefficients of the latter (in $\pi_p(F)$) are replaced by their images in the homology group of F.] It is also asserted that a knowledge of the cohomology rings of M and F, and certain undefined generalizations of the characteristic isomorphisms, lead, in an unstated fashion, to a determination of the additive cohomology groups of $\widetilde{M}$.*
>
> —Steenrod's *Math. Review* (1946) of Hirsch's paper on the transgression [Ste48]

> *It is now abundantly clear that the spectral sequence is one of the fundamental algebraic structures needed for dealing with topological problems.*
>
> —William Massey, 1955 [Mas55, p. 329]

> *Now we illustrate the advantages of commutative multiplication in a fibration formula. This is the [. . .] analogue of the Chevalley–Hirsch–Koszul formula for principal Lie group bundles which was current in 1950 and ignored later in topology. The evident power and simplicity of the CHK formula helped prompt me to the present theory after Armand Borel kindly explained it to me in 1970.*
>
> —Dennis Sullivan, 1977 [Sul77]

Homogeneous spaces are of fundamental importance in geometry and equivariant topology—they are precisely orbits G/K of a transitive smooth action of a Lie group G, equivariant 0-cells—and accordingly the determination of their cohomology was a major research topic in topology from the late 1930s to the mid-1970s. Despite the slogan that cohomology is easy to compute and homotopy is hard, progress toward toward determining $H^*(G/K;\mathbb{Q})$ in general required two major new ideas, sheaf cohomology and spectral sequences (both due to Jean Leray, around 1945), which were complicated and poorly understood. Fortunately, great work was put into understanding and systematizing this early work and its essential features soon began to emerge.

The main ideas of the present work are few:

- The multiplication on a group constrains its rational cohomology to be exterior (hence free graded commutative), in Chapter 1.
- The Serre spectral sequence of a fiber bundle allows one to analyze the cohomology of a fiber bundle in terms of the fiber and the base, in Chapter 2. The related, purely algebraic spectral sequence of a filtered differential graded algebra allows one to compare the cohomology of two algebras by an examination of simpler constituent parts.
- All principal bundles are classified by a map to a universal bundle, in Chapter 5.
- The structure of the cohomology of the universal bundle is constrained by the structure of a spectral sequence. This implies for purely algebraic reason that the rational cohomology ring of a homogeneous space is polynomial (hence free graded commutative).
- Rational cohomology can be computed from a commutative cochain algebra, a "model," in Chapter 4. Surjections onto free objects split, so a free commutative cohomology ring maps as a subring into a commutative model of its own cochain algebra.
- A map of bundles induces a map of spectral sequences, and a related map of commutative models. By comparison, we see in Chapter 8 that the cohomology of a homogeneous space is carried by a very small model.

Each of these ideas is simple but powerful. Thus the historical question of the cohomology of a homogeneous space leads naturally into into a development of several key ideas of algebraic topology.

The key algebraic feature of the theory of differential forms that Leray wanted to emulate in setting up sheaf theory, which he uses to define his spectral sequence, is commutativity. This commutativity was isolated in purely algebraic form by Koszul in his thesis on Lie algebra cohomology, where he observes a spectral sequence always arises from a filtration of a differential graded algebra, such as the de Rham algebra $\Omega^*(M)$ or the singular cochain algebra $C^*(X)$. The spectral sequence pulls apart such an algebraic object one level at a time, and enables one to understand it by understanding its parts; if the filtration comes from a filtration of topological space (the classical examples being simplicial and CW-skeleta), this allows one to understand the cohomology of the space in terms of those of simpler parts. There is a bit of book-keeping involved, but it quickly comes to feel natural. The idea is so essential that there is no purpose to avoiding it, and the author thinks it is best encountered early, so in Chapter 2 we present what we believe is the simplest possible development.

It was rapidly recognized that the key feature of the sheaves Leray used was that their sections, like differential forms but unlike singular cochains, commuted up to sign under multiplication. Henri Cartan, building on unpublished work of Weil and Chevalley, distilled this insight into a conference paper (1950) in which he produces a commutative model computing $H^*(G/K;\mathbb{R})$ and sketches proofs. This model relies on the differential-geometric notion of a connection and some of the structure of a Lie algebra, and at least uses terminology from spectral sequences; it is likely the proofs involved them, but we do not know. In his dissertation, published as an *Annals* paper in 1953, Armand Borel produced a version of this model topologically using a map of fiber bundles, and it is this version we paraphrase here, entirely avoiding structure constants and indices, using the spectral sequence of a bundle one encounters in a second course in algebraic topology (for instance, this one, or the classic book [BT82] of Bott and Tu) and simple and algebraic models.

Both these accounts produce *finitely generated* commutative models of a space. Borel's insight is so bold as to be somewhat shocking. He has already determined one spectral sequence (that of a universal bundle, computed in Section 7.4), and there is a natural mapping into this universal bundle from another bundle we are interested in, determining a mapping of spectral sequences converging to $H^*(G/K)$. The existence of the mapping determines the differentials of certain elements which represent generators of the page E_2 of the spectral sequence of interest, and in particular determines the coboundaries of certain elements of $C^*(G/K;\mathbb{R})$. We know these elements represent elements in cohomology, but we know little about their cup products on the cochain level because the cup product is noncommutative. So we replace the singular cochains $C^*(G/K;\mathbb{R})$ with a graded-commutative differential algebra $A^*(G/K)$. This is still an uncountable object we can define but in no way describe explicitly, but we do have a finite set of commuting elements whose differentials we know. We use this to define an abstract graded-commutative differential algebra C and an injective mapping to $A^*(G/K)$, a finite crystal of pure structure in an uncountable chaos. The algebra C inherits a filtration from $A^*(G/K)$, and so the map $C \longrightarrow A^*(G/K)$, induces a map between the algebraic spectral sequences of their filtrations. The spectral sequence of C does not change on the first few pages $E_0 = E_1 = E_2$, but in the spectral sequence of $A^*(G/K)$, the obscuring mist melts away, until at E_2, every element is in the image of C. This implies by general considerations that the map $C \longrightarrow A^*(G/K)$ in fact induces an isomorphism in cohomology, so that $H^*(C) \cong H^*(G/K;\mathbb{R})$. The primal chaos of cochains was structurally supported all along by a skeleton we understand completely.

Not only is this idea beautiful, but it does not require much algebraic sophistication beyond polynomial and exterior algebras.[4] The author realized this when he was writing his own thesis, and it became a personal goal to present this version of the story, the genesis of many ideas which were to become important in topology and compelling in its own right. His hope is that this writing makes this material more accessible and its essential simplicity clearer.

[4] Homological algebra was being worked out for the first time around the time of Borel's thesis and does not figure. A later version of this story stars the Eilenberg–Moore spectral sequence and hence does explicitly involve Tor, but is independent of this story despite largely sharing its conclusions. A motion toward this history is made in Section 8.8.2.

Chapter 1

The rational cohomology of Lie groups

It was noted in the thirties the cohomology rings of classical Lie groups, over sufficiently easy coefficient rings k, become exterior algebras, and one might wonder whether this holds over Lie groups in general. It has been known since 1941 that it does, due to work of Heinz Hopf exploiting a natural algebraic structure in the (co)homology of a topological group, a development that essentially reduced the study of Lie group cohomology to obtaining torsion information and collating it back into integral cohomology.

We begin by isolating the essential feature of topological groups for our purposes.

Definition 1.0.1. An ***H-space***[1] is a topological space G equipped with a continuous ***product*** map $\mu\colon G \times G \longrightarrow G$ containing an element $e \in G$ neutral up to homotopy: we demand $g \longmapsto \mu(e,g)$ and $g \longmapsto \mu(g,e)$ be homotopic to id_G.

Such a map induces a ***coproduct*** in cohomology, the composition

$$H^*(G) \xrightarrow{H^*(\mu)} H^*(G \times G) \longrightarrow H^*(G) \otimes H^*(G),$$

where the second map arises through the Künneth theorem. We denote the coproduct by μ^*. Because $H^*(\mu)$ and the Künneth map are maps of graded k-algebras, it follows μ^* is a graded algebra homomorphism, and that if $x \in H^n(G)$, then $\mu^*(x) \in \bigoplus H^j(G) \otimes H^{n-j}(G)$.

Suppose as well that G is connected. We know $\mu(-,e) \simeq \mathrm{id}_G$; diagrammatically, this is the homotopy-commutative triangle below on the left, and taking cohomology whilst being casual about Künneth maps yields the commutative diagram on the right.

$$\begin{array}{ccccc} G & \xrightarrow{\approx} & G \times \{e\} & \xrightarrow{i} & G \times G \\ & \searrow^{\mathrm{id}} & & & \downarrow{\scriptstyle \mu} \\ & & & & G \end{array} \qquad \begin{array}{ccccc} H^*(G) & \xleftarrow{\sim} & H^*(G) \otimes H^0(G) & \xleftarrow{H^*(i)} & H^*(G) \otimes H^*(G) \\ & \nwarrow^{\mathrm{id}} & & & \uparrow{\scriptstyle H^*(\mu)} \\ & & & & H^*(G) \end{array}$$

This means the component of $\mu^*(x)$ lying in $H^n(G) \otimes H^0(G)$ is $x \otimes 1$. The same argument run with the identity $\mu(e,-) \simeq \mathrm{id}_G$ yields the component $1 \otimes x$ in $H^0(G) \otimes H^n(G)$. So

$$\mu^*(x) \equiv 1 \otimes x + x \otimes 1 \pmod{\widetilde{H}^*(G) \otimes \widetilde{H}^*(G)}.$$

[1] The choice of H, due to Serre, is in honor of Heinz Hopf.

Recall that the cup product $\smile\colon H^*(G)\times H^*(G)\longrightarrow H^*(G)$ is induced in a similar way by the diagonal map $\Delta\colon G\longrightarrow G\times G$ taking $g\longmapsto (g,g)$; to wit, it can be understood as the composition

$$H^*(G)\otimes H^*(G)\longrightarrow H^*(G\times G)\xrightarrow{\Delta^*} H^*(G).$$

As Δ and μ admit some relations on a topological level, we recover some cohomological identities. Trivially but importantly, $\mu\times\mu$ is a map $\prod^4 G\longrightarrow\prod^2 G$ taking the quadruple (x,y,x,y) to the pair $\big(\mu(x,y),\mu(x,y)\big)=(\Delta\circ\mu)(x,y)$. If we write $\tau\colon G\times G\longrightarrow G\times G$ for the transposition switching the coordinates, then $(x,y,x,y)=(\mathrm{id}\times\tau\times\mathrm{id})(x,x,y,y)=(\mathrm{id}\times\tau\times\mathrm{id})(\Delta\times\Delta)(x,y)$, so

$$\Delta\circ\mu=(\mu\times\mu)\circ(\mathrm{id}\times\tau\times\mathrm{id})\circ(\Delta\times\Delta). \tag{1.2}$$

Taking the cohomology of (1.2), being casual with Künneth maps again, and recalling from Appendix A.2 the sign convention for a tensor product of CGAs, one finds that for all homogeneous $a,b\in H^*(G)$,

$$\mu^*(ab)=\mu^*(a)\mu^*(b),$$

so that $\mu^*\colon H^*(G)\longrightarrow H^*(G)\otimes H^*(G)$ is a ring homomorphism. All this inspires the following definition.

Definition 1.0.3. A ***Hopf algebra*** over k is a graded (not necessarily associative) k-algebra A such that $A^0\cong k$, equipped with an algebra homomorphism $\mu^*\colon A\longrightarrow A\otimes_k A$ such that

$$\mu^*(a)\equiv 1\otimes a+a\otimes 1 \pmod{\widetilde{A}\underset{k}{\otimes}\widetilde{A}}$$

for each homogeneous $a\in A$. (Here $\widetilde{A}\lhd A$ is the augmentation ideal $\bigoplus_{i\geqslant 1}A^i\cong A/A^0$ of elements of positive degree, as defined in Appendix A.2.)

What we have shown is that, given an H-space G, its cohomology ring $H^*(G)$ is naturally a commutative, associative Hopf algebra. The presence of the coproduct imposes severe constraints on the algebra structure, especially with regard to algebra generators. Here is Hopf's structure theorem.

[PROVE WHAT THE MONOGENIC ONES ARE IN POSITIVE CHARACTERISTIC.]

Theorem 1.0.4 (Hopf, $\operatorname{char}k=0$: Hopf's theorem [Hop41, Satz I, p. 23]; Borel, $\operatorname{char}k>0$)**.** *Let A be a commutative, associative Hopf algebra of finite type over a field k. Then it is a tensor product of Hopf algebras on single generators. As algebras these are*

- *exterior algebras $\Lambda[\alpha]$ with $|\alpha|$ odd-dimensional,*
- *polynomial algebras $k[\alpha]$, with $|\alpha|$ even-dimensional if $\operatorname{char}k\neq 2$, and*
- *truncated symmetric algebras $k[\alpha]/(\alpha^{p^j})$ if if $p=\operatorname{char}k>0$, with α even-dimensional if $p>2$.*

Proof [Hat02, Prop. 3C.4, p. 285]. We prove the result for $\operatorname{char}k=0$ by induction on the number n of algebra generators, starting with $n=0$ so the result is trivial. Inductively suppose we have shown the result for n generators and A is generated by $n+1$. Order these algebra generators $x_1,\ldots,x_n,y$ by weakly increasing degree, and let A' be the subalgebra generated by $x_1,\ldots,x_n$. This is actually a Hopf subalgebra, for $\mu^*(x_j)=1\otimes x_j+x_j\otimes 1+(\deg\ <|y|)$, so the last term cannot involve y, and must lie in A'. Since μ^* is an algebra homomorphism, we must have

$\mu^*(A') \leqslant A' \otimes A'$. Because A is a CGA generated by A' and x, there is a surjective k-algebra homomorphism

$$A' \otimes \Lambda[y] \twoheadrightarrow A \quad \text{if } |y| \text{ is odd,}$$
$$A' \otimes k[y] \twoheadrightarrow A \quad \text{if } |y| \text{ is even.}$$

To see A is free, it is enough to prove these maps are injective.

If $|y|$ is odd, suppose $a + by = 0$ in A, where $a, b \in A'$. Then $0 = \mu^*(a + by) \in A \otimes A$ projects under $A \otimes A \twoheadrightarrow A \otimes \Lambda[y]$ to

$$0 = a \otimes 1 + (b \otimes 1)(y \otimes 1 + 1 \otimes y) = \underbrace{(a + by)}_{0} \otimes 1 + b \otimes y = b \otimes y.$$

This can only be zero if b is, but then $0 = a + 0y$, so $a = 0$ and our relation was trivial.

We leave the case $|x|$ even as an exercise. □

Exercise 1.0.5. Finish the proof in the case $|x|$ is even. (*Hint*: Apply μ^* to a relation and examine the image in $A \otimes A/(\widetilde{A}', y^2) \cong A \otimes k[y]/(y^2)$.

Corollary 1.0.6. *Let G be a compact, connected Lie group. Then $H^*(G; \mathbb{Q})$ is an exterior algebra.*

Proof. We already know $H^*(G)$ is a free k-CGA, say on V. If V contained any even-degree elements, then by the theorem, $H^n(G)$ would be nontrivial for arbitrarily large n; but it cannot be, because G is a finite-dimensional CW complex. So V is oddly graded and $H^*(G) \cong \Lambda V$. □

Corollary 1.0.7. *Let G be a Lie group and $G \to E \to B$ a principal G-bundle and suppose $H^*(E) \longrightarrow H^*(G)$ surjects and k is a field of characteristic zero. Then there exists a k-CGA isomorphism*

$$H^*(E) \cong H^*(B) \otimes H^*(E).$$

Proof. By Corollary 2.2.12, one has an $H^*(B)$-module isomorphism $H^*(E) \cong H^*(B) \otimes H^*(G)$. By Corollary 1.0.6, $H^*(G)$ is a free k-CGA, so by Proposition A.4.4, a lifting of $H^*(E) \twoheadrightarrow H^*(G)$ induces a ring isomorphism $H^*(B) \otimes H^*(G) \xrightarrow{\sim} H^*(E)$. □

We can do a bit better in identifying the generators of $H^*(G)$.

Definition 1.0.8. We an element x of a Hopf algebra A ***primitive*** if $\mu^*(x) = 1 \otimes x + x \otimes 1$. Write

$$PA = \{x \in A : x \text{ is primitive}\}$$

for the ***primitive subspace*** and grade this space by $P^r A = PA \cap A^r$. Note that the only primitive in $A^0 \cong k$ can be the identity so that $P^0 A = 0$ and PA is contained in the augmentation ideal $\widetilde{A}$. If $A = H^*(G)$ is the cohomology ring of an H-space G, we abbreviate $PG := PH^*(G)$. Another way to phrase the definition is to say that PA is the kernel of the k-linear homomorphism

$$\psi \colon A \longrightarrow A \otimes A,$$
$$x \overset{\psi}{\longmapsto} \mu^*(x) - (1 \otimes x + x \otimes 1).$$

The ***indecomposable*** elements of an augmented ring A are, informally, those of positive degree that cannot be written as sums of products of lower-degree elements; the idea is to find an analogy for irreducible polynomials for rings with more complex ideal structure. The most convenient definition turns out to be this: the ***module of indecomposables*** is the k-module

$$QA := \widetilde{A}/\widetilde{A}\widetilde{A} \cong \widetilde{A} \underset{A}{\otimes} k$$

where $\widetilde{A}$ is the augmentation ideal and the denominator denoted $\widetilde{A}\widetilde{A}$ is understood to be the module spanned by products ab for $a, b \in \widetilde{A}$ of positive-degree elements. Under this definition we see Q is functorial, since a graded homomorphism $A \longrightarrow B$ takes $\widetilde{A} \longrightarrow \widetilde{B}$ and hence $\widetilde{A}\widetilde{A} \longrightarrow \widetilde{B}\widetilde{B}$. If A is a free k-module, then so is $Q(A)$, so the k-module surjection $\widetilde{A} \twoheadrightarrow Q(A)$ splits by Proposition A.4.1 and we can consider $Q(A)$ (in a badly noncanonical way) as a k-submodule of algebra generators for A. Because it satisfies a product rule, an derivation d on A, like a ring homomorphism, is uniquely determined by its values on such a lifted $Q(A)$, so a linear map on $Q(A)$ determines at most one derivation of A.

There is a natural k-linear composite map

$$P(A) \hookrightarrow \widetilde{A} \twoheadrightarrow \widetilde{A}/\widetilde{A}\widetilde{A} =: Q(A)$$

linking primitives and indecomposables, which is an isomorphism in the case we care about.

Proposition 1.0.9 (Milnor–Moore)**.** *Let A be a commutative, cocommutative Hopf algebra finitely generated as an algebra over a field k. Then this canonical map takes $P(A) \xrightarrow{\sim} Q(A)$. In particular, A is generated by primitive elements.*

Proof. The strong statement is more than we need, but we will prove the result in the case A is a coassociative Hopf algebra over a field k of characteristic $\neq 2$ with underlying algebra an exterior algebra, loosely following Mimura and Toda [MT00, p. 369] for injectivity; this weaker version is due to Hopf and Samelson.

Write $A = \Lambda V$, for V an oddly-graded vector space. That $V \xrightarrow{\sim} Q(A)$ is clear, so we just need to show V can be chosen such that $P(A) = V$. Pick a basis X of V. By anticommutativity, a basis of ΛV is given by monomials $y = x_1 x_2 \cdots x_n$ with $x_i \in X$ of weakly increasing degree. If $n > 1$, then we have

$$\mu^*(y) = \prod \mu^*(x_i) = \prod \big(x_1 \otimes 1 + 1 \otimes x_i + (\cdots)\big) = 1 \otimes y + [x_1 \otimes x_2 \cdots x_n] + \sum a \otimes b,$$

where none of the terms $a \otimes b$ have $a \in \mathbb{Q}x_1$. It follows the term $x_1 \otimes x_2 \cdots x_n$ doesn't cancel, and thus $\mu^*(y) \neq y \otimes 1 + 1 \otimes y$, so $P(A) \leqslant V$.

For the other containment, we induct on $\dim V$. Assume the result is proved for n, and that $\dim V = n+1$. Arrange a homogeneous basis $x_1, \ldots, x_n, y$ of V in weakly increasing degree. By induction, $V' = \mathbb{Q}\{x_1, \ldots, x_n\}$, where we may choose x_j primitive, and it remains to show y is. Since each x_j is primitive, we have $\mu^*(x_j) \leqslant \Lambda[x_j] \otimes \Lambda[x_j]$ for each j, so the coproduct μ^* descends to a coproduct $\overline{\mu^*}$ on $\Lambda V /\!/ \Lambda[x_j]$, and since this is an exterior algebra on n generators, by induction, we have $\overline{\mu^*}(y) = 1 \otimes y + y \otimes 1$ in this quotient, so back in $\Lambda V \otimes \Lambda V$, the difference $\psi(y) := \mu^*(y) - (1 \otimes y + y \otimes 1)$ lies in the ideal $(x_j \otimes 1, 1 \otimes x_j)$. Varying j, we see $\psi(y)$ lies in the the intersection of all these ideals. If we write $x_I := \prod_{i \in I} x_i$, this intersection ideal is that generated by the tensor products $x_I \otimes x_J$ such that $I \amalg J = \{1, \ldots, n\}$ is a partition. In fact, since by definition $\psi(y) \in \widetilde{A} \otimes \widetilde{A}$, it lies in the ideal generated by $x_I \otimes x_J$ with neither I nor J empty. We are then done unless $|y| = \sum_{i=1}^{n} |x_i|$, so assume this equality holds. Then since $\psi(y)$ is homogeneous and the generating elements $x_I \otimes x_J$ already have the right degree, we can write

$$\psi(y) = \sum_{I \amalg J = \{1,\ldots,n\}} a_{I,J} x_I \otimes x_J$$

for some scalars $a_{I,J} \in k$.

The fact that $(\mu^* \otimes \mathrm{id})\mu^* = (\mathrm{id} \otimes \mu^*)\mu^*$, the coassociativity of A, follows for $H^*(G)$ from the associativity of the multiplication on G. It is not hard to see this is equivalent to the condition $(\psi \otimes \mathrm{id})\psi = (\mathrm{id} \otimes \psi)\psi$. Applying this equation to y we obtain

$$\sum a_{I,J} \psi(x_I) \otimes x_J = \sum a_{I,J} x_I \otimes \psi(x_J),$$

where the sum runs over partitions $I \amalg J = \{1, \ldots, n\}$ with $I \neq \varnothing \neq J$. These equations expand to

$$\sum a_{I,J} \sum_{I_1, I_2} x_{I_1} \otimes x_{I_2} \otimes x_J = \sum a_{I,J} \sum_{J_1, J_2} x_I \otimes x_{J_1} \otimes x_{J_2},$$

where $I \amalg J = \{1, \ldots, n\}$ as before and in the sums on either side, one has $I_1 \amalg I_2 = I$ and $J_1 \amalg J_2 = J$, and $I, J, I_1, I_2, J_1, J_2 \neq \varnothing$. Fix a partition $I_1 \amalg I_2 \amalg J = \{1, \ldots, n\}$. The coefficients of $x_{I_1} \otimes x_{I_2} \otimes x_J$ on the left-hand side and the right, which must consequently be equal, are $a_{I,J}$ and $a_{I_1, I_2 \amalg J}$. These equalities show all $a_{I,J}$ are equal to some single scalar $a \in k$, so

$$\psi(y) = a \sum_{I,J \neq \varnothing} x_I \otimes x_J = a\psi(x_1 \cdots x_n),$$

or $\psi(y - ax_1 \cdots x_n) = 0$. Thus $x_1, \ldots, x_n, y - ax_1 \cdots x_n$ is a set of primitive generators of A. □

Remark 1.0.10. An analogous result holds in characteristic 2 with the weaker assumption on A that it not necessarily be an exterior algebra, but merely admit a simple system of generators (see Definition A.2.4). The proof is correspondingly much more difficult.

We will later need as well the fact that a map of H-spaces induces a map of primitives in cohomology.

Proposition 1.0.11. *Let $\phi\colon K \longrightarrow G$ be a homomorphism of H-spaces. Then the map $\phi^*\colon H^*(G) \longrightarrow H^*(K)$ in cohomology takes $PG \longrightarrow PK$.*

Proof. To ask ϕ be a homomorphism is, by definition, to require $\mu_G \circ (\phi \times \phi)$ and $\phi \circ \mu_K$ be homotopic maps $K \times K \longrightarrow G$. In cohomology, then, if $z \in PG$ is primitive, we have

$$\mu_K^* \phi^* z = (\phi^* \otimes \phi^*)\mu_G^* z = (\phi^* \otimes \phi^*)(1 \otimes z + z \otimes 1) = 1 \otimes \phi^* z + \phi^* z \otimes 1. \qquad \square$$

There is a further theorem determining $\dim PG$.

Theorem 1.0.1 (Hopf [Hop40, p. 119])**.** *Let G be a compact, connected Lie group and T a maximal torus. Then the total Betti number $h^\bullet(G) = 2^{\dim T}$.*

Proof [Sam52]. By the preceding theorem, $H^*(G; \mathbb{Q})$ is an exterior algebra, so from Appendix A.2.3 we see $h^\bullet(G) = 2^l$ for some $l \in \mathbb{N}$. To see that $l = \dim T$, consider the squaring map $s\colon g \longmapsto g^2$ on G. Since $s = \mu \circ \Delta$, it follows that for a primitive $a \in H^*(G)$ one has

$$s^* a = \Delta^* \mu^* a = \Delta^*(1 \otimes a + a \otimes 1) = 1 \smile a + a \smile 1 = 2a,$$

so if $[G] \in H^{\dim G}(G)$ is the fundamental class, the product of l independent primitives, one has $s^*[G] = 2^l[G]$. Thus the degree of s is 2^l. On the other hand, restricting to the abelian subgroup $T \cong (\mathbb{R}/\mathbb{Z})^{\dim T}$, it is easy to see the s-preimage of a generic element of T contains $2^{\dim T}$ points, which, since s is orientation-preserving, should each be counted with multiplicity 1. By a standard theorem on degree [Hat02, Ex. 3.3.8, p. 258] we then know $2^{\dim T} = \deg s = 2^l$, so $l = \dim T$. □

These results also let us obtain a classical topological fact usually proven through other means.

Corollary 1.0.12 ([BtD85, Prop. V.(5.13), p. 225])**.** *The second homotopy group $\pi_2 G$ of a compact Lie group G is trivial.*

Proof. The universal compact cover $\widetilde{G}$ of G (see Theorem B.4.5) satisfies $\pi_2\widetilde{G} \cong \pi_2 G$ by the long exact homotopy sequence of a bundle Theorem B.1.4, and $\widetilde{G} \cong A \times K$ for A a torus and K simply connected. Using the long exact homotopy sequence of the short exact sequence $\mathbb{Z}^n \to \mathbb{R}^n \to T^n$, one sees $\pi_2 A = 0$, and since $\pi_1 K = 0$, successively applying the Hurewicz theorem, the universal coefficient theorem, and Hopf's theorem, one finds $\pi_2 K \cong H_2 K \cong H^2 K = 0$, so $\pi_2\widetilde{G} \cong \pi_2 A \times \pi_2 K = 0$. □

Remark 1.0.13. The multiplication on a Lie group G induces a product on $H_*(G;\mathbb{Q})$, the *Pontrjagin product*, making it a Hopf algebra as well, the *homology ring*, which is dual to $H^*(G;\mathbb{Q})$. It is this ring that Hopf originally discovered the structure of, though the way he put it was that the homology ring of G was isomorphic to that of a product $\prod S^{2n_j-1}$ of odd-dimensional spheres. Serre noted later [FHT01, p. 216] that this was actually due to a *rational homotopy equivalence*: there is a map $\prod S^{2n_j-1} \longrightarrow G$ inducing isomorphisms

$$\pi_*\Big(\prod S^{2n_j-1}\Big) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_*(G) \otimes \mathbb{Q}$$

on rational homotopy groups. Because the rational Hurewicz map

$$\pi_*\Big(\prod S^{2n_j-1}\Big) \otimes \mathbb{Q} \longrightarrow H_*\Big(\prod S^{2n_j-1};\mathbb{Q}\Big)$$

is an isomorphism when restricted to the span $\bigoplus \mathbb{Q} \cdot [S^{2n_j-1}]$ of the fundamental classes of the factor spheres, the image of the Hurewicz map $\pi_*(G) \otimes \mathbb{Q} \longrightarrow H_*(G;\mathbb{Q})$ contains the homological primitives $P_*(G) = PH_*(G)$. In Remark 2.2.23, we will show that this means these primitives are in the image of the transgression in the homological Serre spectral sequence of any G-bundle.

Chapter 2

Spectral sequences

One of the main tools in our development is the spectral sequence. This is an algebraic gadget with a reputation for ferocity that we maintain is undeserved. While it is common in topology to be able to prove a spectral sequence exists without being able to compute its differentials explicitly, the cohomology of homogeneous spaces offers many beautiful examples where the sequence is completely computable.

This section introduces the Serre spectral sequence relating the cohomology rings of the constituent spaces $F \to E \to B$ of a fiber bundle, or more generally a fibration. In order that the exposition be self-contained, we prove the structure we need in the later sections of this chapter, but we do not recommend reading it immediately; while it is important culturally to know at some point what is going on, and we will eventually need some details of its construction in Section 8.1.2, our initial applications do not require these details, and there is enough to assimilate that it is reasonable to go at it in stages, learning to use the machine before lifting up the hood to see how it goes.

For reasons of digestibility, we start the section with the statement of Serre spectral sequence itself and some applications. We will need the filtration spectral sequence of an abstract filtered differential graded algebra later, of which the Serre spectral sequence is one particular case, so we develop this, with full proofs, in a long appendix to this chapter. There is value to understanding why the machine works, but it is not immediately useful for our purposes, and the reader is advised to defer reading these proofs until the tension becomes unbearable.

We believe this is a good way to introduce oneself to this machine, there are many recountings of this story, and we do not claim ours is optimal. The author recommends the discussion in his advisor's book [BT82, Ch. 3] as still the clearest introduction he has seen to this material.

2.1. The idea of a spectral sequence

A spectral sequence is a tool that allows us to understand an algebraic object in terms of its constituent parts. The particular example we will use, takes a differential graded algebra A and recovers the associated graded algebra $\mathrm{gr}_\bullet H^*(A)$ of the cohomology ring $H^*(A)$, as defined in Section 2.5, at the end of a computation whose first steps are forming the simpler associated graded algebra $\mathrm{gr}_\bullet A$ with respect to some filtration, and taking *its* cohomology $H^*(\mathrm{gr}_\bullet A)$. This seems like it is "just computing cohomology with extra steps," but it is often useful if the initial A is too complicated—say, too large—to be understood directly.

For example, the singular cochain algebra $C^*(X)$ of a CW complex X will be uncountable if

$\dim X \geqslant 1$, but in terms of the CW skeleta X^p, recall that there are associated *cellular cochains*

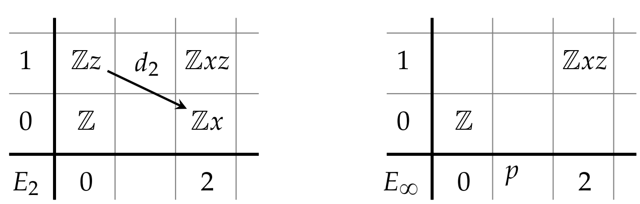
$$\mathrm{Cell}^p(X) := H^p(X^p, X^{p-1}) \cong \widetilde{H}^p(X^p/X^{p-1})$$

which can be identified as free groups on the p-cells of the CW structure, and are thus finitely generated in our cases of interest. The cup product of cochains induces a product on the direct sum of these groups and there is a differential δ given by the connecting maps in the long exact sequence of a triple (X^{p+1}, X^p, X^{p-1}), and one shows in a first course in algebraic topology that the cohomology of the algebra $\mathrm{Cell}^\bullet(X)$ is just $H^*(X)$. This calculation is actually exactly what the spectral sequence of the previous paragraph returns if we feed it $C^*(X)$, filtered by ideals corresponding to the p-skeleta X^p.

If X is the total space of a fiber bundle $F \to X \to B$ and we instead use a filtration of $C^*(X)$ induced from the p-skeleta B^p of the base, we will get a computation that starts, under reasonable circumstances, with $H^*(B) \otimes H^*(F)$, proceeds in a well-determined manner, and returns $H^*(X)$ at the end. This will enable us, in the first place, to often determine $H^*(X)$ in terms of $H^*(F)$ and $H^*(B)$, which we will use to compute the cohomology of the classical Lie groups, and later to compute the cohomology of $H^*(B)$ in terms of $H^*(X)$ and $H^*(B)$ which we will use to determine the cohomology of a classifying space. Later still, we will use *maps* of spectral sequences to determine the cohomology of a homogeneous space, which fits into a *system* of bundles in such a way that all of the information of the spectral sequence is calculable.

In more detail, a spectral sequence, for us, will be a sequence $(E_r)_{r \geqslant 0}$ of differential algebras such that each algebra is the cohomology of the previous: $E_{r+1} = H^*(E_r)$. Particularly, each algebra is a subquotient of the previous, so they can be considered as "decreasing" in a certain sense. In the cases we consider, there will always be a number N such that $d_r = 0$ for all $r \geqslant N$, so we will have $E_r \cong E_{r+1} \cong E_{r+2} \cong \cdots$. We will write E_∞ for this last page.

So far, this is a finite sequence of rings. These additionally will be bigraded: $E_r = \bigoplus_{p,q \geqslant 0} E_r^{p,q}$ as an abelian group, and the multiplication will add the bidegrees: so that on any given page the product of an element of bidegree (p,q) and one of (p', q') will have bidegree $(p+p', q+q')$. The bigrading seems at first glance to complicate things, since now each page is an infinite-by-infinite array of groups—and it certainly does encumber the notation—but in practice being able to separate out all this information into many components simplifies life, as each of these pieces will be a finitely-generated abelian group we have a good handle on, and each ring will be generated by finitely many elements. Since each differential d_r is a derivation, it will be determined by finitely many of these values, and this will actually make computations much more tractable. Here is a picture of a spectral sequence we will encounter later (Figure 2.2.18), that corresponding to the Hopf fibration $S^1 \to S^3 \to S^2$.



The left diagram is meant to indicate that

$$E_2 = \mathbb{Z}[x,z]/(x^2, z^2), \qquad \text{where } x \in E_2^{2,0} \quad \text{and} \quad z \in E_2^{0,1}.$$

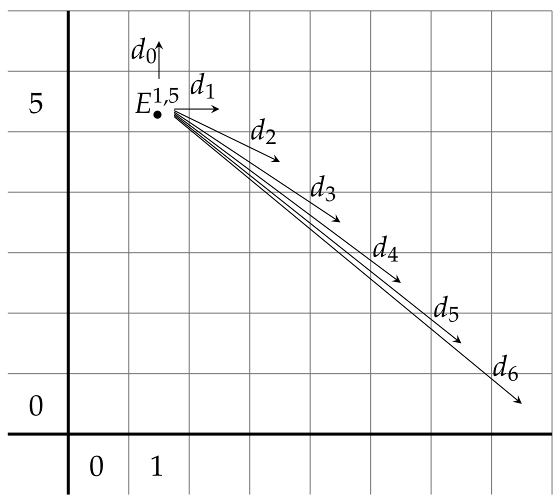
The arrow d_2 indicates that $d_2(z) = x$, and the absence of an arrow from 1 and x indicates that $d_2(1) = 0$ and $d_2(x) = 0$. The differentials are in fact derivations, so for example one can deduce

$$d_2(xz) = d_2(x)z + (-1)^2 x \cdot d_2(z) = 0 \cdot z + x \cdot x = 0$$

as well. The cohomology as a group, is thus $\ker d_2 / \operatorname{im} d_2 = \mathbb{Z}\{1, x, xz\}/\mathbb{Z}x \cong \mathbb{Z}\{1, xz\}$, as we see in the right diagram for $E_3 = E_\infty$. Implicit in the discussion is the fact that the rest of the differentials, d_r for $r \geqslant 3$, are all zero.

In this picture, we see d_2 goes one step down and two right. In general, each differential d_r has bidegree $(1-r, r)$, meaning it runs from a square (p, q) to square $\big(p+r, q-(r-1)\big)$, as seen in Figure 2.1.1

Figure 2.1.1: The differentials out of $E_\bullet^{1,5}$



Here is a formal statement of the spectral sequence of a filtered differential graded algebra; the proof will be deferred to Section 2.6.

Theorem 2.1.2. *(Koszul). Let $(C^\bullet, d, \tilde{\iota})$ be a filtered differential $\mathbb{N}$-graded algebra such that the associated filtration of $H^n(C^\bullet)$ is finite for each n. Then there is an associated* ***filtration spectral sequence*** *in which*

- $(E_0, d_0) = (\operatorname{gr}_\bullet C^\bullet, \operatorname{gr}_\bullet d)$,
- $E_1 \cong H^*(\operatorname{gr}_\bullet C^\bullet)$,
- $E_\infty^{p,q} \cong \operatorname{gr}_p H^{p+q}(C^\bullet)$.

We call this the ***filtration spectral sequence*** *of the filtered* DGA *$(C^\bullet, d, \tilde{\iota})$. It is* ***first-quadrant spectral sequence*** *in that $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$. All pages become differential algebras under the bigrading $E_r^{p,q}$ induced from the bigrading $E_0^{p,q} := \operatorname{gr}_p C^{p+q}$ of $E_0 = \operatorname{gr}_\bullet C^\bullet$ and the product induced from that of C, with differential d_r of bidegree $(r, 1-r)$. Moreover, the product on each page is induced by that on the last. This sequence is functorial in homomorphisms of filtered* DGAs.

Our examples will mostly be concrete and topological, but as a purely algebraic application, here is a proof of the algebraic Künneth corollary A.3.3 over a field.

Proof. As we will not need a notation for coboundaries, we will write $B^\bullet$ for instead for the differential graded algebras with k-flat cohomology. Take $C = A^\bullet \otimes_k B^\bullet$, bigraded by $C^{p,q} = A^p \otimes_k B^q$, with the differential $d = d_A \otimes \mathrm{id} + (-1)^p \mathrm{id} \otimes d_B$. We apply Theorem 2.1.2 with the filtration given by $F_p C = A^{\geqslant p} \otimes_k B^\bullet$. Then we have $E_0 = \mathrm{gr}_\bullet C \cong C$ on the level of graded groups by inspection (or Corollary 2.6.8) and $d_0 = \mathrm{gr}_\bullet d = (-1)^p \mathrm{id} \otimes d_B$, so that

$$E_1 = H^*_{d_0}(A^\bullet \underset{k}{\otimes} B^\bullet) = A^\bullet \underset{k}{\otimes} H^*_{\pm d_B}(B^\bullet) = A^\bullet \underset{k}{\otimes} H^*(B^\bullet).$$

If $z \in B^q$ represents a class in $H^q(B^\bullet)$, then for any $a \in A^p$ we have $d(a \otimes z) = d_A a \otimes z \pm a \otimes d_B z = d_A a \otimes z$, and it follows $d_1 = \delta_A \otimes \mathrm{id}$, so that

$$E_2^{p,q} \cong \frac{\ker\big(A^p \otimes H^q(B^\bullet) \longrightarrow A^{p+1} \otimes H^q(B^\bullet)\big)}{\mathrm{im}\big(A^{p-1} \otimes H^q(B^\bullet) \longrightarrow A^p \otimes H^q(B^\bullet)\big)}.$$

But $H^q(B^\bullet)$ is flat, so this is $(\ker d_A / \mathrm{im}\, d_A) \otimes H^q(B^\bullet) = H^p(A^\bullet) \otimes H^q(B^\bullet)$. □

2.2. The Serre spectral sequence

Most of our examples of spectral sequences will arise from a fibration $F \to E \xrightarrow{\pi} B$ with B a CW complex, as gestured at in the previous section. Let B^p be the p-skeleton of B. Then $(E^p) := (\pi^{-1} B^p)$ an increasing filtration of E; set $E^p = \varnothing$ for $p < 0$. Associated to each pair (E, E^p) is a short exact sequence

$$0 \to C^*(E, E^p) \longrightarrow C^*(E) \longrightarrow C^*(E^p) \to 0 \tag{2.2.1}$$

of cochain complexes, where for simplicity we suppress the coefficient group k. Because $E^{p-1} \subseteq E^p$, each restriction $C^*(E) \longrightarrow C^*(E^{p-1})$ factors through $C^*(E^p)$, so the increasing topological filtration (E^p) leads to a *decreasing* algebraic filtration

$$F_p C^*(E) = C^*(E, E^{p-1})$$

of $C^*(E)$.[1] We have $\bigcap F_p C^*(E) = 0$, for each singular simplex $\sigma\colon \Delta^n \longrightarrow E$ has image in some E^p.[2] The associated filtration of $H^*(E)$ is given by $F_p H^*(E) = \mathrm{im}\big(H^*(E, E^{p-1}) \to H^*(E)\big)$. Assume for convenience that the action of $\pi_1 B$ on $H^*(F)$ is trivial. Then turning the crank of the associated filtration spectral sequence of Theorem 2.1.2, one arrives at the following.

Theorem 2.2.2. *Let $F \to E \to B$ be a fibration such that $\pi_1 B$ acts trivially on $H^*(F;k)$. There exists a first-quadrant spectral* ***Serre spectral sequence*** *$(E_r, d_r)_{r \geqslant 0}$ of k-DGAs with*

$$\begin{aligned} E_0^{p,q} &= C^{p+q}(E, E^{p-1}; k), \\ E_2^{p,q} &= H^p\big(B; H^q(F;k)\big), \\ E_\infty^{p,q} &= \mathrm{gr}_p H^{p+q}(E;k), \end{aligned}$$

for the filtrations (E^p) and $F_p H^*(E)$ indicated above. If $H^*(F;k)$ is a free k-module (for instance, if k is a field), we may also write $E_2 \cong H^*(B;k) \otimes_k H^*(F;k)$. This construction is functorial in fibrations $E \to B$ and in rings k, in that a map of fibrations or of rings induces a map of spectral sequences.

[1] The mismatch of p and $p-1$ is initially jarring, but worth it to guarantee $F_0 C^*(E) = C^*(E)$.

[2] The image of $\Delta^n \xrightarrow{\sigma} E \to B$ is compact, and a compact subset of a CW complex can only can meet only finitely cells lest it contain an infinite discrete set.

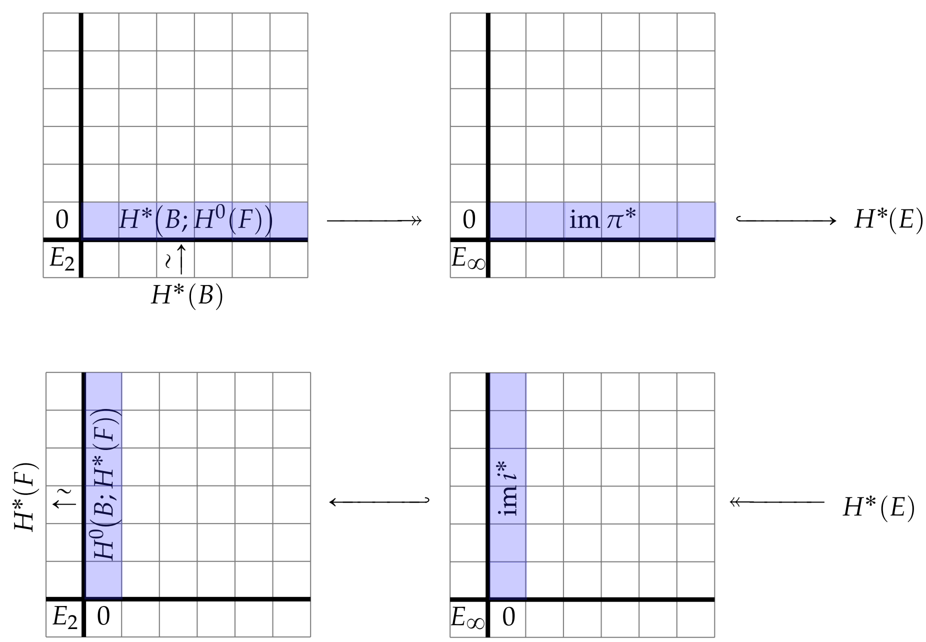
Most of this is immediate, but the proof of the characterization of the E_2 page is nontrivial,[3] and we defer it to Section 2.9. Critically for us in all that follows, this version of the formulation applies to principal bundles.

Proposition 2.2.3. *Let G be a path-connected group. If $G \to E \to B$ is a principal G-bundle, then $\pi_1 B$ acts trivially on $H^*(G)$.*

Proof. The transition functions are given by right multiplication r_g by elements of G, as discussed in Appendix B.3.1. Since G is path-connected, each r_g is homotopic to $r_1 = \mathrm{id}_G$, so the action of $\pi_1 B$ on $H^*(G)$ is trivial. □

It is important to us to be able to identify the maps in cohomology induced by fiber inclusion and projection to the base.

Figure 2.2.4: The maps induced by $F \xrightarrow{i} E \xrightarrow{\pi} B$ in the Serre spectral sequence



Proposition 2.2.5. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration such that $\pi_1 B$ acts trivially on $H^*(F)$. The fiber projection $i^*\colon H^*(E) \longrightarrow H^*(F)$ is realized by the left-column* **edge map** *$E_\infty^{\bullet,\bullet} \twoheadrightarrow E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet}$ in Theorem 2.2.2: to wit, we can write*

$$\mathrm{gr}_\bullet\, H^*(E) \xrightarrow{\sim} E_\infty^{\bullet,\bullet} \twoheadrightarrow E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet} \xrightarrow{\sim} H^*(F).$$

Likewise, the base lift $\pi^*\colon H^*(B) \longrightarrow H^*(E)$ is realized by the bottom-row edge map $E_2^{\bullet,0} \twoheadrightarrow E_\infty^{\bullet,0} \hookrightarrow E_\infty^{\bullet,\bullet}$:

$$H^*(B) \xrightarrow{\sim} E_2^{\bullet,0} \twoheadrightarrow E_\infty^{\bullet,0} \hookrightarrow E_\infty^{\bullet,\bullet} \xrightarrow{\sim} \mathrm{gr}_\bullet\, H^*(E).$$

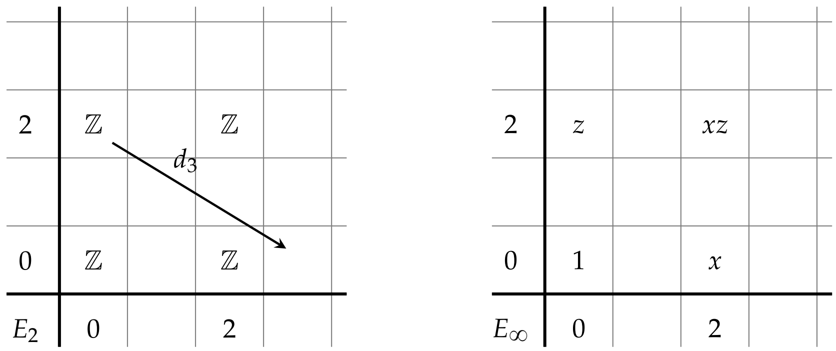
[3] *On the many occasions in graduate courses when I have carried out the E_2 calculation for the Serre spectral sequence, both the students and I have agreed that the material I presented could surely be reorganized into an actual proof of the desired theorem . . .*

—Edgar Brown, Jr. [BJ94]

Here is a picture of the situation;[4] the proof is again deferred so that we may immediately embark on some examples and applications.

Example 2.2.6. Consider a sphere bundle over a sphere $S^2 \to E \to S^2$. Since $H^*(S^2) = \mathbb{Z}[x]/(x^2)$ for $x = [S^2]$ in degree 2, which is a free abelian graded group, we have $E_2^{p,q} = H^p\big(S^2; H^q(S^2)\big) = H^p(S^2) \otimes H^q(S^2)$, as appears in Figure 2.2.7.

Figure 2.2.7: The Serre spectral sequence of $S^2 \to E \to S^2$



The nonzero squares (p,q) are labeled by their inhabiting group and the zero groups are unmarked. The differentials out of the bottom row are zero, as they head into the fourth quadrant, so the only potentially nonzero differentials begin in the second row and go down to the zeroth. But $\operatorname{bideg} d_3 = (-2,3)$, so these differentials land in odd columns, whereas only even ones are inhabited. Thus the spectral sequence collapses at $E_\infty = \operatorname{gr}_\bullet H^*(E)$.

Now we try to reconstruct $H^*(E)$ from its associated graded. We know $H^0(E) \cong \mathbb{Z}$ because E must be path-connected. The filtration has only one term, so we can also recover this from looking at the $p+q=0$ diagonal of the spectral sequence. Explicitly,

$$\mathbb{Z} = \operatorname{gr}_0 H^0(E) = F_0H^0(E)/F_1H^0(E) = H^0(E)/\{0\} = H^0(E).$$

We know $H^4(E) \cong \mathbb{Z}$ because E is a 4-manifold, but in terms of the filtration, we have unknown terms $F_p = F_pH^4(E)$, with successive quotients as indicated below:

$$H^4(E) \underbrace{\geqslant}_{0} F_1 \underbrace{\geqslant}_{0} F_2 \underbrace{\geqslant}_{\mathbb{Z}} F_3 \underbrace{\geqslant}_{0} F_4 \underbrace{\geqslant}_{0} 0$$

It follows that $0 = F_4 = F_3$ and hence that that $\mathbb{Z} \cong F_2/F_3 = F_2 = F_1 = F_0 = H^4(E)$, as projected. As for $H^2(E)$, we have

$$H^2(E) \underbrace{\geqslant}_{\mathbb{Z}} F_1 \underbrace{\geqslant}_{0} F_2 \underbrace{\geqslant}_{\mathbb{Z}} 0,$$

so that $\mathbb{Z} = F_2 = F_1$ and $\mathbb{Z} = H^2(E)/F_1 = H^2(E)/\mathbb{Z}$. Since these groups are abelian, $H^2(E) \cong \mathbb{Z} \oplus \mathbb{Z}$.

[4] We intend to provide diagrams for spectral sequences despite space constraints.

> *Unhappily the authors continue the conspiracy of silence according to which the rectangular diagrams, used by all the experts, never appear in print.*
>
> —Mac Lane, reviewing Cartan and Eilenberg's *Homological Algbra* [Mac56]

Now let us see what we can say about the multiplication. If we write $H^*(F) = \mathbb{Z}[z]/(z^2)$ for the cohomology of the fiber S^2 and $H^*(B) = \mathbb{Z}[x]/(x^2)$ for the cohomology of the base, then

$$E_\infty \;=\; E_2 \;\cong\; \mathbb{Z}[x]\big/(x^2) \otimes \mathbb{Z}[z]\big/(z^2) \;\cong\; \mathbb{Z}[x,z]/(x^2, z^2)$$

as a bigraded ring. As far as $H^*(E)$ itself goes, from Proposition 2.2.5, we can identify x with $\pi^*[S^2] \in H^2(E)$, and pick an element $\tilde{z} \in H^2(E) = F_0$ representing $z = \tilde{z} + F_1$. From Proposition 2.2.5 again, $i^*\tilde{z} = [S^2]$ in the cohomology of the fiber S^2. Since

$$xz = (x + F_3)(\tilde{z} + F_1) = x \smile \tilde{z} + F_3$$

in the associated graded and $F_3 = 0$, it follows that $x \smile \tilde{z} = [E]$ generates $H^4(E)$. Since $x^2 \in F_3 = 0$, it follows $x \smile x = 0$ in $H^*(E)$ and not just in E_∞. As for z^2, we know

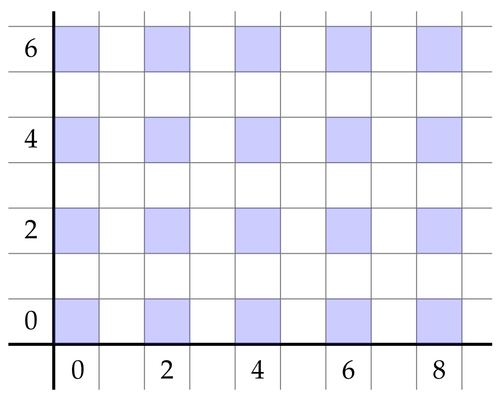
$$0 = z^2 = (\tilde{z} + F_1)(\tilde{z} + F_1) = \tilde{z} \smile \tilde{z} + F_1$$

in the associated graded, but this means only that $\tilde{z} \smile \tilde{z} \in F_1 = F_2$. Since $[E]$ lies in $F_2H^4(E)$, this actually doesn't tell us anything about $\tilde{z} \smile \tilde{z}$.

Indeed, we chose $\tilde{z}$ as a representative of $z \in H^2(E)/\mathbb{Z}x$, so for any $n \in \mathbb{Z}$, the element $\tilde{z} + nx$ serves equally well as a generator of $H^2(E)$. This element squares to $\tilde{z} \smile \tilde{z} + 2n[E]$, since $x \smile x = 0$, so choosing n appropriately we can replace $\tilde{z}$ with $\tilde{z}'$ such that $\tilde{z}' \smile \tilde{z}'$ is either 0 or $[E]$.[5]

This example shows both the strengths and the limitations of this technique. That E_2 is E_∞ was helpful; when this happens, one says the spectral sequence ***collapses*** at E_2. We can generalize the collapse of the example substantially.

Figure 2.2.8: Even support implies collapse



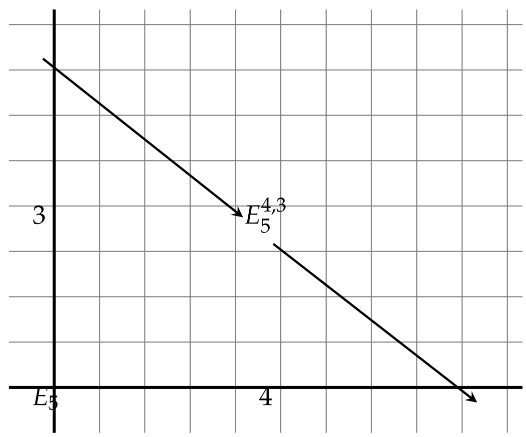
Corollary 2.2.9. *Let $F \to E \to B$ be a fibration such that the action of $\pi_1 B$ on $H^*(F)$ is trivial and $H^*(B)$ and $H^*(F)$ are both concentrated in even degrees. Then the spectral sequence collapses at E_2.*

Proof. If $H^*(B)$ and $H^*(F)$ are both concentrated in even degrees, then so is $E_2 = H^*\big(B; H^*(F)\big)$ concentrated in even total degree, as in Figure 2.2.8. Since the differentials d_r increase total degree by 1, mapping from even diagonals to odd and vice versa, they must all be trivial, so the sequence collapses at E_2. □

[5] Indeed, these are both options. If $E = S^2 \times S^2$, then by the Künneth theorem B.1.2 we can arrange that $\tilde{z}^2 = 0$. We will not show this, but the other option is realized by $E = (S^3 \times S^2)/S^1$, where S^1 acts on S^2 by rotation about a fixed axis and on $S^3 \subsetneq \mathbb{C}^2$ by the diagonal action (complex scalar multiplication).

Thus, for example, the analysis of Example 2.2.6 carries through to any bundle of the form $S^{2q} \to E \to S^{2p}$ for $p, q > 0$, so that $H^*(E) \cong \mathbb{Z}\{1, x, z, xz\}$ as a graded group for $x = \pi^*[S^{2p}]$ and z such that $i^*z = [S^{2q}]$, and we have $x \smile x = 0$. If $p \neq q$, we also have $z \smile z = 0$ since $H^{4q}(E) = 0$, so that $H^*(E) \cong H^*(S^{2p}) \otimes H^*(S^{2q})$ as a graded ring. This genre of reasoning, that something must stabilize at a certain page—or vanish before a certain page, lest it survive to E_∞—goes by the trade name of "***lacunary considerations***." One uses such considerations as frequently as possible because they are usually far simpler than actually computing differentials. Occasionally this kind of spatial reasoning enables one to understand what happens in a spectral sequence without having done any algebra at all.

Figure 2.2.10: The differentials to and from $E_5^{4,3}$ leave the first quadrant



Another simple example of a lacunary consideration is the following:

Proposition 2.2.11. *Let (E_r, d_r) be a first-quadrant spectral sequence. If $p < r$ and $q < r - 1$, then $E_r^{p,q} = E_\infty^{p,q}$.*

Proof. Because the bidegree of d_r is $(r, 1 - r)$, the domain $E_r^{p-r,q+r-1}$ of the component of d_r with codomain $E_r^{p,q}$ lies in the second quadrant, and the codomain $E_r^{p+r,q+1-r}$ of the component of d_r with domain $E_r^{p,q}$ lies in the fourth quadrant. See Figure 2.2.10. Since these quadrants are inhabited only by zero groups, the differentials in and out of $E_r^{p,q}$ are zero, so $E_r^{p,q} = E_{r+1}^{p,q}$. All later differentials out of this square must also be zero for the same reason. □

We notice that in the examples $S^{2q} \to E \to S^{2p}$, $i^* \colon H^*(E) \longrightarrow H^*(S^{2q})$ was surjective and also the spectral sequence collapsed. This is no coincidence.

Corollary 2.2.12. *Let $F \xrightarrow{i} E \to B$ be a fibration such that the action of $\pi_1(B)$ on $H^*(F)$ is trivial and $H^*(F)$ is a flat k-module. Then i^* is surjective if and only if the spectral sequence of the bundle collapses at E_2.*[6]

Proof. Recall from Proposition 2.2.5 that the fiber projection $i^* \colon H^*(E) \longrightarrow H^*(F)$ factors as be realized as $H^*(E) \twoheadrightarrow H^*(E)/F_1 = E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet}$. This map will be surjective if and only if $E_\infty^{0,\bullet} = \cdots = E_3^{0,\bullet} = E_2^{0,\bullet}$, which means that $E_3^{0,\bullet} = E_2^{0,\bullet} \cap \ker d_2 = E_2^{0,\bullet}$, so that $d_2 E_2^{0,\bullet} = 0$, similarly that $d_3 E_3^{0,\bullet} = 0$ and so on: all differentials vanish on the left column.

[6] We do not discuss the general case where $\pi_1(B)$ potentially acts nontrivially on $H^*(F)$, but in general $E_2^{0,\bullet} \cong H^*(F)^{\pi_1(B)}$, so in fact if i^* is surjective, then $\pi_1(B)$ *must* act on $H^*(F)$ trivially.

This is the case by definition if the sequence collapses at E_2. For the converse implication, note that by our assumptions, $E_2 \cong H^*(B) \otimes_k H^*(F)$, and d_2 vanishes on $H^*(B)$ by lacunary considerations. If i^* is surjective, then as we have discussed, the differentials vanish on the left columns $E_\bullet^{0,\bullet}$. Since d_2 is an antiderivation vanishing on tensors of the form $1 \otimes z$ and $x \otimes 1$ both, it is identically zero, so $E_3 = E_2 \cong H^*(B) \otimes H^*(F)$. But d_3 on $H^*(B)$ by necessity and on $H^*(F)$ by assumption, so one has $d_3 = 0$ as well. By induction, $E_2 = E_\infty$. □

Something even stronger can be said.

Theorem 2.2.13 (Leray–Hirsch)**.** *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration such that the action of $\pi_1(B)$ on $H^*(F)$ is trivial, $H^*(F)$ is a free k-module, and i^* is surjective. Then $H^*(E) \cong H^*(B) \otimes H^*(F)$ as an $H^*(B)$-module.*

This theorem, due to Leray and Hirsch, can be viewed as a strengthening of the Künneth Theorem B.1.2. The proof can be seen as a less structured version of that of Proposition A.4.4.

Proof. From Corollary 2.2.12 we see that $\operatorname{gr}_\bullet H^*(E) = E_\infty \cong H^*(B) \otimes H^*(F)$ as a bigraded algebra, but it is not *a priori* clear what bearing this has on the original multiplicative structure. Select a graded k-module basis (z_j) for $H^*(F)$ and lift the elements $1 \otimes z_j \in E_\infty^{0,\bullet} = \operatorname{gr}_0 H^*(E)$ back to elements $\tilde{z}_j$ of $H^*(E)$. Then $M = \pi^* H^*(B)\{\tilde{z}_j\}$ is a filtered graded $H^*(B)$-submodule of $H^*(E)$, and there is by Proposition 2.2.5 a natural $H^*(B)$-module homomorphism $\psi \colon E_\infty \longrightarrow M$. This homomorphism clearly preserves the filtration induced from the grading of $H^*(B)$, so $\operatorname{gr}_\bullet \psi \colon H^*(B) \otimes H^*(F) \longrightarrow H^*(B) \otimes H^*(F)$ is defined, and as it takes $1 \otimes z_j \longmapsto 1 \otimes z_j$ by construction, it is an $H^*(B)$-module isomorphism. Thus, by Corollary 2.5.2, so is ψ. □

Exercise 2.2.14. Derive the topological Künneth theorem over a field k by applying Theorem 2.2.13 to the projections of $X \times Y$.

Remark 2.2.15. [Explain the significance of the Künneth theorem and the zig-zag argument per Loring's book as Leray's motivation for spectral sequences, possibly with an original Leray quote. Borel quote: "The starting point is an argument which occurs repeatedly in [1945a]. Its first goal was to prove that the 'forms on a space' (see 6) obey some of the rules of exterior differential calculus (cf. the introductory remarks in [1945b] quoted above in 5). According to [1950a] p. 9 or [1959c], p.10, it is the analysis of this argument which led Leray to the cohomological invariants of a continuous map, described initially in [1946b]."]

Theorem 2.2.16 (Leray [Ler50][find theorem number])**.** *Let $F \to E \to B$ be a fibration and k a ring such $H^*(B;k)$ contains no 2-torsion, the action of $\pi_1(B)$ on $H^*(F;k)$ is trivial, and $H^*(F) \cong k[\vec{z}]/(\vec{z})^2$, where each degree $|z_j|$ is even and positive; in other words, let F have the cohomology of a product of connected even-dimensional spheres. Then $H^*(E) \cong H^*(B) \otimes H^*(F)$ as an $H^*(B)$-module.*

Proof. By the preceding Leray–Hirsch theorem 2.2.13 it is enough to show the spectral sequence collapses at E_2, and by Corollary 2.2.12 to show that all differentials vanish on $H^*(F)$. Since this group is spanned by monomials $z^J = \prod_{j \in J} z_j$ in the generators, it is enough to show each $d_r(1 \otimes z_j) = 0$. Suppose inductively that $d_{r-1} = 0$, so that $E_r \cong E_2 \cong H^*(B) \otimes H^*(F)$. We can write

$$d_r(1 \otimes z_\ell) = \sum x_J \otimes z^J, \qquad x_J \in H^*(B).$$

Since d_r lowers total degree by one, for every term such that $x_J \neq 0$ and all $j \in J$ we have $|z_\ell| > |z_\ell| - 1 \geqslant z_j$, so that z_ℓ does not appear as a factor of any term in $d_r(1 \otimes z_\ell)$. But then

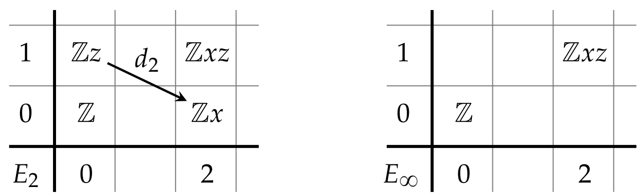
$$0 = d_r(0) = d_r(1 \otimes z_\ell^2) = 2 \sum x_J \otimes z_\ell z^J$$

since d_r is a derivation. Since z_ℓ is not a factor of z^J, the factor $z_\ell z^J$ is nonzero. The monomials in the z_j form a basis of $H^*(F)$, so it follows each x_J is 2-torsion; but by assumption, there is no 2-torsion, so $d_r(1 \otimes z_\ell) = \sum x_J \otimes z^J = 0$ itself, concluding the induction. □

After all this collapse, it is about time for an example with a nontrivial differential.

Example 2.2.17. Consider the Hopf fibration $S^1 \to S^3 \to S^2$, obtained by letting $S^1 < \mathbb{C}^\times$ act diagonally by complex multiplication on $S^3 < \mathbb{C} \times \mathbb{C}$ and modding out to get $\mathbb{CP}^1$. Of course $H^*(S^2) \cong \mathbb{Z}[x]/(x^2)$ for $x = [S^2]$ and $H^*(S^1) \cong \Lambda[z]$ for $z = [S^1]$, which is free abelian, so that $E_2 \cong H^*(S^2) \otimes H^*(S^1)$. See Figure 2.2.18.

Figure 2.2.18: The Serre spectral sequence of $S^1 \to S^3 \to S^2$



In this case, we already know the end result should be $E_\infty = H^*(S^3) = \Lambda[y]$ for $y \in H^3(S^3)$. The only potentially nonzero differential is $d_2 \colon \mathbb{Z}z \longrightarrow \mathbb{Z}x$, whose kernel will be $H^1(S^3) = 0$ and whose cokernel will be $H^2(S^3) = 0$; there is no need to worry about the associated graded because each diagonal $p + q = n$ has at most one nonzero entry. It follows d_2 is an isomorphism and hence $d_2(z) = \pm x$.

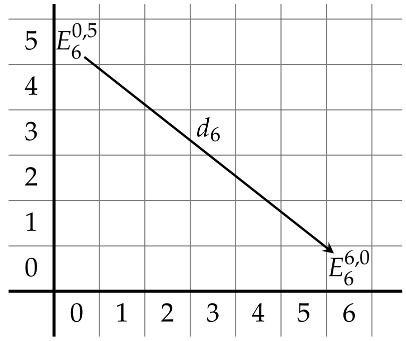
We will use a generalization of this calculation in Section 7.1 to calculate $H^*(\mathbb{CP}^\infty)$.

The d_2 in the previous example stretching from the left column to the bottom row is the first example of an important phenomenon that will play heavily in our computation of the cohomology of a homogeneous space. It admits the following characterization. In the long exact homotopy sequence of the Hopf fibration $S^1 \to S^3 \to S^2$, the boundary map $\partial \colon \pi_2(S^2) \longrightarrow \pi_1(S^1)$ is an isomorphism. Recall that this sequence can be identified with the long exact sequence of the pair (S^3, S^1), where S^1 is thought of as the fiber over some point $* \in S^2$, and that this long exact sequence is connected to the long exact homology sequence via the Hurewicz map. Modulo torsion, the cohomology long exact sequence is dual to this long exact sequence.

Exercise 2.2.19. Use Hurewicz maps to check that the dual of ∂ is $\delta \colon H^1(S^1) \xrightarrow{\sim} H^2(S^3, S^1)$ and the map $\pi^* \colon H^2(S^2) \cong H^2(S^2, *) \to H^2(S^3, S^1)$ is an isomorphism.

Then d_2 is determined as $d_2 = (\pi^*)^{-1} \circ \delta$. In general $(\pi^*)^{-1} \circ \delta$ is not a well-defined map, but a relation on $H^{r+1}(B) \times H^r(F) \cong E_2^{r+1,0} \times E_2^{0,r}$. We will show momentarily that this relation describes via representatives in E_2 the ***transgression*** maps $d_{r+1} \colon E_{r+1}^{0,r} \longrightarrow E_{r+1}^{r+1,0}$ for each $r \geqslant 2$.

Figure 2.2.20: The transgression



We will discuss the transgression in both the filtration and Serre spectral sequences and prove the following result in Section 2.8.

Proposition 2.2.21. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration with all spaces path-connected and such that the action of $\pi_1 B$ on $H^*(F)$ is trivial. An element $[z] \in H^r(F) = E_2^{0,r}$ (Definition 2.8.1) represents an element of $E_{r+1}^{0,r}$, and hence transgresses to the class in $E_{r+1}^{r+1,0}$ represented by some $[b] \in H^{r+1}(B)$, if and only if there exists $c \in C^r(E)$ in the singular cochain group such that $i^*c = z$ and $\delta c = \pi^* b$. This is the picture:*

$$\begin{array}{ccc} & C^q(E) & \xrightarrow{i^*} & Z^q(F) \\ & \downarrow{\scriptstyle \delta} & \swarrow{\scriptstyle \tau} & \\ Z^{q+1}(B) & \xrightarrow[\pi^*]{} & Z^{q+1}(E), & \end{array} \qquad \begin{array}{ccc} c & \longmapsto & z \\ \downarrow & \swarrow & \\ b & \longmapsto & \delta c. \end{array}$$

We will ultimately need this cochain-level description to prove Theorem 8.1.5, but there is an illuminating way of understanding the transgression which does not require us to descend this far. Recall from Theorem B.1.4 that associated to a bundle $F \xrightarrow{i} E \xrightarrow{\pi} B$ is an exact triangle of homotopy groups

$$\pi_*(F) \longrightarrow \pi_*(E) \longrightarrow \pi_*(B) \xrightarrow{\deg -1} \pi_*(F).$$

Thus there is a degree-shifting map linking the homotopy groups of the base and fiber. Viewing $F = E|_*$ as a specific fiber over a point $* \in B$, this sequence arises from the long exact sequence of relative homotopy groups associated to the pair (E, F),

$$\pi_*(F) \longrightarrow \pi_*(E) \longrightarrow \pi_*(E, F) \xrightarrow{\deg -1} \pi_*(F),$$

via the homotopy lifting property. The long exact sequence of a pair

$$H^*(F) \xrightarrow{\deg +1} H^*(E, F) \longrightarrow H^*(E) \longrightarrow H^*(F).$$

is one of the Eilenberg–Steenrod axioms, but it no longer will do in general to substitute $\widetilde{H}^*(B) = H^*(B, *)$ for $H^*(E, F)$. If it did, we would always have a degree-shifting cohomological map like the transgression linking the base and the fiber. Nevertheless, π is a map of pairs $(E, F) \longrightarrow (B, *)$, so one has map of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^q(F) & \xrightarrow{\delta} & H^{q+1}(E,F) & \longrightarrow & H^{q+1}(E) & \xrightarrow{i^*} & H^{q+1}(F) & \longrightarrow \cdots \\
& \uparrow & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow & \\
\cdots \longrightarrow & H^q(*) & \longrightarrow & H^{q+1}(B,*) & \xrightarrow{\sim} & H^{q+1}(B) & \longrightarrow & H^{q+1}(*) & \longrightarrow \cdots
\end{array}$$

Proposition 2.2.22. *The transgression is given by the composite relation $(\pi^*)^{-1} \circ \delta$.*

Proof. A pair $\big([b],[z]\big) \in H^{q+1}(B) \times H^q(F)$ stands in the relation $(\pi^*)^{-1} \circ \delta$, by definition, if $\pi^*[b] = \delta[z]$ in $H^{q+1}(E,F)$. But $\delta[z] \in H^{q+1}(E,F)$ is by definition the class of δc for any cochain $c \in C^q(E)$ such that $i^*c = z$. Thus, if elements (z,c,b) satisfy the specification put forth in Proposition 2.2.21, then $\pi^*[b] = [c] = \delta[z]$. Conversely, if $\pi^*[b] = \delta[z]$, then the proof of Proposition 2.2.21 shows that the extension c of z can be chosen so that $\pi^* b = \delta c$ on the nose. □

Thus the transgressed classes in $H^{q-1}(F)$ can be imagined as the images of the connecting homomorphism $\eta = (\pi^{-1})^* \circ \delta$ in a fictitious long exact sequence

$$H^*(F) \xrightarrow{\eta} H^*(B) \longrightarrow H^*(E) \xrightarrow{i^*} H^*(F)$$

of a bundle corresponding to the long exact sequence of homotopy groups. The transgressive elements can be said, morally speaking, to be those for which such a sequence holds.

Remark 2.2.23. There is an analogous Serre spectral sequence of a bundle in *homology*, whose differentials are of degree $(-r, r-1)$, and a (partially defined) transgression $H_r(B) \longrightarrow H_{r-1}(B)$. Dually to our definition in cohomology, the transgressed elements of $H_q F$ are images of transgressive elements of $H_{q+1} B$ under an incompletely-defined map τ_* in the dual fictitious long exact sequence

$$H_*(B) \xrightarrow{\tau_*} H_*(F) \longrightarrow H_*(E) \longrightarrow H_*(B).$$

Because the Hurewicz homomorphism $\pi_*(X,A) \longrightarrow H_*(X,A)$ from homotopy groups to homology groups discussed in Theorem B.1.1 is natural, it pieces together into a map from the homotopy long exact sequence of a pair (E,F) to the homology long exact sequence of that pair. It follows from the existence of this map of long exact sequences and the long exact homotopy sequence of a bundle (Theorem B.1.4) that everything in the image of the Hurewicz map $\pi_* F \longrightarrow H_* F$ is the image of the transgression in every fibration with fiber F, a fact we will have cause to comment on again in Section 7.4. [Flesh this out.] Moreover, when k is a field, the cohomology transgression $\tau\colon H^{q-1}(F) \longrightarrow H^q(B)$ and the homology transgression $\tau_*\colon H_q(B) \longrightarrow H_{q-1}(F)$ are *dual* [Ral]. [Flesh this out.]

Remarks 2.2.24. (*a*) Although we will also have occasion to invoke the spectral sequence of a filtered DGA again in Section 7.4, Theorem 8.1.5, and Appendix C.3, from here on out, "spectral sequence" *simpliciter* will connote the cohomological Serre spectral sequence of a bundle. It will be deployed with sufficient frequency that we allow ourselves also to abbreviate it *SSS*.

(*b*) This spectral sequences applies more generally, even if instance that $\pi_1 B$ fails to act trivially on $H^*(F)$, with the concession that the coefficients $H^*(F)$ must instead be taken as a sheaf of groups or, at the most concrete, a $k[\pi_1 B]$-module.

(*c*) In the event the fibration $F \to E \xrightarrow{\pi} B$ is in fact a fiber bundle, as it will be in all cases that actually concern us, the Serre spectral sequence is isomorphic from E_2 on to the *Leray spectral sequence* of the map π, which we will introduce in Appendix C.2 to complete our account of Borel's original 1953 proof of Theorem 8.1.14.

(*d*) We have stated Serre's theorem for singular simplicial cohomology, but he initially stated it for singular cubical homology and cohomology, and it goes through essentially unchanged for Alexander–Spanier cohomology, Čech cohomology, or cohomology with A_{PL}-cochains as we introduce in Section 4.2. The skeletal filtration $F_pC^*(E) = \ker\big(C^*(E) \to C^*(\pi^{-1}B^p)\big)$ is actually due to Kudo. Writing $I = [0,1]$ for the unit interval, $I^n \longrightarrow I^p$ for the projection from a cube onto the first p coordinates, and $\pi\colon E \longrightarrow B$ for the fibration in question, Serre's filtration is

$$F^pC_n^{\mathrm{cube}}(E) := \{c\colon I^n \to E \mid \pi \circ c\colon I^n \to E \to B \text{ factors through } I^n \to I^p\}.$$

2.3. Sample applications

Starting in Chapter 3 and throughout the book we will see more than enough examples of the Serre spectral sequence to build a healthy intuition, but before we do this the author wanted to give some example of its broad applicability. We begin with a number of results Leray announced in the *Comptes Rendus* notes where he publicized his creation to the world and a notable early result of Borel and Serre before citing some results from Serre's thesis. This material is not needed for the main development.

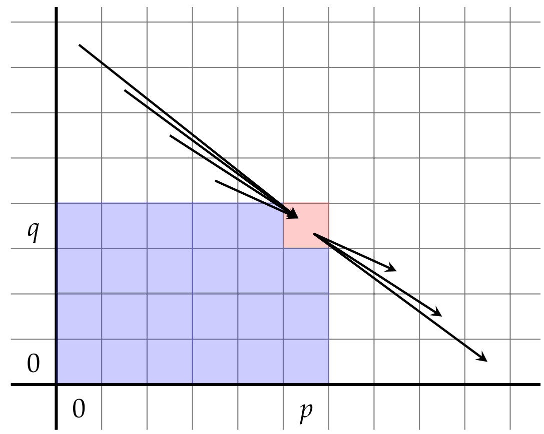
The following theorem was the first successful application of spectral sequences by anyone but Leray. In late 1949, Borel and Serre resolved what had been taken to be a hard problem in one afternoon.

Theorem 2.3.1. *If $F \to \mathbb{R}^n \xrightarrow{\pi} B$ is a fiber bundle over a CW complex B with path-connected fiber F, then $\widetilde{H}^*(B) \cong 0 \cong \widetilde{H}^*(F)$.*

We say the spaces F and B are ***acyclic*** in this case.

Proof. Since $\mathbb{R}^n$ is connected, B must be as well. The homotopy long exact sequence of the bundle contains the fragment $\pi_1(\mathbb{R}^n) \to \pi_1 B \to \pi_0 F$, so B is simply-connected, and Theorem 2.2.2 applies. Since $\mathbb{R}^n$ is n-dimensional, B is a CW complex of dimension at most n, and F is a deformation retract of an open subset $\pi^{-1}(U) \approx U \times F$ for contractible open $U \subsetneq B$, so $H^{\geqslant n+1}B = 0 = H^{\geqslant n+1}F$. Let $p,q \leqslant n$ be maximal such that $H^p(B)$ and $H^q(F)$ are nonzero; we need to show $p = q = 0$. By the universal coefficient theorem B.1.1, we have $E_2^{p,q} = H^p\big(B; H^q(F)\big) \cong H^p(B) \otimes H^q(F) \neq 0$. This is the red square Figure 2.3.2. Now we consider the E_2 page of the Serre spectral sequence.

Figure 2.3.2: Only blue is inhabited, so red does not support a differential



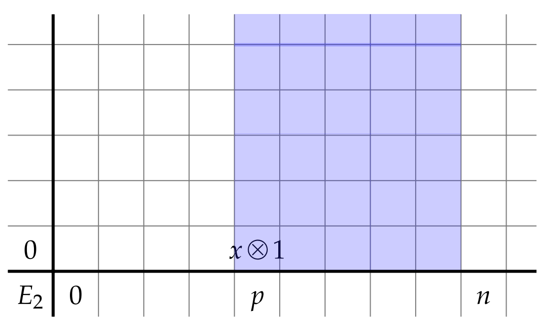
Since $H^{>p}(B) = 0 = H^{>q}(F)$, the only potentially inhabited squares lie in the rectangle $[0, p] \times [0, q]$, shown in blue in Figure 2.3.2. But differentials to and from the (p, q)-square end outside of this rectangle, so we must have $E_\infty^{p,q} = E_2^{p,q} \neq 0$. But $H^{p+q}(\mathbb{R}^n) = 0$, so $p = q = 0$. $\square$

This is actually weaker than the original statement, which unfortunately uses a bit too much background for the proof to be self-contained.

Theorem 2.3.3 (Borel–Serre)**.** *Let $F \to \mathbb{R}^n \to B$ be a bundle with compact fiber F. Then F is a point and B is $\mathbb{R}^n$.*

Proof. Since $\mathbb{R}^n$ is locally path-connected, so is F, and the quotient map reducing each path-component of F to a point defines another fibration $\varpi \colon \mathbb{R}^n \longrightarrow B'$. Since the base B' is path-connected, this is another fiber bundle, this time with connected fiber F_0.

Figure 2.3.4: A contradictory permanent cycle



We first show F_0 is a point by contradiction. Note that for a sufficiently small neighborhood U of any point of B we have $\varpi^{-1}(U) \approx U \times F_0$ an open subset of $\mathbb{R}^n$, and since F_0 is assumed not to be a point, it follows from dimension theory that B' has topological dimension $\leqslant n - 1$. Now we consider the Leray spectral sequence of the bundle $F_0 \to \mathbb{R}^n \to B'$ in *Čech cohomology with compact supports* $\check{H}_c^*$, as derived in Appendix C.2. This works algebraically the same way as the Serre spectral sequence of the bundle but has

$$E_2^{p,q} = \check{H}_c^p\big(B; \check{H}_c^q(F; \mathbb{R})\big) \cong \check{H}_c^p(B; \mathbb{R}) \underset{\mathbb{R}}{\otimes} \check{H}_c^q(F_0; \mathbb{R})$$

and converges to $\check{H}_c(\mathbb{R}^n;\mathbb{R})$, which is $\mathbb{R}$ in dimension n and zero in all other dimensions. It follows the total degree n of E_2 is nonzero so for some $p \leqslant n-1$ there is a nonzero element of $E_2^{p,n-p}$. In particular, for some p there is a nonzero element x of $\check{H}_c^p(B;\mathbb{R})$. Let p be minimal such this holds. Since F_0 is compact connected, we have $\check{H}_c^0(F_0;\mathbb{R}) \cong \check{H}^0(F_0;\mathbb{R}) \cong \mathbb{R}$, represented by the constant function 1. Now, as seen in Figure 2.3.4, the element $x \otimes 1$ is of minimum total degree in E_2 (alternately), it receives no differentials from the nonzero region, so it must persist to E_∞. But $\check{H}_c^p(\mathbb{R}^n;\mathbb{R}) = 0$, a contradiction.

We have shown each component F_0 of the original fiber F is a point. As F is compact, it follows it is a finite discrete set, so that $\mathbb{R}^n$ is the universal cover of B. It follows $\pi_1(B)$ is a finite group acting freely freely on $\mathbb{R}^n$. If $\pi_1(B) \neq 1$, then by Cauchy's theorem, there is an element $\gamma \in \pi_1(B)$ of some prime order p, generating a free $\mathbb{Z}/p$-action on $\mathbb{R}^n$. Compactifying $\mathbb{R}^n$ with a point at infinity, we get a $\mathbb{Z}/p$-action on S^n with precisely one fixed point. But this is impossible by Smith theory [Hsi75, p. 50], which shows that the fixed point set $X = (S^n)^{\mathbb{Z}/p}$ must have $H^*(X;\mathbb{F}_p) \cong H^*(S^m;\mathbb{F}_p)$ for some sphere S^m (with $m < n$). It follows that $\pi_1(B) = 1$, so F is connected. □

Corollary 2.3.5 (Leray [Ler46a]). *Let $F \to E \to B$ be a fibration such that the action of $\pi_1 B$ on $H^*(F)$ is trivial and $H^*(F)$ is a free k-module. Suppose further that F and B are of finite type. Then the Poincaré series satisfy*

$$p(E) \leqslant p(B)p(F),$$

in the sense that each coefficient of $p(B)p(F) - p(E)$ is nonnegative, with equality if and only if the fiber inclusion $F \hookrightarrow E$ is surjective in cohomology. More specifically, there is a series $b(t) \in \mathbb{N}[[t]]$ such that

$$p(E) + (1+t)b(t) = p(B)p(F) \text{ in } \mathbb{N}[[t]].$$

Proof. We take $k = \mathbb{Q}$. Then $E_2 = H^*(B;\mathbb{Q}) \otimes H^*(F;\mathbb{Q})$ in the Serre spectral sequence of $F \to E \to B$, showing $p(E_2) = p(B)p(F)$. The rank of each $E_r^{p,q}$, and hence the Poincaré polynomial, can only decrease by E_∞, and it can only *fail* to decrease if $E_2 \cong E_\infty$; that is the case if and only if $H^*(E;\mathbb{Q}) \twoheadrightarrow H^*(F;\mathbb{Q})$, by Corollary 2.2.12.

On the level of graded vector spaces, through the selection of arbitrary graded linear complements, we have the following isomorphisms:

$$\begin{aligned} E_2 &\cong \ker d_2 \oplus E_2/\ker d_2, \\ \ker d_2 &\cong \operatorname{im} d_2 \oplus E_3. \end{aligned}$$

Since d_2 descends to a graded isomorphism $E_2/\ker d_2 \xrightarrow{\sim} \operatorname{im} d_2$ of degree one, it follows

$$p(E_2) = p(\ker d_2) \oplus t^{-1} p(\operatorname{im} d_2) = p(E_3) + (1+t^{-1})p(\operatorname{im} d_2).$$

Set $b_2(t) = t^{-1}p(\operatorname{im} d_2) \in \mathbb{N}[[t]]$, so that we get $p(E_2) = p(E_3) + (1+t)b_2(t)$. A similar analysis provides for each $r \geqslant 2$ a series $b_r(t) \in \mathbb{N}[[t]]$ such that $p(E_r) = p(E_{r+1}) + (1+t)b_r(t)$. Now, in each fixed total degree n, the sequence (E_r^n) stabilizes at a finite $r = r(n)$, so the n^{th} coefficient of $b_s(t)$ is zero for $s \geqslant r(n)$. Hence it makes sense to take the limit as $r \to \infty$ of the equations $p(E_2) = p(E_{r+1}) + (1+t)\sum_{s=2}^r b_s(t)$. □

The Serre spectral sequence allows a vast generalization of the covering result Proposition B.2.5.

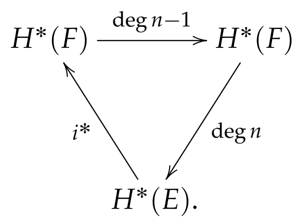
Proposition 2.3.6. *Let $F \to E \to B$ be a fiber bundle such that the action of $\pi_1 B$ on $H^*(F)$ is trivial and $h^\bullet(B)$ and $h^\bullet(F)$ are finite. Then the Euler characteristics of these spaces satisfy $\chi(E) = \chi(F)\chi(B)$.*

Proof. Consider $E_2 = H^*(B) \otimes H^*(F)$ as a single complex with $\deg(H^p B \otimes H^q F) = p + q$. With this grading, $\chi(E_2) = \chi(B)\chi(F)$. By repeated application of Proposition A.3.1, one finds

$$\chi(E_2) = \chi(E_3) = \cdots = \chi(E_\infty) = \chi(E).$$

□

Proposition 2.3.7. *Given a fibration $F \xrightarrow{i} E \to B$ such that $H^*(B) \cong H^*(S^n)$ and $\pi_1(B)$ acts trivially on $H^*(F)$, there exists a* ***Wang exact sequence***



Exercise 2.3.8 (Leray [Ler46a]). Prove Proposition 2.3.7, consulting Figure 2.3.10 and emulating the proof of Proposition 2.3.11.

[Add Leray's G/S^1 proof as best we can reconstruct it.]

2.3.1. Sphere bundles

Figure 2.3.9: The Gysin sequence

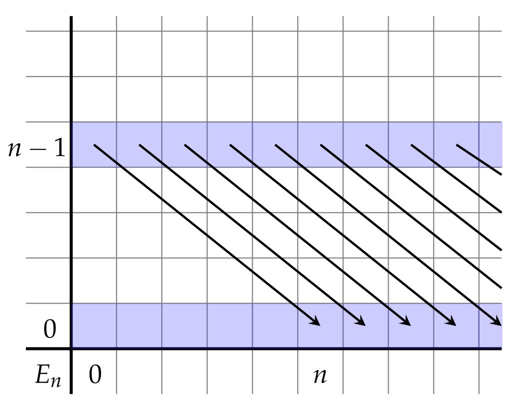
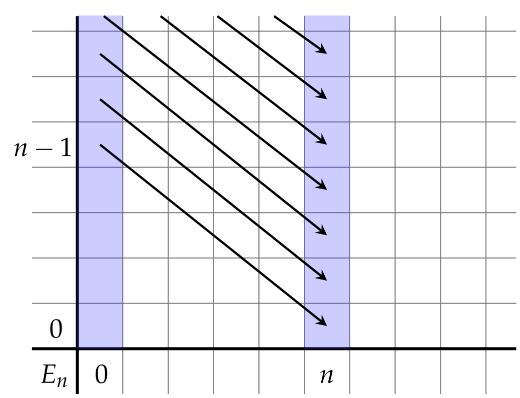
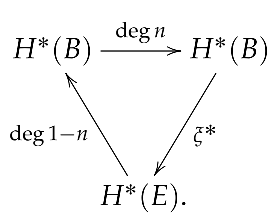


Figure 2.3.10: The Wang sequence



Proposition 2.3.11 (Gysin, in homology [Gys41]; Steenrod, in cohomology [Stea, §11])**.** *Given a fibration $F \to E \xrightarrow{\xi} B$ such that $H^*(F) \cong H^*(S^{n-1})$ and $\pi_1(B)$ acts trivially on $H^*(F)$, there exists a long exact* ***Gysin sequence*** *of graded groups*



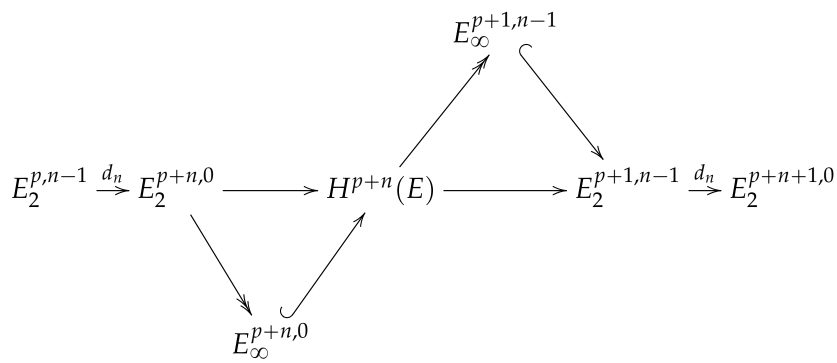
The map $H^*(B) \longrightarrow H^*(B)$ is linear up to a sign and the map ξ_* of degree $1-n$ satisfies $\xi_*\big(\xi^*(b) \smile x\big) = b \smile \xi_*(x)$.

In the most important case, F is actually a sphere.

Proof (Leray [Ler46a]). From Figure 2.3.9, it is clear that the only potentially nontrivial differential is d_n, so $E_n = E_2$ and $E_{n+1} = E_\infty$. The kernel and cokernel of d_n are respectively $E_\infty^{\bullet,n-1}$ and $E_\infty^{\bullet,0}$; in sequence form,

$$0 \to E_\infty^{p,n-1} \longrightarrow E_2^{p,n-1} \longrightarrow E_2^{p+n,0} \longrightarrow E_\infty^{p+n,0} \to 0$$

is exact for each p. But since $E_\infty = \operatorname{gr}_\bullet H^*(E)$, we have $E_\infty^{p+n,0} \cong F_1 H^{p+n}(E)$ and $E_\infty^{p+1,n-1} \cong H^{p+n}(E)/F_1 H^{p+n}(E)$, so we can splice these sequences end-to-end: the horizontal file of



is exact. Further, $E_2^{p,0} \cong H^p(B) = E_2^{p,n-1}$ for all p, so we can roll up this sequence into an exact triangle as claimed in the statement of the theorem. □

Exercise 2.3.12. Verify the map $H^*(B) \longrightarrow H^*(E)$ arising from our identifications is indeed ξ^* and the map ξ_* has the claimed $H^*(B)$-linearity property.

We say a sphere bundle $S^{n-1} \to E \xrightarrow{\xi} B$ is ***oriented*** with respect to k if the conditions of the theorem hold. Thus all sphere bundles are oriented with respect to $k = \mathbb{F}_2$ or if π_B preserves the orientation class $[S^{n-1}] \in H^{n-1}(S^{n-1})$, and not generally. Note that the map $H^*(B) \longrightarrow H^*(B)$ comes from the transgression d_n, which takes $b \otimes [F] \longmapsto (-1)^{|b|} b \cdot d_n[F] \otimes 1$, so it is right multiplication by $(-1)^{|b|} d_n[F]$. Thus in a sense $d_n[F] = \tau[F]$ is the only cohomology invariant of an orientable sphere bundle $\xi\colon E \to B$.

Definition 2.3.13. When $S^{n-1} \to E \to B$ is a $\mathbb{Z}$-orientable sphere bundle, the class $\tau[F] \in H^n(B;\mathbb{Z})$ is called the ***Euler class*** and written $e(\xi)$. When $S^{n-1} \to E \to B$ is any sphere bundle, the class $\tau[F] \in H^n(B;\mathbb{F}_2)$ is called the n^{th} ***Stiefel–Whitney class*** and written $w_n(\xi)$.

Since the Serre spectral sequence is functorial in bundle maps, so are these classes: that is, if $(\bar{f}, f)\colon (E' \xrightarrow{\xi'} B') \longrightarrow (E \xrightarrow{\xi} B)$ is a map of oriented sphere bundles, then

$$f^* e(\xi) = e(\xi'),$$
$$f^* w_n(\xi) = w_n(\xi').$$

The coefficient homomorphism induced by $\mathbb{Z} \longrightarrow \mathbb{F}_2$ induces a map of spectral sequences sending a generator of $H^{n-1}(S^{n-1};\mathbb{Z})$ to a generator of $H^{n-1}(S^{n-1};\mathbb{F}_2)$, so in fact $w_n = e \mod 2$.

Note that if n is odd, so that F is an even-dimensional cohomology sphere and k is chosen so that $H^*(B;k)$ does not have 2-torsion, then by Theorem 2.2.16, the Euler class vanishes. We will produce a consquence of this fact in Section 7.5. For now, note that the mapping cylinder $M\xi := B \amalg (E \times I) \big/ \big((e,1) \sim \xi(e)\big)$ of $\xi \colon E \longrightarrow B$ is itself a fiber bundle over B with fibers the cones $\{b\} \cup_\xi (E_b \times I) \approx D^n$ over $E_b \approx S^{n-1}$, naturally containing $\xi \colon E \times \{0\} \longrightarrow B$ as a subbundle. We can apply to this inclusion the following "fiber-relative" Serre spectral sequence.

Theorem 2.3.14. *Let $F \to E \to B$ be a fibration and E' a subspace of E such that $E' \hookrightarrow E \twoheadrightarrow B$ is also a fibration, with fiber F' such that $\pi_1 B$ acts trivially both on $H^*(F;k)$ and $H^*(F';k)$. There exists a first-quadrant fiber-relative Serre spectral sequence $(E_r, d_r)_{r \geqslant 0}$ of generally nonunital k-DGAs with*

$$E_2^{p,q} = H^p\big(B; H^q(F,F';k)\big),$$
$$E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(E,E';k).$$

If $H^*(F,F';k)$ is a free k-module, we may also write $E_2 \cong H^*(B;k) \otimes_k H^*(F,F';k)$. This construction is functorial in maps of fibrations $(F,F') \to (E,E') \to B$ of pairs.

Proof. We collapse each fiber of E' by attaching the mapping cylinder of $\pi' \colon E \longrightarrow B$ to E along E'. As B is a retract of $E \cup_{E'} M\pi'$ via the inclusion of B on the free end of $M\pi'$, the exact sequence of the pair $(E \cup M\pi', B)$ is a split short exact sequence

$$0 \to H^*(E \cup M\pi', B) \longrightarrow H^*(E \cup M\pi') \longrightarrow H^*(B) \to 0.$$

As $M\pi'$ deformation retracts to B and CB to the cone point, it follows $E \cup M\pi' \cup_B CB \simeq E \cup_{E'} CE'$. Thus $H^*(E \cup M\pi', B) \cong \widetilde{H}^*(E \cup CE') \cong H^*(E,E')$, so $H^*(E \cup M\pi') \cong H^*(E,E') \oplus H^*(B)$.

The inclusion of B as a retract induces a map of fibrations $(F \cup CF' \to E \cup M\pi' \to B) \longrightarrow (* \to B \to B)$, inducing a map of spectral sequences which includes $H^*(B) \cong H^*\big(B; H^*(*)\big)$ in $\widehat{E}_2 = H^*\big(B; H^*(F \cup CF')\big)$ as the complement of $E_2 = H^*(B; \widetilde{H}^*(F \cup CF')) \cong H^*\big(B; H^*(F,F')\big)$. As the spectral sequence of the trivial bundle $* \to B \to B$ collapses, its image in $\widehat{E}_\bullet$ does as well, representing the image of $H^*(B) \longrightarrow H^*(E \cup M\pi')$ on $\widehat{E}_\infty$. It follows the spectral subsequence $E_\bullet$ converges to $H^*(E,E')$ as claimed. ☐

Applying this tool to the relative spectral sequence $(D^n, S^{n-1}) \to (M\xi, E) \xrightarrow{(\hat{\xi},\xi)} B$, so long as $\pi_1 B$ acts trivially on $H^{n-1}(S^{n-1})$ we have

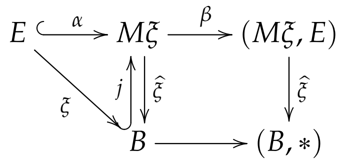
$$E_2 = H^*\big(B; H^*(D^n, S^{n-1})\big) = H^*\big(B; \widetilde{H}^*(S^n)\big) \cong H^*(B) \otimes H^n(S^n) = E_\infty \cong H^*(M\xi, E),$$

since the spectral sequence has only the one nonzero row. It follows there is an element $u \in H^0\big(B; H^n(D^n, S^{n-1})\big)$ such that $\Phi \colon b \longmapsto \hat{\xi}^*(b) \smile u$ is $H^*(B)$-linear isomorphism $H^*(B) \longrightarrow H^{*+n}(M\xi, E)$. This u is called the ***Thom class***. By construction, the inclusion $(D^n, S^{n-1}) \longleftrightarrow (M\xi, E)$ determined by the inclusion of any fiber $S^{n-1} \longleftrightarrow E$ induces a surjection taking u to a generator of $H^n(D^n, S^{n-1})$. We can equally well view u as an element of the cohomology of the ***Thom space***

$$T\xi := M\xi/E,$$

which we can think of a sort of bundle of discs all sharing one point at infinity. The Thom construction is easily seen to be functorial in orientable sphere bundles, since the mapping cylinder is, and it follows the Thom class is as well.

Note from the long exact sequence of the pair $(M\xi, E)$ and commutativity of the diagram



that $\operatorname{im}\beta^* = \ker\alpha^* = \widehat{\xi}^*(\ker\xi^*)$ in positive degree. But $\ker\xi^*$ is the ideal of $H^*(B)$ generated by $e(\xi)$, while

$$\beta^*\widetilde{H}^*(M\xi, E) = \beta^* \operatorname{im}\Phi = \beta^*(\operatorname{im}\widehat{\xi}^* \smile u) = \widehat{\xi}^* H^*(B) \smile u.$$

It follows that $u = \pm j^* e$.

Proposition 2.3.15. *The Euler class is the restriction of the Thom class to the zero section.*

Proof. That u and e are functorial in orientable sphere bundles, the equation $u = \pm j^* e$ can be seen as an equality of natural transformations between the identity functor on orientable S^{n-1}-bundles $\xi : E \to B$ and the set-valued functor $\xi \mapsto B \mapsto H^n(B)$. Thus it will be enough to check the sign on one example.

[To be written...] □

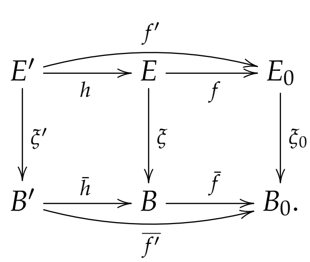
2.3.2. Homotopy groups of spheres and Eilenberg–Mac Lane spaces

[Add results on rational homotopy of spheres and on loop spaces]

2.4. A natural lemma on bundles

In this section, we use the Serre spectral sequence to prove a lemma on cohomology of bundles we will use repeatedly to good effect. It seems analogous to the Theorem 2.2.13 that if $F \to E \to B$ is a bundle such that $H^*(E) \to H^*(F)$ is surjective, then $H^*(E) \cong H^*(B) \otimes H^*(F)$ as an $H^*(B)$-module. There is a proof by Larry Smith [Smi67, Cor. 4.4, p. 88] using the Eilenberg–Moore spectral sequence as well as the Serre spectral sequence, but the following proof only uses what we have already developed.

Let F be a topological space and $\xi_0 \colon E_0 \to B_0$ an F-bundle. From the category of F-bundles and F-bundle maps, we can form a slice category $F\text{-}\mathsf{Bun}/\xi_0$ of F-bundles ***over*** ξ_0 as follows. An object of $F\text{-}\mathsf{Bun}/\xi_0$ is an F-bundle ξ equipped with a bundle map $\xi \to \xi_0$; a morphism between objects $\xi' \to \xi_0$ and $\xi \to \xi_0$ is a bundle map $\xi' \to \xi$ making the expected triangle commute. Such a map entails the following commuting prism:



(2.4.1)

Note that the maps between total spaces yield two functors

$$\begin{aligned} F\text{-}\mathsf{Bun}/\xi_0 &\longrightarrow H^*(E_0)\text{-}\mathsf{CGA}: \\ (E \to B) &\longmapsto H^*(E); \\ (E \to B) &\longmapsto H(B) \underset{H^*(B_0)}{\otimes} H^*(E_0). \end{aligned}$$

If $H^*(E_0) \longrightarrow H^*(F_0)$ is surjective, we claim these functors are naturally isomorphic.

Theorem 2.4.1. *Let $\xi_0 \colon E_0 \to B_0$ be an F-bundle such that the fiber inclusion $F \longhookrightarrow E_0$ is H^*-surjective, such that $H^*(F)$ is a free k-module, and such that $\pi_1 B_0$ acts trivially on $H^*(F)$. Then the fiber inclusions of all F-bundles over ξ_0 are H^*-surjective, and there is a natural ring isomorphism*

$$H^*(E) \xleftarrow{\sim} H^*(B) \underset{H^*(B_0)}{\otimes} H^*(E_0)$$

of functors $F\text{-}\mathsf{Bun}/\xi_0 \longrightarrow H^*(E_0)\text{-}\mathsf{CGA}$. Diagrammatically, the commutative diagram (2.4.1) gives rise to

$$\begin{array}{ccc} H^*(E') & \xleftarrow{\quad h^* \quad} & H^*(E) \\ \uparrow \wr & & \uparrow \wr \\ H^*(B') \otimes_{H^*(B_0)} H^*(E_0) & \xleftarrow[\bar{h}^* \otimes \mathrm{id}]{} & H^*(B) \otimes_{H^*(B_0)} H^*(E_0). \end{array}$$

Verbally, if a fiber inclusion is surjective in cohomology, then cohomology takes pullbacks to pushouts.

Proof. By the definition of a bundle map, the fiber inclusion $F \longhookrightarrow E_0$ factors as $F \hookrightarrow E \to E_0$, so the assumed surjectivity of $H^*(E_0) \to H^*(E) \to H^*(F)$ implies surjectivity of the factor $H^*(E) \longrightarrow H^*(F)$.

Because of these surjections, the spectral sequences of these bundles stabilize at their E_2 pages by Corollary 2.2.12. Applying H^* to the right square of the assemblage (2.4.1) yields

$$\begin{array}{ccc} H^*(E) & \xleftarrow{f^*} & H^*(E_0) \\ \uparrow \scriptstyle{\xi^*} & & \uparrow \scriptstyle{\xi_0^*} \\ H^*(B) & \xleftarrow[\bar{f}^*]{} & H^*(B_0), \end{array}$$

which manifests on the E_2 page as

$$\begin{array}{ccc} H^*(B) \otimes H^*(F) & \xleftarrow{\bar{f}^* \otimes \mathrm{id}} & H^*(B_0) \otimes H^*(F) \\ \uparrow \scriptstyle{\mathrm{id} \otimes 1} & & \uparrow \scriptstyle{\mathrm{id} \otimes 1} \\ H^*(B) & \xleftarrow[\bar{f}^*]{} & H^*(B_0). \end{array}$$

The commutativity of the left square means there is an induced map of rings

$$\begin{array}{ccc} H^*(B) \underset{H^*(B_0)}{\otimes} H^*(E_0) & \longrightarrow & H^*(E), \\ b \otimes x & \longmapsto & \xi^*(b) f^*(x), \end{array}$$

whose E_2 manifestation is the canonical $H^*(B)$-module isomorphism

$$H^*(B) \underset{H^*(B_0)}{\otimes} \big[H^*(B_0) \otimes H^*(F)\big] \xrightarrow{\sim} H^*(B) \otimes H^*(F).$$

Since this E_2 map is a bijection, the ring map is an $H^*(E_0)$-algebra isomorphism.

For naturality, note that the ring map $h^*\colon H^*(E) \longrightarrow H^*(E')$ is completely determined its restrictions to its tensor-factors $H^*(B)$ and $H^*(E_0)$. The left square and top triangle of (2.4.1) imply the commutativity of the squares

$$\begin{array}{ccc} H^*(E') & \xleftarrow{h^*} & H^*(E) \\ \uparrow{\scriptstyle (\xi')^*} & & \uparrow{\scriptstyle \xi^*} \\ H^*(B') & \xleftarrow[\bar{h}^*]{} & H^*(B), \end{array} \qquad \begin{array}{ccc} H^*(E') & \xleftarrow{h^*} & H^*(E) \\ \uparrow{\scriptstyle (f')^*} & & \uparrow{\scriptstyle f^*} \\ H^*(E_0) & = & H^*(E_0), \end{array}$$

so that these factor maps are respectively $\bar{h}^*\colon H^*(B) \longrightarrow H^*(B')$ and $\mathrm{id}_{H^*(E_0)}$. □

2.5. Filtered objects

The rest of this chapter constitutes what should be seen as an *appendix* to the preceding sections of the chapter, to fill in missing technical details and be referred back to as necessary. We will eventually need some level of explicitness in describing the transgression and the construction of a filtration spectral sequence, but the choice of how much to take on faith lies with the conscience of the reader.

In all that follows, k will be an ungraded commutative ring with unity. A ***filtered module*** is a pair $(C, F_\bullet)$, where C is a k-module and $F_\bullet$ is an infinite descending sequence

$$\cdots = F_{-1} = F_0 = C \geqslant F_1 \geqslant F_2 \geqslant \cdots$$

of k-submodules. We also write $F_p = F_pC$.[7] One can equivalently repackage this information as a $\mathbb{Z}$-graded k-module $\bigoplus F_\bullet C := \bigoplus_{p\in\mathbb{Z}} F_p$ equipped with an injective endomorphism i of degree -1 which is an isomorphism in nonpositive degrees. We denote either of these equivalent phrasings, slightly abusively, by (C, i). Say a filtration is ***Hausdorff*** if $\bigcap_{p\in\mathbb{Z}} F_pC = 0$, and ***finite*** if $F_pC = 0$ for p sufficiently large. The k-module

$$\mathrm{gr}_\bullet\, C := \operatorname{coker} i = \bigoplus_{p\geqslant 0} F_pC/F_{p+1}C$$

is the ***associated graded*** module of (C, i). A ***filtered k-algebra*** (C, i) is a k-algebra C such that (C, i) is a filtered group and $F_p \cdot F_q \leqslant F_{p+1}$ for all p, q. In this case $\mathrm{gr}_\bullet\, C$ becomes a graded k-algebra, with multiplication defined on individual degrees by

$$\mathrm{gr}_p\, C \times \mathrm{gr}_q\, C \longrightarrow \mathrm{gr}_{p+q}\, C,$$
$$(x + F_{p+1}) \cdot (y + F_{q+1}) := xy + F_{p+q+1}.$$

A map $f\colon B \longrightarrow C$ is said to ***preserve filtrations*** (B, ι) and (C, i) if $f(F_pB) \leqslant F_pC$. We write such a map as $f\colon (B, \iota) \longrightarrow (C, i)$. Such a map induces an associated graded map $\mathrm{gr}_\bullet\, f\colon \mathrm{gr}_\bullet\, B \longrightarrow \mathrm{gr}_\bullet\, C$. We have the following recurring result on such maps.

[7] In general usage, filtrations (F_pC) are *not* required to stabilize in negative degrees or to be ***exhaustive*** in the sense that $\bigcup_{p\in\mathbb{Z}} F_pC = C$. Since we will never have cause to use such a general filtration, we include these more restrictive hypotheses in our definition off the bat.

Proposition 2.5.1. *Let $f\colon (B,\iota) \longrightarrow (C,i)$ be a filtration-preserving cochain map of filtered groups and suppose that both filtrations are finite. Then if $\operatorname{gr}_\bullet f$ is an isomorphism, so also must be f itself.*

Proof. Fix a filtration degree p sufficiently large that $F_{p+1}B = 0 = F_{p+1}B$. We have a map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{p+1}B & \longrightarrow & F_pB & \longrightarrow & \operatorname{gr}_p B & \longrightarrow & 0 \\ & & \big\downarrow f & & \big\downarrow f & & \wr\big\downarrow \operatorname{gr}_\bullet f & & \\ 0 & \longrightarrow & F_{p+1}C & \longrightarrow & F_pC & \longrightarrow & \operatorname{gr}_p D & \longrightarrow & 0 \end{array}$$

of short exact sequences; by the five lemma, it follows $F_pf\colon F_pB \longrightarrow F_pC$ is an isomorphism. This begins a decreasing induction on p, which terminates in $f\colon B \xrightarrow{\sim} C$ when $p = 0$. □

We can also define ***filtered graded k-modules*** $(C^\bullet, i)$. These are simply direct sums $C^\bullet = \bigoplus_{n\in\mathbb{Z}} C^n$ of filtered k-modules (C^n, i_n) in each degree, equipped with total filtration $F_pC^\bullet = \bigoplus_n F_pC^n$. Such an object is said to be ***finite in each degree*** (more commonly, *bounded*) if the filtration $F_\bullet C^n$ in each degree is finite. For maps of graded filtered groups, applying Proposition 2.5.1 individually in each degree, one finds the following.

Corollary 2.5.2. *Let $f\colon (B^\bullet,\iota) \longrightarrow (C^\bullet,i)$ be a filtration-preserving cochain map of filtered graded groups. Suppose that both filtrations are finite in each degree. Then if $\operatorname{gr}_\bullet f$ is an isomorphism, so also must be f itself.*

It is also useful to know that if an associated graded object is free, the original object must be.

Proposition 2.5.3 ([McC01, Example 1.K, p. 25])**.** *Let $(A^\bullet, i)$ be a filtered graded k-algebra, free as a k-module. If $\operatorname{gr}_\bullet A^\bullet$ is a free bigraded k-CGA, then $\operatorname{gr}_\bullet A^\bullet \cong A^\bullet$ as a singly graded k-CGA.*

Proof. Select free bihomogeneous generators $x \in E_\infty^{p,q}$ of E_∞, and for each of these fix a representative $y \in F_pH^{p+q}(A)$ Then the assignment $x \longmapsto y$ extends to a filtration-preserving map of graded CGAs $E_\infty \longrightarrow H^*(A)$. The induced map of associated graded algebras $E_\infty = \operatorname{gr} E_\infty \longrightarrow \operatorname{gr} H^*(A) = E_\infty$ takes each generator $x \longmapsto x$, and hence is an isomorphism, so by Corollary 2.5.2, the map $E_\infty \longrightarrow H^*(A)$ is an isomorphism as well. □

A ***filtered differential k-module*** is a triple $(C, d, \tilde{\imath})$ such that (C, d) is a differential group, $(C, \tilde{\imath})$ a filtered group, and d preserves the filtration in the sense that $dF_p \leqslant F_p$. A homomorphism of filtered differential k-modules is a cochain map commuting with the filtration. In this case the differential d descends to a differential d_0 on $\operatorname{gr}_\bullet C$, inducing a short exact sequence of differential k-modules

$$0 \to \bigoplus_{p\in\mathbb{Z}} F_pC \xrightarrow{\tilde{\imath}} \bigoplus_{p\in\mathbb{Z}} F_pC \overset{\tilde{\jmath}}{\twoheadrightarrow} \operatorname{gr}_\bullet C \to 0,$$

where $\tilde{\imath}$ is the degree-(-1) map we have identified with the filtration. This induces a triangular exact sequence

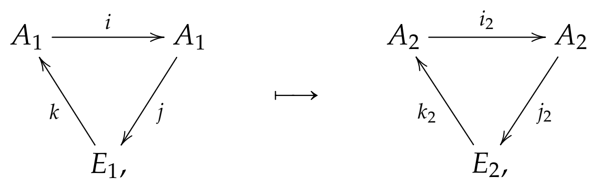
$$\begin{array}{ccc} \bigoplus H(F_pC) & \xrightarrow{\quad i \quad} & \bigoplus H(F_pC) \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & H_{d_0}(\operatorname{gr}_\bullet C) & \end{array}$$

of cohomology groups. Such a triangle is traditionally called an ***exact couple***. If we set $d_1 = jk$, then $d_1^2 = j(kj)k = 0$, so d_1 is a differential on $H_{d_0}(\mathrm{gr}_\bullet C)$. Note for later that $H(\mathrm{gr}_\bullet C)$ is naturally graded by $H(\mathrm{gr}_\bullet C)^p := H^*(\mathrm{gr}_p C)$ and the map i induces a filtration $F_p H(C) := i^p H(F_p C)$ on $H(C)$.

Remark 2.5.4. Note that a map of filtered differential k-modules induces a map of short exact sequences and a map of exact couples in cohomology (a triangular prism), and so in particular a map of differential k-modules between the E components.

2.6. The filtration spectral sequence

There is a functor



taking an exact couple to the ***derived couple*** whose objects are $A_2 = iA_1$ and $E_2 = H(E_1, d_1)$, and whose maps are given by $i_2 = (i \restriction iA_1)$ and $j_2 \colon ia \longmapsto [ja]$ and $k_2 \colon [e] \longmapsto ke$.

Exercise 2.6.1. Check these maps are well-defined and that the derived couple is again exact.

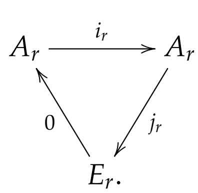
Exercise 2.6.2. Check that a map of exact couples induces a map of derived couples.

One iterates this process, and the sequence (E_r, d_r) of differential groups so derived is called the ***spectral sequence of the exact couple***. Each E_r is traditionally called a ***page***.[8] A ***homomorphism of spectral sequences*** is a sequence $\big(\psi_r \colon (\widetilde{E}_r, \tilde{d}_r) \longrightarrow (E_r, d_r)\big)_{r \geqslant n}$ of cochain maps of differential groups such that each ψ_{r+1} for $r \geqslant n$ is induced by ψ_r, which is to say $\psi_{r+1} = H(\psi_r)$. From Remark 2.5.4 and Exercise 2.6.2, it follows that a map of filtered differential groups induces a map of exact couples and iteratively a map of spectral sequences.

In all our applications in this book, the initial exact couple (A_1, E_1) will be that from Section 2.5, namely $\big(\bigoplus_p H(F_p C), H(\mathrm{gr}_\bullet C)\big)$, induced by a filtered differential group $(C, \tilde{d}, \tilde{\imath})$. In this case, the p^{th} graded component of $A_r = i^r A_0$ is the image of $i^r = H(\tilde{\imath})^r \colon H(F_p C) \longrightarrow H(F_{p-r} C)$. Since our filtrations all have $C = F_0 C = F_{-1} C = \cdots$, for $r > p$, the map i_r is an injection on the p^{th} component. If the filtration $F_p H(C)$ is *finite*, say with $F_r H(C) = 0$, then A_r is the direct sum of the graded components

$$\cdots = H(C) = H(C) \geqslant iH(F_1 C) \geqslant i^2 H(F_2 C) \geqslant \cdots \geqslant i^{r-1} H(F_{r-1} C) \geqslant 0 = 0 = \cdots,$$

which can be identified with the filtrands $F_p H(C)$, and i_r is injective on every component since everything is now a submodule of $H(C)$. Thus the r^{th} triangle becomes a short exact sequence



[8] Even more traditionally, it was called a *term*. The author is not sure when the switch started.

As $j_r k_r = j_r 0 = 0$, it follows $E_r = E_{r+1} = \cdots$. We will call this terminal value the ***limiting page***, and denote it E_∞. By exactness,

$$E_\infty = E_r \cong \frac{A_r}{i_r A_r} = \bigoplus_p \frac{F_p H(C)}{F_{p+1} H(C)} = \operatorname{gr}_\bullet H(C).$$

In such a situation, when $E_\infty \cong \operatorname{gr}_\bullet H(C)$, we say that (E_r, d_r) ***converges*** to $H(C)$. One sometimes writes this as $E_r \implies H(C)$.

There is another important way to look at this spectral sequence: since we ultimately want to use it to understand the cohomology of C, we should understand the differentials in terms of C itself. Let's first try in terms of E_1. Each page, by definition, is the cohomology of the previous, so that for instance $E_{r+1} = (\ker d_r)/(\operatorname{im} d_r)$; and here in turn $\ker d_r$ and $\operatorname{im} d_r$ are subgroups of $E_r = (\ker d_{r-1})/(\operatorname{im} d_{r-1})$. Thus, the preimages of $\ker d_r$ and $\operatorname{im} d_r$ under the quotient map $\pi\colon \ker d_{r-1} \twoheadrightarrow E_r$, both contain $\ker d_{r-1}$, and by the third isomorphism theorem, we still have $(\pi^{-1} \ker d_r)/(\pi^{-1} \operatorname{im} d_r) \cong E_{r+1}$. Iteratively pulling all the kernels and images back to E_1, we get a sequence of subgroups

$$\widetilde{\operatorname{im} d_1} \leqslant \widetilde{\operatorname{im} d_2} \leqslant \widetilde{\operatorname{im} d_3} \leqslant \cdots \leqslant \widetilde{\ker d_3} \leqslant \widetilde{\ker d_2} \leqslant \widetilde{\ker d_1}$$

of E_1 such that $E_{r+1} = \widetilde{\ker d_r}/\widetilde{\operatorname{im} d_r}$. We can now define $E_\infty := \bigcap \widetilde{\ker d_r}/\bigcup \widetilde{\operatorname{im} d_r}$, which is defined independent of the convergence of the sequence.

Let us try to characterize these subgroups.

- An element $e \in E_1$ lies in $\widetilde{\operatorname{im} d_1}$, meaning it represents the trivial class $[0]_2 \in E_2$, if is in $\operatorname{im} jk$, or equivalently, by exactness, if $e \in j(\ker i)$.

- An element $e \in E_1$ lies in $\widetilde{\ker d_1}$, meaning it represents an element of E_2, if it is in $\ker jk$, or equivalently, by exactness, if $e \in k(\operatorname{im} i)$.

- An element $[e]_2 \in E_2$, meaning it represents the trivial class $[0]_3 \in E_3$, if it is in $\operatorname{im} j_2 k_2$, meaning $[e]_2 = [ji^{-1}ke']_2$ for some $e' \in k^{-1}(\operatorname{im} i)$. This means $e - ji^{-1}ke' \in \widetilde{\operatorname{im} d_1} = j(\ker i)$. Thus $e = ji^{-1}ke' + ja$ for some $a \in \ker i$, so $e \in j \ker i^2$. Conversely, if $e = ja$ and $i^2 a = 0$, then $ia = ke'$ for some $e' \in \widetilde{\ker d_1}$ by exactness and $e = ji^{-1}ke' \in j_2 k_2 [e']_2$.

- An element $[e]_2 \in E_2$ lies in $\widetilde{\ker d_2}$, meaning it represents an element of E_3, if $j_2 k_2 [e]_2 = [0]_2$, or in other words, if $ji^{-1}ke \in \widetilde{\operatorname{im} d_1} = j(\ker i)$.[9] Thus $ji^{-1}ke = ja$ for some $a \in \ker i$, so $i^{-1}ke - a \in \ker j = \operatorname{im} i$ and $ke \in \operatorname{im} i^2$. Conversely, if $ke \in \operatorname{im} i^2$, then $ji^{-1}ke = 0$.

Exercise 2.6.3. Show by induction that $\widetilde{\operatorname{im} d_r} = j(\ker i^r)$ and $\widetilde{\ker d_r} = k^{-1}(\operatorname{im} i^r)$.

Thus the operation $ji^{-r}k$, defined on elements of $\widetilde{\ker d_r}$, descends to become d_{r+1}. Now we lift this description back to the associated graded group $E_0 := \operatorname{gr}_\bullet C$. An element e_p of $E_1^p = H(F_p C/F_{p+1} C)$ is represented by a cocycle in $\operatorname{gr}_p C$, which is an element $c_p + F_{p+1}$ such that dc_p represents 0 in $\operatorname{gr}_p C$, or in other words $dc_p \in F_{p+1}$. Such an element c_p represents 0 in E_1 if it lies in $F_{p+1} + dF_p$. Let us agree to write for these groups of representatives

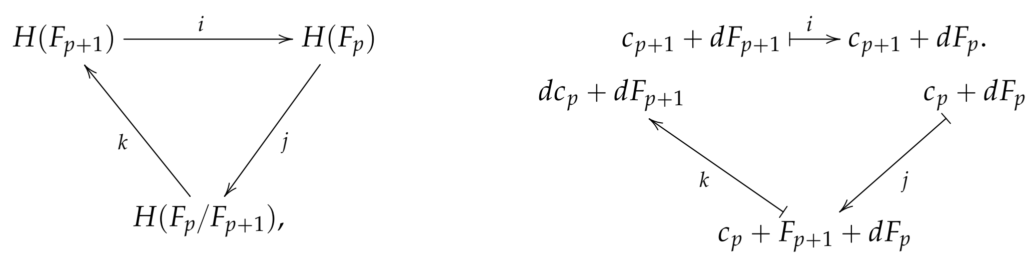
$$Z_0^p := \{c_p + F_{p+1} \in \operatorname{gr}_p C : dc_p \in F_{p+1}\},$$
$$B_0^p := \{dc_p + F_{p+1} \in \operatorname{gr}_p C : c_p \in F_p\}.$$

9 The particular preimage $i^{-1}ke$ taken does not affect the calculation.

The maps i, j, k in the original exact triangle arise from the long exact sequence

$$0 \to \bigoplus F_p \xrightarrow{\tilde{\imath}} \bigoplus F_p \longrightarrow \mathrm{gr}_\bullet C \to 0,$$

taking, in individual graded components,



[FIX ARROWHEAD LOCATIONS HERE]

Note that i, j, k respectively change the p-grading by $-1, 0, 1$, so that the p-degree of d_r, which is induced from $ji^{-(r-1)}k$ on $\widetilde{\ker d_{r-1}}$, is $0 + (r-1) + 1 = r$. Moreover, since the connecting map k takes a class represented by $c \in C$ to one represented by dc, we see each d_r is induced by the original differential d.

Write Z_r and B_r respectively for the subgroups of E_0 comprising representatives of $\widetilde{\ker d_r} \leqslant E_1 = H(E_0)$ and of $\widetilde{\mathrm{im}\, d_r} \leqslant E_1$.

- From Exercise 2.6.3, an element $e \in E_1^p$ lies in $\widetilde{\mathrm{im}\, d_r}$ if it can be written as ja with $i^r a = 0$ for some $a \in A_1$. That is, there is $c_p + dF_p$ such that $e = c_p + dF_p + F_{p+1}$ and c_p represents zero in $H(F_{p-r})$, meaning $c_p = dc_{p-r}$ for some $c_{p-r} \in F_{p-r}$.

- From Exercise 2.6.3, an element $e \in E_1^p$ lies in $\widetilde{\ker d_r}$ if ke can be written as $i^r a$ for some $a \in A_1$. If $c_p \in F_p$ represents e, that is, $dc_p + dF_{p+1} \in H(F_{p+1})$ is $c_{p+r+1} + dF_{p+1}$ for some cocycle $c_{p+r+1} \in F_{p+r+1}$.

Summing up, for $r \geqslant 0$ we have

$$Z_r^p = \{c_p + F_{p+1} \in \mathrm{gr}_p C \ : \ dc_p \in F_{p+r+1}\},$$
$$B_r^p = \{dc_{p-r} + F_{p+1} \in \mathrm{gr}_p C \ : \ c_{p-r} \in F_{p-r}\},$$

the cosets of elements that d respectively sends forward $r+1$ steps or has sent forward r steps. Note how our definitions of Z_0 and B_0 were contrived to make this still true for $r = 0$; in fact the expressions still make sense for $r = -1$, yielding respectively $\mathrm{gr}_p C$ and 0. To produce more succinct expressions, we adopt the notation $F_{p \to q} := \{c_p \in F_p : dc_p \in F_q\}$. Expressed in terms of elements of C, then, we see that for $r \geqslant -1$,

$$Z_r^p = \frac{F_{p \to p+r+1} + F_{p+1}}{F_{p+1}} \cong \frac{F_{p \to p+r+1}}{F_{p+1 \to p+r+1}},$$
$$B_r^p = \frac{dF_{p-r \to p} + F_{p+1}}{F_{p+1}} \cong \frac{dF_{p-r \to p}}{dF_{p-r \to p+1}}, \qquad (2.6.4)$$
$$E_{r+1}^p \cong \frac{Z_r^p}{B_r^p} \cong \frac{F_{p \to p+r+1}}{dF_{p-r \to p} + F_{p+1 \to p+r+1}}.$$

To determine E_∞, we extend the notation by setting $F_\infty := \bigcap_{p\in\mathbb{Z}} F_p$ and $F_{-\infty} := \bigcup_{p\in\mathbb{Z}} F_p$. Then one has

$$F_{p\to\infty} := \bigcap_{r\geqslant 0} F_{p\to p+r} = F_p \cap d^{-1}F_\infty \qquad \text{and} \qquad F_{-\infty\to p} := \bigcup_{r\geqslant 0} F_{p-r\to p} = F_p \cap \bigcup dF_{-\infty},$$

so quotients involving these expressions will really be about the complex $\overline{C} := F_{-\infty}/F_\infty$, its induced differential $\bar{d}$, and the induced filtrations of $\overline{C}$, $\ker \bar{d}$, $\operatorname{im} \bar{d}$, and $H(\overline{C})$. Taking $r \to \infty$ in (2.6.4), we have

$$Z^p_\infty = \bigcap Z^p_r = \frac{F_{p\to\infty} + F_{p+1}}{F_{p+1}} \cong \frac{F_{p\to\infty}}{F_{p+1\to\infty}} \cong \frac{F_p \ker \bar{d}}{F_{p+1}\ker \bar{d}},$$

$$B^p_\infty = \bigcup B^p_r = \frac{dF_{-\infty\to p} + F_{p+1}}{F_{p+1}} \cong \frac{dF_{-\infty\to p}}{dF_{-\infty\to p+1}} \cong \frac{F_p \operatorname{im} \bar{d}}{F_{p+1}\operatorname{im} \bar{d}},$$

$$E^p_\infty = \frac{Z^p_\infty}{B^p_\infty} \cong \frac{F_p \ker \bar{d}}{F_p \operatorname{im} \bar{d} + F_{p+1}\ker \bar{d}} \cong \operatorname{gr}_p H(\overline{C}).$$

When we assume our filtrations are exhaustive ($F_{-\infty} = C$) and Hausdorff ($F_\infty = 0$), so that $\overline{C} = C$, we get the better expressions

$$Z^p_\infty \cong \operatorname{gr}_p \ker d,$$
$$B^p_\infty \cong \operatorname{gr}_p \operatorname{im} d,$$
$$E^p_\infty \cong \operatorname{gr}_p H(C).$$

Remark 2.6.5. **N.B.** that this is not the indexing convention used by most authors. It is common to define the spectral sequence of a filtration directly, without exact couples, and in this case it is natural to use Z^p_r for our $F_{p\to p+r}$ and B^p_r for our $dF_{p-r\to p}$. Under these conventions, our formula for E^p_r transforms to the standard expression $Z^p_r/(B^p_{r-1} + Z^{p+1}_{r-1})$.

Now let us consider a ***filtered differential graded algebra*** $(C^\bullet, d, \tilde{\imath})$. This is a filtered differential group such that $(C^\bullet, d)$ is a DGA and $(C^\bullet, \tilde{\imath})$ is a filtered graded group. A ***homomorphism of filtered differential graded algebras*** is a filtration-preserving DGA map. In the resulting exact couple $\big(\bigoplus H^*(C^\bullet_p), H^*(\operatorname{gr}_\bullet C^\bullet)\big)$, one has $i\colon H^n(F^\bullet_{p+1}) \longrightarrow H^n(F^\bullet_p)$ and $j\colon H^n(F^\bullet_p) \longrightarrow H^n(F^\bullet_p/F^\bullet_{p+1})$ of degree zero, but connecting map $k\colon H^n(F^\bullet_p/F^\bullet_{p+1}) \longrightarrow H^{n+1}(F^\bullet_{p+1})$ of degree 1. It is standard to define a ***complementary grading*** $q := n - p$ so that $F^n_p = F^{p+q}_p$. Then we get the statement we made at the beginning of this chapter:

Theorem 2.1.2. *(Koszul). Let $(C^\bullet, d, \tilde{\imath})$ be a filtered differential $\mathbb{N}$-graded algebra such that the associated filtration of $H^n(C^\bullet)$ is finite for each n. Then there is an associated* ***filtration spectral sequence*** *in which*

- $(E_0, d_0) = (\operatorname{gr}_\bullet C^\bullet, \operatorname{gr}_\bullet d)$,
- $E_1 \cong H^*(\operatorname{gr}_\bullet C^\bullet)$,
- $E^{p,q}_\infty \cong \operatorname{gr}_p H^{p+q}(C^\bullet)$.

*We call this the **filtration spectral sequence** of the filtered* DGA $(C^\bullet, d, \tilde{\imath})$. *It is **first-quadrant spectral sequence** in that* $E_r^{p,q} = 0$ *if* $p < 0$ *or* $q < 0$. *All pages become differential algebras under the bigrading* $E_r^{p,q}$ *induced from the bigrading* $E_0^{p,q} := \operatorname{gr}_p C^{p+q}$ *of* $E_0 = \operatorname{gr}_\bullet C^\bullet$ *and the product induced from that of* C*, with differential* d_r *of bidegree* $(r, 1-r)$*. Moreover, the product on each page is induced by that on the last. This sequence is functorial in homomorphisms of filtered* DGA*s.*

Proof. Everything follows from the previous discussion except the statements about convergence, bidegrees, and product structure. Because the filtrations are finite in each degree, the convergence result follows in each degree separately from the previous discussion.

For bidegrees of d_r, first note that d_0 is just the internal differential of $E_0 = \operatorname{gr}_\bullet C$ by definition, which is of bidegree $(p,q) = (0,1)$. The first exact couple (A_1, E_1, i, j, k) is the long exact sequence associated to the short exact sequence of chain complexes $0 \to \bigoplus F_pC \xrightarrow{\tilde{\imath}} \bigoplus F_pC \xrightarrow{\tilde{\jmath}} \operatorname{gr}_\bullet C \to 0$. As we said before the proposition, i, j, k respectively increase the complex degree n by $1, 0, 0$, and we saw before they increase p by $-1, 0, 1$. Thus their respective (p,n)-bidegrees are $(-1,0)$, $(0,0)$, $(1,1)$, so their (p,q)-bidegrees are $(-1,1)$, $(0,0)$, $(1,0)$. Recalling that the differential d_r is represented by $e \longmapsto ji^{-(r-1)}ke \mod \widetilde{\operatorname{im} d_r}-1$ on representatives $e \in \widetilde{\ker d}_{r-1} \leqslant E_1$, we see $\operatorname{bideg}(d_r) = (r, 1-r)$ and $\deg(d_r) = 1$.

As for the multiplication, we consult (2.6.4). If $a \in F_{p \to p+r+1}C^n$ and $b \in F_{p' \to p'+r+1}C^{n'}$, then

$$ab \in F_p \cdot F_{p'} \leqslant F_{p+p'} \quad \text{and} \quad d(ab) = da \cdot b + (-1)^p a \cdot db \in F_{p+r+1} \cdot F_{p'} + F_p \cdot F_{p'+r+1} \leqslant F_{p+p'+r+1},$$

so $ab \in F_{p+q \to p+p'+r+1}C^{n+n'}$ has the right filtration behavior and algebra degree, and it the fact the multiplication on each page is induced by that on the last will be clear once we check this putative multiplication on $E_r^\bullet$ is well-defined. To do so, we need to see that we could have chosen another representative congruent to a modulo $dF_{p-r} + F_{p+1 \to p+r+1}$ (and similarly for b, but the argument is symmetric); for this it is enough to note $F_{p+1 \to p+r+1} \cdot F_{p' \to p'+r+1} \leqslant F_{p+p'+1 \to p+p'+r+1}$ and $dF_{p \to r} \cdot F_{p' \to p'+r+1} \leqslant dF_{p+p'-r \to p'+p'}$.

Since the multiplication adds filtration degrees p and algebra degrees n, it adds the complementary degree $q = n - p$ as well, so each $E_r^{\bullet,\bullet}$ is a bigraded algebra. That d_r is a derivation on E_r follows from the fact that it is induced from d. □

Exercise 2.6.6. Check that indeed

$$F_{p+1 \to p+r+1} \cdot F_{p' \to p'+r+1} \leqslant F_{p+p'+1 \to p+p'+r+1},$$
$$dF_{p \to r} \cdot F_{p' \to p'+r+1} \leqslant dF_{p+p'-r \to p'+p'}.$$

Given a differential bigraded algebra $(A^{\bullet,\bullet}, d)$, the ***horizontal filtration***, is given by

$$F_pA^{\bullet,\bullet} := \bigoplus_{i \geqslant p} A^{i,\bullet}.$$

The algebra is also a filtered DGA if in the decomposition $d = \sum_{\ell \in \mathbb{Z}} d^\ell$ into component maps (see Appendix A.3.1) one has $d^\ell = 0$ for $\ell < 0$. In this case, the theorem applied to $(A^{\bullet,\bullet}, d, i)$ yields a spectral sequence $(E_r, d_r)^{\bullet,\bullet}$ with $E_0 \cong \operatorname{gr}_\bullet A^{\bullet,\bullet}$ again. The filtration of $H^n(A^{\bullet,\bullet})$ is clearly finite in each total degree $n = p + q$ since the filtration $F^pA^n = \bigoplus_{i \geqslant p} A^{i,n-i}$ already is.

Corollary 2.6.7. *Let* $(A^{\bullet,\bullet}, d, i)$ *be a filtered, nonnegatively-bigraded* DGA*. Then in the spectral sequence associated to the horizontal filtration one has*

- $(E_0, d_0) \cong (A^{\bullet,\bullet}, d^0)$,
- $E_1 \cong \bigoplus_{p \in \mathbb{N}} H^*(A^{p,\bullet}, d^0), \quad d_1 = H^*_{d^0}(d^1)$,
- $E_2 \cong H^*_{d^1} H^*_{d^0}(A^{\bullet,\bullet})$,
- $E_\infty \cong \operatorname{gr}_\bullet H^*(A^{\bullet,\bullet})$.

In one recurrent situation, we can say even more about E_2.

Corollary 2.6.8. *If $(A^{\bullet,\bullet}, d, i) \cong (A^{\bullet,0}, d^1) \otimes (A^{0,\bullet}, d^0)$ is free as a k-module and i is the horizontal filtration, then*

- $E_0 \cong A, \qquad d_0 = \mathrm{id} \otimes d^0$,
- $E_1 \cong A^{\bullet,0} \otimes H^*_{d^0}(A^{0,\bullet}), \qquad d_1 = d^1 \otimes \mathrm{id}$,
- $E_2 \cong H^*_{d^1}(A^{\bullet,0}) \otimes H^*_{d^0}(A^{0,\bullet})$,
- $E_\infty \cong \operatorname{gr}_\bullet H^*(A)$.

Remark 2.6.9. The algebraic Künneth Theorem A.3.2 of this chapter and the universal coefficient Theorem B.1.1 of the appendices both are special cases of general filtration spectral sequences that still exist if we do not assume that the modules in question are free over the base ring k or that k is a principal ideal domain.

2.7. Fundamental results on spectral sequences

A common way to understand the cohomology ring of a filtered DGA is to engage in wishful thinking: one finds another spectral sequence that one would *like* to approximate that of the DGA in question, contrives a map between the idealized sequence and the actual sequence, and shows it yields an isomorphism on a late enough page. The theoretical justification behind this chicanery has at most two steps.

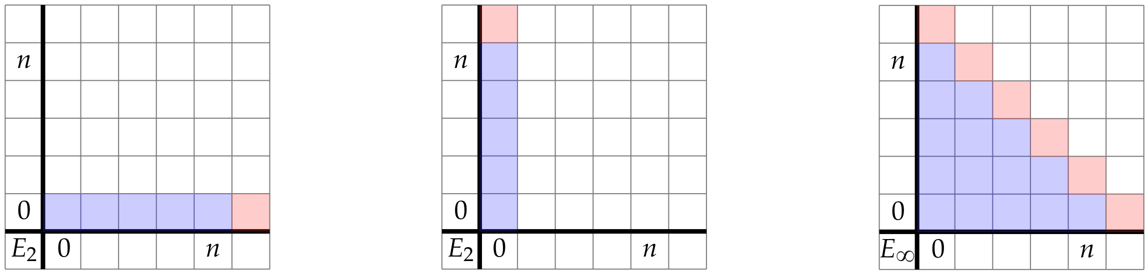
Theorem 2.7.1 (Zeeman–Moore, [MT00, Thm. VII.2.4, p. 375])**.** *Let $(\psi_r)\colon ({}'E_r, {}'d_r) \longrightarrow (E_r, d_r)$ be a map of bigraded spectral sequences of k-modules such that $E_2 \cong E_2^{\bullet,0} \otimes E_2^{0,\bullet}$ and ${}'E_2 \cong {}'E_2^{\bullet,0} \otimes {}'E_2^{0,\bullet}$ decompose as tensor products. Consider the following three conditions:*

- *$(B)_N$: $\psi_2^{p,0}$ is an isomorphism for $p < N$ and an injection for $p = N$.*
- *$(F)_N$: $\psi_2^{0,q}$ is an isomorphism for $q < N$ and an injection for $q = N$.*
- *$(E)_N$: $\psi_\infty^{p,q}$ is an isomorphism for $p + q < N$ and an injection for $p + q = N$.*
- *$(E)_N^+$: $\psi_r^{p,q}$ is, for all $r \geqslant 2$, an isomorphism for $p + q < N$ and an injection for $p + q = N$.*

There are the following implications:

- *$(F)_N$ and $(B)_N$ together imply $(E)_N^+$.*
- *$(F)_{N-1}$ and $(E)_N$ together imply $(B)_N$.*
- *$(B)_{N+1}$ and $(E)_N$ together imply $(F)_N$.*

Figure 2.7.1: The conditions $(\mathrm{B})_n$, $(\mathrm{F})_n$, $(\mathrm{E})_n$ in Zeeman's theorem, isomorphism blue, injection red



We will use this result to prove the Borel transgression theorem Theorem 7.4.5 and then again in Borel's derivation Section 8.1.2 of the Cartan algebra. Given an isomorphism of E_2 pages or E_∞ pages then shows that the inducing map of DGAs was a quasi-isomorphism.

Proposition 2.7.2. *Let $f\colon A \longrightarrow B$ be a map of filtered DGAs and $(\psi_r)\colon ({}'E_r, {}'d_r) \longrightarrow (E_r, d_r)$ the associated map of filtration spectral sequences. Suppose that both filtrations are finite in each degree (as defined in Section 2.5) If ψ_r is an isomorphism for any $r \geqslant 0$, then $f^*\colon H^*(A) \longrightarrow H^*(B)$ is an isomorphism.*

Proof. If any ψ_r is an isomorphism, then since ${}'d_r\psi_r = \psi_r d_r$, it follows that all later ψ_r and ψ_∞ are isomorphisms. By Corollary 2.6.7, ψ_∞ is the isomorphism $\mathrm{gr}_\bullet f^*\colon \mathrm{gr}_\bullet H^*(A) \longrightarrow \mathrm{gr}_\bullet H^*(B)$. For any given total degree n, we can apply Corollary 2.5.2 to the map $\psi^n_\infty\colon \mathrm{gr}_\bullet H^n(A) \longrightarrow \mathrm{gr}_\bullet H^n(B)$ to conclude $H^n(f)$ is an isomorphism. □

Here is a useful splitting-type result for spectral sequences.

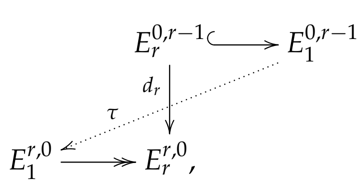
Proposition 2.7.3 ([McC01, Example 1.K, p. 25])**.** *Let $(A, d, \tilde{\imath})$ be a filtered differential k-algebra, free as a k-module, and (E_r, d_r) the associated spectral sequence. If E_∞ is a free k-CGA , then $E_\infty \cong H^*(A, d)$ as a k-CGA.*

Proof. This is just an application of Proposition 2.5.3. □

2.8. The transgression

Early on in the history of bigraded spectral sequences of the form discussed above, it was noticed that the maps $d_r\colon E_r^{0,r-1} \longrightarrow E_r^{r,0}$ from the left column to the bottom row (Figure 2.2.20) have a special importance.

Definition 2.8.1 (Koszul, 1950 [Kos50, Sec. 18])**.** Let (E_r, d_r) be the filtration spectral sequence of a filtered DGA $(C^\bullet, d, \tilde{\imath})$. If $z \in E_2^{0,r-1}$ is in the kernel of each d_p for $p < r$, so that $d_r z \in E_r^{0,r}$ is defined (that is, if z survives long enough to be in the domain of an edge homomorphism), then z is said to ***transgress***. The ***transgression*** is the dotted arrow τ in the diagram



described as the relation on $E_1^{r,0} \times E_1^{0,r-1}$ given by $x\,\tau\,z \;:\Longleftrightarrow\; [x]_{E_r} = d_r z$ and $z \in E_r^{0,r-1}$.

It is not ruled out by this convention that $0 \,\tau\, z$; the important part is just that $d_p z = 0$ for $p \leqslant |z|$. It is wrong but conventional to write $x = \tau(z)$ and think of the transgression as a partially-defined map $z \longmapsto \tau z$ on $E_1^{0,r-1}$, either ignoring the ambiguity inherent in viewing $d_r z$ as an element of E_1 or else removing it by singling out a specific preimage of $d_r z$ in E_1, which is sometimes called *choosing a transgression*.

We may rephrase this in terms of the filtered DGA as follows.

Proposition 2.8.2. *Let (E_r, d_r) be the filtration spectral sequence of a filtered* DGA *$(A^\bullet, d, i)$. An element $z \in E_1^{0,r-1} = H^{r-1}(F_0C^\bullet/F_1C^\bullet, d_0)$ transgresses to $\tau z \in E_1^{r,0} = H^r(F_rC^\bullet/F_{r+1}C^\bullet, d_0)$ if and only if there exists $c \in F_0C^{r-1}$ such that z represents c and $dc \in F_rC^r$ represents τz.*

When dealing with the Leray or Serre spectral sequences, which on the E_0 and E_1 pages still can depend on the sheaf resolution or cohomology theory chosen, it is more conventional to conceive of the transgression as a relation on the E_2 page. The description at the cochain level remains unchanged by this.

Historical remarks 2.8.3. According to the concluding notes in Greub *et al.* [GHV76], instances of transgressions were first identified by Shiing-Shen Chern [Che46] and Guy Hirsch [Hir48] before Koszul observed the pattern and coined the term “transgression” in his thesis work.

The filtration spectral sequence is first described in Koszul's *Comptes Rendus* note [Kos47a], and is extracted from Leray's earlier work as described in a 1946 *Comptes Rendus* notice [Ler46a]. Koszul was the first other person to work through and understand Leray's post-war topological output, and was the chief instigator of the simplifications that made spectral sequences accessible to the rest of the mathematical community [Mil00]. The term *filtration* itself and its isolation was due to Cartan. Exact couples are due to Massey [Mas52, Mas53].

2.9. Proofs regarding the Serre spectral sequence

In this section we prove Theorem 2.2.2 and its elaborations.

Theorem 2.2.2. *Let $F \to E \to B$ be a fibration such that $\pi_1 B$ acts trivially on $H^*(F;k)$. There exists a first-quadrant spectral* ***Serre spectral sequence*** *$(E_r, d_r)_{r\geqslant 0}$ of k-*DGA*s with*

$$E_0^{p,q} = C^{p+q}(E, E^{p-1}; k),$$
$$E_2^{p,q} = H^p\big(B; H^q(F;k)\big),$$
$$E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(E;k),$$

for the filtrations (E^p) and $F_pH^*(E)$ indicated above. If $H^*(F;k)$ is a free k-module (for instance, if k is a field), we may also write $E_2 \cong H^*(B;k) \otimes_k H^*(F;k)$. This construction is functorial in fibrations $E \to B$ and in rings k, in that a map of fibrations or of rings induces a map of spectral sequences.

Proof. The existence of the sequence is given by Theorem 2.1.2. The convergence will follow if we can show $F_pH^n(E) = 0$ for $p > n$, but this is so because $\pi_{\leqslant p-1}(E, E^{p-1}) = \pi_{\leqslant p-1}(B, B^{p-1}) = 0$ by the homotopy lifting property.

The functoriality of the spectral sequence in bundle maps follows from the fact any map $B \longrightarrow B'$ between CW complexes can be homotoped to a cellular map f with $f(B^p) \subseteq (B')^p$. By the homotopy lifting property, the resulting map $\widetilde{f}\colon E \longrightarrow E'$ of total spaces will be homotopic to the original, but now will satisfy $\widetilde{f}(E^p) \subseteq (E')^p$.[10] Thus $\widetilde{f}_* C_*(E^{p-1}) \leqslant C_*\big((E')^{p-1}\big)$, so if $c' \in C^*(E')$ annihilates $C_*\big((E')^{p-1}\big)$, then $\widetilde{f}^* c' = c' \circ \widetilde{f}_*$ annihilates $C_*(E^{p-1})$, meaning $f^*(F_p C') \leqslant F_p C$. Now use that the filtration spectral sequence is functorial in filtration-preserving DGA maps.

The functoriality in kmaps follows from the fact a coefficient group homomorphism ϕ: $kk \longrightarrow$
kk' induces homomorphisms $C^n(E;$
$kk) \longrightarrow C^n(E;$
$kk')$ and if ϕ is a map of rings, these is a khomomorphism with respect to cup product, obviously preserving the filtration.

The nontrivial part of the proof involves identifying the E_2 page. The page E_0 is the associated graded algebra $\mathrm{gr}_\bullet\, C^*(E)$ with summands $C^*(E, E^{p-1})/C^*(E, E^p)$. Considekthe map of complexes (2.2.1) induced by the inclusion of E^{p-1} in E^p, the Snake Lemma identifies these summands with $C^*(E^p, E^{p-1})$. Thus $E_1 \cong \bigoplus_p \widetilde{H}^*(E^p/E^{p-1})$. Since B^p is formed from B^{p-1} by attaching p-cells along their boundaries and a fibration over a contractible space is trivial, we have a further identification

$$\widetilde{H}^*(E^p/E^{p-1}) \cong \mathrm{Hom}\big(\mathrm{Cell}_p(E), H^*(F;kk)\big) =: C^p_{\mathrm{Cell}}\big(B; H^*(F;kk)\big).$$

Once we verify that the differential d_1 can be identified with the cellular coboundary operator δ_{Cell} and the product with the cup product, it will follow immediately that $E_2 \cong H^*\big(B; H^*(F;$
$kk)\big)$ as bigraded
kk-modules and it will only remain to verify that the product structure on E_2 agrees up to sign with the cup product on $H^*\big(B; H^*(F;$
$kk)\big)$.

[To be written...] □

Proposition 2.2.5. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration such that $\pi_1 B$ acts trivially on $H^*(F)$. The fiber projection $i^*\colon H^*(E) \longrightarrow H^*(F)$ is realized by the left-column edge map $E_\infty^{\bullet,\bullet} \twoheadrightarrow E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet}$ in Theorem 2.2.2: to wit, we can write*

$$\mathrm{gr}_\bullet\, H^*(E) \xrightarrow{\sim} E_\infty^{\bullet,\bullet} \longrightarrow\!\!\!\!\!\rightarrow E_\infty^{0,\bullet} \hookrightarrow E_2^{0,\bullet} \xrightarrow{\sim} H^*(F).$$

Likewise, the base lift $\pi^*\colon H^*(B) \longrightarrow H^*(E)$ is realized by the bottom-row edge map $E_2^{\bullet,0} \twoheadrightarrow E_\infty^{\bullet,0} \hookrightarrow E_\infty^{\bullet,\bullet}$:

$$H^*(B) \xrightarrow{\sim} E_2^{\bullet,0} \longrightarrow\!\!\!\!\!\rightarrow E_\infty^{\bullet,0} \hookrightarrow E_\infty^{\bullet,\bullet} \xrightarrow{\sim} \mathrm{gr}_\bullet\, H^*(E).$$

Proof [McC01, p. 147]. We have a commutative square

$$\begin{array}{ccccc}
F & = & F & \longrightarrow & * \\
\| & & \downarrow & & \downarrow \\
F & \xrightarrow{i} & E & \xrightarrow{\pi} & B \\
\downarrow & & \downarrow & & \| \\
* & \longrightarrow & B & = & B
\end{array}$$

[10] This could also be achieved with a functorial CW replacement, for example the one replacing a space with its total singular simplicial complex.

where each column (and row) is a fibration, with the original fibration in the middle column, and the maps between columns are fiber-preserving. These maps induce maps of spectral sequences, which we can denote as

$$ {}^F E_r \longleftarrow E_r \longleftarrow {}^B E_r. $$

The middle spectral sequence is the Serre spectral sequence of the original fibration, while ${}^F E_r$ is that of $F \to F \to *$, which collapses at ${}^F E_2 = H^*\big(*; H^*(F)\big) = H^*(F)$, and ${}^B E_r$ is that of $* \to B \to B$, which also collapses instantly, at ${}^B E_2 = H^*\big(B; H^*(*)\big) = H^*(B)$. On E_2 pages, the induced maps are $E_2(i^*)\colon E_2 \longrightarrow {}^F E_2$, which is the left-column projection $H^*\big(B; H^*(F)\big) \longrightarrow H^0\big(B; H^*(F)\big) \cong H^*(F)$, and $E_2(\pi^*)\colon {}^B E_2 \longrightarrow E_2$, which is the bottom-row inclusion $H^*(B) \longrightarrow H^*\big(B; H^0(F)\big)$, the maps we would like to descend to the maps $i^* = \mathrm{gr}_\bullet\, i^*$ and $\pi^* = \mathrm{gr}_\bullet\, \pi^*$ on E_∞ pages. The maps between E_∞ pages are

$$\begin{array}{ccccc} H^*(F) & & & & \\ \uparrow \wr & & & & \\ {}^F E_2 & \xleftarrow{\mathrm{gr}_\bullet\, \pi^*} & \mathrm{gr}_\bullet\, H^*(E) & \xleftarrow{\mathrm{gr}_\bullet\, i^*} & {}^B E_2 \\ & & & & \uparrow \wr \\ & & & & H^*(B), \end{array}$$

by the fact that the isomorphism of final page E_∞ with $\mathrm{gr}_\bullet\, H^*(E)$ is natural. But that shows that these maps descend from the E_2 column and row maps as claimed. □

We will make extensive use of the transgression in the Serre spectral sequence of a bundle in the last two chapters. On the E_2 level, an edge homomorphism d_r takes (a submodule of) $H^{r-1}(F)$ to (a quotient of) $H^r(B)$, but we will need to know what this means on the cochain level, so we need a slightly more topological description.

Proposition 2.2.21. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration with all spaces path-connected and such that the action of $\pi_1 B$ on $H^*(F)$ is trivial. An element $[z] \in H^r(F) = E_2^{0,r}$ (Definition 2.8.1) represents an element of $E_{r+1}^{0,r}$, and hence transgresses to the class in $E_{r+1}^{r+1,0}$ represented by some $[b] \in H^{r+1}(B)$, if and only if there exists $c \in C^r(E)$ in the singular cochain group such that $i^*c = z$ and $\delta c = \pi^* b$. This is the picture:*

$$\begin{array}{ccccc} & & C^q(E) & \xrightarrow{i^*} & Z^q(F) \\ & \overset{\tau}{\swarrow} & \downarrow \delta & & \\ Z^{q+1}(B) & \xrightarrow[\pi^*]{} & Z^{q+1}(E), & & \end{array} \qquad \begin{array}{ccc} & c & \longmapsto\ z \\ \swarrow & \downarrow & \\ b & \longmapsto & \delta c. \end{array}$$

Proof. Recall that the Serre spectral sequence is the filtration spectral sequence associated to the filtration $F_p C^*(E) = C^*(E, E^{p-1})$,[11] of the singular cochain algebra, where $E^{p-1} := \pi^{-1} B^{p-1}$ and (B^p) is a CW structure on B with one 0-cell.

Consulting Proposition 2.8.2, $c' \in C^r(E)$ represents a transgressive element if and only if $c' \in F_0 C^r(E) = C^r(E)$ and $\delta c' \in F_{r+1} C^{r+1}(E) = C^{r+1}(E, E^r)$. Of course $\delta(\delta c') = 0$, so $\delta c'$ represents a class in $H^{r+1}(E, E^r)$. Since π satisfies the homotopy lifting property with respect to spheres, $\pi_*(E, E^r) \longrightarrow \pi_*(B, B^r)$ is an isomorphism, and $\pi_{\leqslant r}$ and hence $H_{\leqslant r}$ vanish on (E, E^r) and (B, B^r)

[11] Again, the surprising $p-1$ ensures that $F_0 C^*(E) = C^*(E)$.

since B^r is the r-skeleton of B, so by the Hurewicz theorem B.1.1, $H_{r+1}(E, E^r) \longrightarrow H_{r+1}(B, B^r)$ can be identified with the isomorphism $\pi_{r+1}(E, E^r) \xrightarrow{\sim} \pi_{r+1}(B, B^r)$, and by the universal coefficient theorem B.1.1, $\pi^*\colon H^{r+1}(B, B^r) \longrightarrow H^{r+1}(E, E^r)$ is also an isomorphism. Thus there is $b \in C^{r+1}(B, B^r)$ such that $\pi^*[b] = [\delta c'] \in H^{r+1}(E, E^r)$, meaning $\pi^* b - \delta c'$ is some coboundary $\delta c''$ for $c'' \in C^{r+1}(E, E^r)$. Set $c := c' + c''$; then and c presents the same class as c' in $E_0^{0,r}$ and $\delta c = \pi^* b$ for $b \in Z^{r+1}(B)$. Evidently, since $\delta c \in C^r(E, E^r)$, its restriction to $E^r = \pi^{-1}(B^r)$ and hence $F = \pi^{-1}(B^0)$ is zero, so $i^* c = z$ represents a class of $H^r(F)$.

Conversely, suppose $b' \in Z^{r+1}(B)$ is such that $\pi^* b'$ represents $\delta c'$ for some $c' \in C^r(E)$ such that $i^* c' = z$ is a cocycle in $C^r(F)$. In the long exact cohomology sequence of the pair (B, B^r) we have the fragment $H^{r+1}(B, B^r) \to H^{r+1}(B) \to H^{r+1}(B^r) = 0$, so b' differs by a coboundary from a cocycle $b \in Z^{r+1}(B, B^r)$, say $b = b' + \delta b''$. Pulling back, $\pi^* b = \pi^* b' + \pi^* \delta b'' = \delta(c' + \pi^* b'')$, where $c := c' + \pi^* b''$ satisfies $i^* c = i^* c' + (\pi i)^* b'' = z$. □

Chapter 3

The cohomology of the classical groups

The rational cohomology of a compact Lie group G is as simple as anyone has any right to expect, and this simplicity can be seen as caused either by the multiplcation on G or by the existence of invariant differential forms (again a consequence of the multiplication). The Serre spectral sequence will allow us to compute the rational cohomology of the classical groups, a major achievement in the 1930s, in a few pages. We will cite general references for this material throughout the chapter, and diligently recount historical origins when we know them. Proofs, however, unless explicitly noted otherwise, have been dredged from the author's own memories or created anew. We start out with $k = \mathbb{Q}$, which destroys torsion off the bat, but much can be said with $\mathbb{Z}$ and torsion coefficients, and these computations give nice examples of the Serre spectral sequence, so we include them.

The general structure of the work does not require the results of this chapter, but the example computations in later sections all do.

3.1. Complex and quaternionic unitary groups

Note that $\mathrm{U}(n)$ acts by isometries on $\mathbb{C}^n$, so that it preserves the unit sphere S^{2n-1}. If we view this action as a left action on the space $\mathbb{C}^{n\times 1}$ of column vectors, the first column of an element g of $\mathrm{U}(n)$ determines where it takes the standard first basis vector $e_1 = (1, \vec{0})^\top \in S^{2n-1}$, so the stabilizer of e_1 is the subgroup

$$\begin{bmatrix} 1 & \vec{0} \\ \vec{0}^\top & \mathrm{U}(n-1) \end{bmatrix}$$

of elements with first column e_1, which we will identify with $\mathrm{U}(n-1)$. Since the first vector of $g \in \mathrm{U}(n)$ can be any element of S^{2n-1}, the action of $\mathrm{U}(n)$ on S^{2n-1} is transitive, so the orbit–stabilizer theorem yields a diffeomorphism $\mathrm{U}(n)/\mathrm{U}(n-1) \cong S^{2n-1}$, which is in fact a fiber bundle

$$\mathrm{U}(n-1) \longrightarrow \mathrm{U}(n) \longrightarrow S^{2n-1}.$$

Similarly, the action of $\mathrm{Sp}(n)$ on $\mathbb{H}^n$, preserving the unit sphere S^{4n-1}, gives rise to a fiber bundle

$$\mathrm{Sp}(n-1) \longrightarrow \mathrm{Sp}(n) \longrightarrow S^{4n-1},$$

and the action of $\mathrm{O}(n)$ on $\mathbb{R}^n$, preserving S^{n-1}, gives rise to bundles

$$\mathrm{O}(n-1) \longrightarrow \mathrm{O}(n) \longrightarrow S^{n-1},$$
$$\mathrm{SO}(n-1) \longrightarrow \mathrm{SO}(n) \longrightarrow S^{n-1}.$$

The SSSs of these bundles allow us to recover the cohomology of the classical groups.

Proposition 3.1.1. *The integral cohomology of the unitary group* $\mathrm{U}(n)$ *is given by*

$$H^*\big(\mathrm{U}(n);\mathbb{Z}\big) \cong \Lambda[z_1, z_3, \ldots, z_{2n-1}], \quad \deg z_j = j.$$

This can be seen as saying that in the SSSs of the bundles (right angles down) in the diagram

$$\begin{array}{ccccccccccc} \mathrm{U}(1) & \longrightarrow & \mathrm{U}(2) & \longrightarrow & \mathrm{U}(3) & \longrightarrow & \cdots & \longrightarrow & \mathrm{U}(n) & \longrightarrow & \mathrm{U}(n+1) \\ \downarrow\wr & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^1 & & S^3 & & S^5 & & \cdots & & S^{2n-1} & & S^{2n+1}, \end{array} \qquad (3.1.2)$$

the simplest possible thing happens, and the cohomology of each object is the tensor product of those of the objects to the left of it and below it.

Proof. The proof starts with the case $\mathrm{U}(1) \cong S^1$, so that $H^*(S^1) \cong \Lambda[z_1]$. Inductively assume $H\big(\mathrm{U}(n)\big) \cong \Lambda[z_1, z_3, \ldots, z_{2n-1}]$ as claimed. We have a fiber bundle

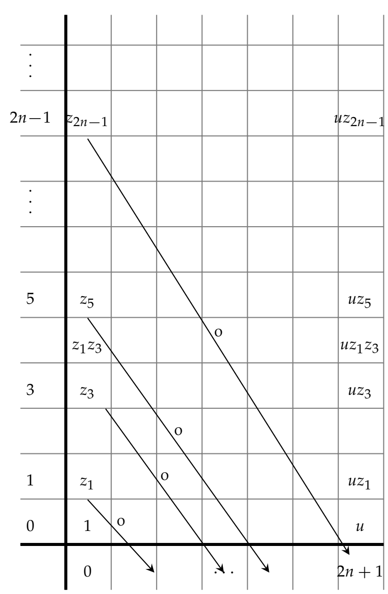
$$\mathrm{U}(n) \longrightarrow \mathrm{U}(n+1) \longrightarrow S^{2n+1},$$

where the cohomology of the fiber and base are known, so the impulse is to use Theorem 2.2.2. Since the cohomology of the fiber is free abelian by assumption, the E_2 page is given by

$$E_2^{\bullet,0} \otimes E_2^{0,\bullet} = \Lambda[u_{2n+1}] \otimes \Lambda[z_1, z_3, \ldots, z_{2n-1}],$$

and the sequence is concentrated in columns 0 and $2n+1$. Since the bidegree of the differential d_r is $(r, 1-r)$, the only differential that could conceivably be nonzero is $d = d_{2n+1}$, of bidegree $(2n+1, -2n)$.

Figure 3.1.3: The Serre spectral sequence of $\mathrm{U}(n) \to \mathrm{U}(n+1) \to S^{2n+1}$



But this d sends the square $E_{2n+1}^{0,q} = H^q\big(\mathrm{U}(n)\big)$ in the leftmost column into the fourth quadrant, so $dz_j = 0$ for all j. Because d satisfies the product rule and sends all generators of E_{2n+1} into the fourth quadrant, it follows $d = 0$. Thus $E_2 = E_\infty = \Lambda[z_1, z_3, \ldots, z_{2n-1}, u_{2n+1}]$.

A priori, this is only the the associated graded algebra of $H^*\big(\mathrm{U}(n+1)\big)$, but since E_∞ is an exterior algebra, by Proposition 2.7.3, there is no extension problem. □

The same proof, applied to the bundles $\mathrm{Sp}(n-1) \to \mathrm{Sp}(n) \to S^{4n-1}$ and starting with $\mathrm{Sp}(1) \approx S^3$, yields the cohomology of the symplectic groups. The diagram associated to this induction is

$$\begin{array}{ccccccccccc} \mathrm{Sp}(1) & \longrightarrow & \mathrm{Sp}(2) & \longrightarrow & \mathrm{Sp}(3) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Sp}(n) & \longrightarrow & \mathrm{Sp}(n+1) \\ \downarrow\wr & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^3 & & S^7 & & S^{11} & & \cdots & & S^{4n-1} & & S^{4n+3}. \end{array} \tag{3.1.4}$$

Proposition 3.1.5. *The integral cohomology of the symplectic group* $\mathrm{Sp}(n)$ *is given by*

$$H^*\big(\mathrm{Sp}(n);\mathbb{Z}\big) \cong \Lambda[z_3, z_7, \ldots, z_{4n-1}], \quad \deg z_j = j.$$

The cohomology of the special unitary groups is closely related to that of the unitary groups.

Proposition 3.1.6. *The integral cohomology of the special unitary group* $\mathrm{SU}(n)$ *is given by*

$$H^*\big(\mathrm{SU}(n);\mathbb{Z}\big) \cong \Lambda[z_3, \ldots, z_{2n-1}], \quad \deg z_j = j.$$

Proof. The determinant map yields a split short exact sequence

$$1 \to \mathrm{SU}(n) \hookrightarrow \mathrm{U}(n) \xrightarrow{\det} S^1 \to 1; \tag{3.1.7}$$

a splitting is given by $z \longmapsto \mathrm{diag}(z, \vec{1})$. This semidirect product structure means $\mathrm{U}(n)$ is topologically a product $\mathrm{SU}(n) \times S^1$, and it follows from the Künneth theorem B.1.2 that

$$H^*(\mathrm{SU}(n)) \;\cong\; H^*(\mathrm{U}(n))/\!\!/H^*(S^1) \;=\; \Lambda[z_1, z_3, \ldots, z_{2n-1}]/(z_1) \;=\; \Lambda[z_3, \ldots, z_{2n-1}].$$ □

The information we have accumulated makes it easy to cheaply acquire as well the cohomology the complex and quaternionic Stiefel manifolds: the idea is just, in the diagram (3.1.2), to stop before one gets to $\mathrm{U}(1)$.

Proposition 3.1.8. *The integral cohomology of the complex Stiefel manifolds* $V_j(\mathbb{C}^n) = \mathrm{U}(n)/\mathrm{U}(n-j)$ *is*

$$H^*\big(V_j(\mathbb{C}^n);\mathbb{Z}\big) = \Lambda[z_{2(n-j)+1}, \ldots, z_{2n-3}, z_{2n-1}].$$

The integral cohomology of the quaternionic Stiefel manifolds $V_j(\mathbb{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(n-j)$ *is given by*

$$H^*\big(V_j(\mathbb{H}^n);\mathbb{Z}\big) = \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-5}, z_{4n-1}].$$

Proof. The spectral sequences of the bundles (3.1.2) dealt with in Proposition 3.1.1 all collapsed at the E_2 page, so that in particular the maps $H^*\mathrm{U}(n) \longrightarrow H^*\mathrm{U}(n-1)$ are surjective and the iterated map $H^*\mathrm{U}(n) \longrightarrow H^*\mathrm{U}(n-j)$ is surjective by induction: explicitly, it is the projection

$$\Lambda[z_1, z_3, \ldots, z_{2(n-j)-1}] \otimes \Lambda[z_{2(n-j)+1}, \ldots, z_{2n-1}] \longrightarrow\!\!\!\!\!\rightarrow \Lambda[z_1, z_3, \ldots, z_{2(n-j)-1}],$$

with kernel $(z_1, z_3, \ldots, z_{2(n-j)-1})$ the extension of the augmentation ideal of the second factor. One has more or less definitionally the fiber bundle

$$\mathrm{U}(n-j) \longrightarrow \mathrm{U}(n) \longrightarrow V_j(\mathbb{C}^n), \tag{3.1.9}$$

whose SSS collapses at E_2 by Section 8.3.1 since we have just shown the fiber projection is surjective. Thus the base pullback $H^*V_j(\mathbb{C}^n) \longrightarrow H^*\mathrm{U}(n)$ is injective and $H^*V_j(\mathbb{C}^n)$ is an exterior subalgebra of $H^*\mathrm{U}(n)$ whose augmentation ideal extends to the kernel $(z_{2(n-j)+1}, \ldots, z_{2n-1})$ of the fiber projection. We see $H^*V_j(\mathbb{C}^n)$ can only be as claimed.

The proof for $H^*V_j(\mathbb{H}^n)$ is entirely analogous. □

3.2. Real difficulties

The degeneration of spectral sequences that occurs for unitary and symplectic fails for the orthogonal groups, because in the analogue of the iterated fiber decompostion (3.1.2) of the orthogonal groups, one encounters spheres of adjacent dimension, which could lead to nontrivial differentials. Indeed, this does lead to rather complicated 2-torsion, so we pass to simpler coefficient rings. Even with this simplification, there seems to be a certain unavoidable difficulty in handling $H^*\mathrm{SO}(n)$, forcing case distinctions and a rather explicit calculation of a map of homotopy groups. The proofs here are, in the author's own opinion, cleaner and more scrutable than those in the source material, but he would not claim they make an easy read. The reader can be forgiven for skipping to the next chapter at this point, but it seems only right to say what can be explained about $H^*\mathrm{SO}(n)$ and $H^*\mathrm{Spin}(n)$ at this point, and we will need this material for examples later.

To proceed, we require on a lemma [MT00, Cor. 3.13, p. 121] about the cohomology of a Stiefel manifold $V_2(\mathbb{R}^n)$. The proof here is a hybrid of Mimura and Toda's and that in online notes by Bruner, Catanzaro, and May [BCM]. Recall our notational conventions from Appendix A.2.1.

Lemma 3.2.1. *The real Stiefel manifold $V = V_2(\mathbb{R}^n)$ (for $n \geqslant 4$) has*

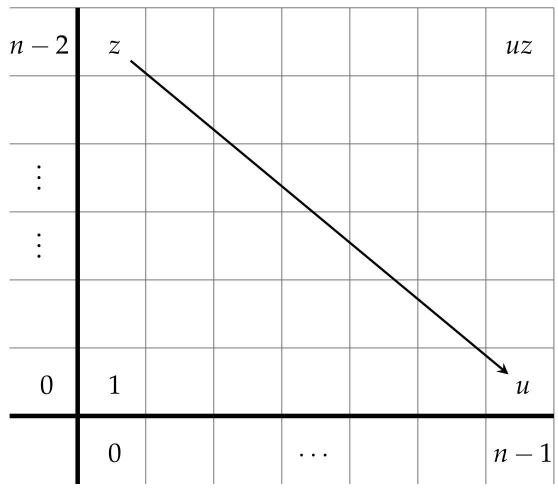
$$H_{n-2}(V) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ \mathbb{Z}/2 & n \text{ odd}. \end{cases}$$

Proof. If we define $V_2(\mathbb{R}^n) := \mathrm{SO}(n)/\mathrm{SO}(n-2)$ as the set of pairs of orthogonal elements of S^{n-1}, or equivalently $n \times 2$ matrices with orthonormal columns, then projection q to the first column is the projection of a bundle

$$S^{n-2} \longrightarrow V_2(\mathbb{R}^n) \xrightarrow{q} S^{n-1}.$$

The associated Serre spectral sequence is as in Figure 3.2.2, and it is clear the lone potentially nonzero differential is $H^{n-2}(S^{n-2}) \xrightarrow{d} H^{n-1}(S^{n-1})$.

Figure 3.2.2: The differential d_{n-1} in the Serre spectral sequence of $S^{n-2} \to V_2(\mathbb{R}^n) \to S^{n-1}$

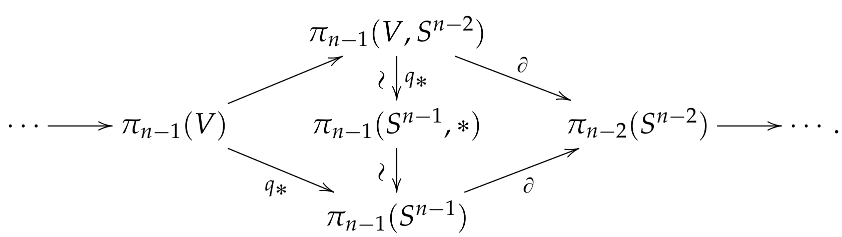


In particular, we have $H^j(V) = 0$ for $j < n-2$, and $H_j(V) = 0$ as well by the universal coefficient Theorem B.1.1. Since we have assumed $n \geqslant 4$, it follows from the long exact homotopy sequence of the bundle (Theorem B.1.4) that V is simply-connected, so by the Hurewicz Theorem B.1.1, $\pi_{n-2}(V) \cong H_{n-2}(V)$, and we can concern ourselves with this group instead. The long exact homotopy sequence of Theorem B.1.4 contains the subsequence

$$\underbrace{\pi_{n-1}(S^{n-1})}_{\cong \mathbb{Z}} \xrightarrow{\partial} \underbrace{\pi_{n-2}(S^{n-2})}_{\cong \mathbb{Z}} \longrightarrow \pi_{n-2}(V) \longrightarrow \underbrace{\pi_{n-2}(S^{n-1})}_{0},$$

showing $\pi_{n-2}(V) \cong \mathbb{Z}/\operatorname{im}\partial$, so our task is now to identify $\operatorname{im}\partial$. Since $\pi_{n-1}(S^{n-1})$ is cyclic, it is enough to know what ∂ does to a generator.

Recall that the long exact homotopy sequence of the bundle $S^{n-2} \to V \to S^{n-1}$ is derived from the long exact homotopy sequence of the pair (V, S^{n-2}) through the isomorphism induced by the map of pairs $q\colon (V, S^{n-2}) \longrightarrow (S^{n-1}, *)$, as follows:



The top ∂ takes the class represented by a map of pairs $\iota\colon (D^{n-1}, S^{n-2}) \longrightarrow (V, S^{n-2})$ to the homotopy class of the restriction $\iota \restriction S^{n-2}$. Since the vertical maps are isomorphisms, such an ι will represent a generator just if $q\iota\colon (D^{n-1}, S^{n-2}) \longrightarrow (S^{n-1}, *)$ represents a generator of $\pi_{n-1}(S^{n-1}, *)$. We turn to constructing this ι.

It will be convenient to consider $V = V_2(\mathbb{R}^n)$ as a quotient $\mathrm{O}(n)/\mathrm{O}(n-2)$.[1] Write $p_2\colon \mathrm{O}(n) \longrightarrow V_2(\mathbb{R}^n)$ for the natural projection of a matrix to the first two columns, realizing this quotient description, and $p_1\colon O(n) \longrightarrow S^{n-1}$ for projection to the first column alone. Note that $p_1 = qp_2$

[1] We introduced it as $\mathrm{SO}(n)/\mathrm{SO}(n-2)$, but this is the same; any $g \in \mathrm{O}(n)$ extending an orthonormal 2-frame $(v, w) \in V_2(\mathbb{R}^n)$ can be made into an element of $\mathrm{SO}(n)$ by multiplying the last column by ± 1.

and that p_1 can be seen as the evaluation map $g \longmapsto ge_1$ taking an automorphism $g \in \mathrm{O}(n) < \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^n)$ to its value at the standard basis vector $e_1 = (1, \vec{0})^\top$ of $\mathbb{R}^n$. The preimage of e_1 under p_1 is the stabilizer $\mathrm{Stab}(e_1) < \mathrm{O}(n)$, a block-diagonal $\{1\} \times \mathrm{O}(n-1)$ which we write as $\mathrm{O}(n-1)$. The image $p_2\big(\mathrm{O}(n-1)\big) \approx S^{n-2}$ of this subgroup in V is the fiber of the bundle $S^{n-2} \to V \to S^{n-1}$ over e_1. Summarizing:

$$\overbrace{\big(\mathrm{O}(n), \mathrm{O}(n-1)\big) \underset{p_2}{\longrightarrow} (V, S^{n-2}) \underset{q}{\longrightarrow} (S^{n-1}, e_1)}^{p_1}.$$

The map ι arises from the natural map $S^{n-1} \longrightarrow \mathrm{O}(n)$ taking a unit vector v to the reflection $r_v \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ through the hyperplane $v^\perp < \mathbb{R}^n$ orthogonal to v. Write $r \colon D^{n-1} \longrightarrow \mathrm{O}(n)$ for the restriction of this map to the northern hemisphere $D^{n-1} = \{v \in S^{n-1} : v \cdot e_1 \geqslant 0\}$ of S^{n-1}. Note that the composition $p_1 r$ takes $v \longmapsto p_1(r_v) = r_v(e_1)$. If $S^{n-2} = \partial D^{n-1}$ is the equator, made up of those unit vectors perpendicular to e_1, then we claim r takes S^{n-2} to $\mathrm{O}(n-1)$:

$$(D^{n-1}, S^{n-2}) \xrightarrow{\ r\ } \big(\mathrm{O}(n), \mathrm{O}(n-1)\big).$$

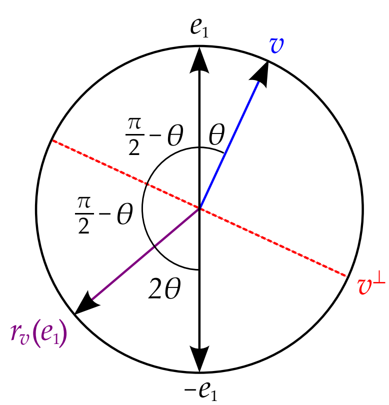
To see this, note that if $v \in S^{n-2}$, so that v is perpendicular to e_1, then e_1 is in the hyperplane $v^\perp$ fixed by r_v, so $(p_1 r)(v) = r_v(e_1) = e_1$. That means the first column of r_v is e_1, so that $r_v \in \mathrm{O}(n-1)$.

We set $\iota = p_2 r \colon (D^{n-1}, S^{n-2}) \longrightarrow (V, S^{n-2})$. To see ι represents a generator of $\pi_{n-1}(V, S^{n-2})$, we show

$$q\iota = qp_2 r = p_1 r \colon (D^{n-1}, S^{n-2}) \longrightarrow (S^{n-1}, e_1)$$

represents a generator of $\pi_{n-1}(S^{n-1}, e_1)$ by demonstrating it takes the interior $D^{n-1} \backslash S^{n-2}$ homeomorphically onto $S^{n-1} \backslash \{e_1\}$. Let $v \in D^{n-1} \backslash S^{n-2}$. If $v = e_1$, then $r_{e_1}(e_1) = -e_1$, and otherwise v and e_1 together span a 2-plane, which cuts S^{n-1} in a circle and $v^\perp$ in a line, and $(p_1 r)(v) = r_v(e_1)$ lies in this plane; see Figure 3.2.3. Since $p_1 r$ preserves these circles, it is be enough to show that the restriction of $p_1 r$ to each open upper semicircle is injective, but this is the case because if the nonzero angle $\theta = \measuredangle(e_1, v)$ lies in the interval $(-\pi/2, \pi/2)$, then $\measuredangle\big(r_v(e_1), -e_1\big) = 2\theta$ lies in $(-\pi, 0) \cup (0, \pi)$.

Figure 3.2.3: The reflection of e_1 through $v^\perp$



Now, since ι represents a generator of $\pi_{n-1}(V, S^{n-2})$, the restriction $\chi = (\iota \restriction S^{n-2}) \colon S^{n-2} \longrightarrow S^{n-2}$ represents a generator of $\mathrm{im}\, \partial$. Write $S^{n-3} \subsetneq S^{n-2}$ for the set of those unit vectors v perpendicular to both e_1 and e_2. For such a v, the reflection r_v will leave the first two coordinates of an element of $\mathbb{R}^n$ invariant, so $\chi(S^{n-3}) = \big\{(e_1, e_2)\big\} \in V$. Since $r_v = r_{-v}$, the same argument as for $p_1 r$ shows that χ takes the interiors of both north and south hemispheres homeomorphically onto $S^{n-2}\backslash\{e_2\}$, so restrictions to these hemispheres are maps

$$\tau_\pm \colon (D^{n-2}, S^{n-2}) \longrightarrow (S^{n-2}, e_2)$$

representing generators of $\pi_{n-2}(S^{n-2}, e_2) \cong \pi_{n-2}(S^{n-2})$ such that $[\chi] = [\tau_+] + [\tau_-]$. These generators are closely related: if

$$\begin{aligned} \alpha \colon S^{n-2} &\longrightarrow S^{n-2}, \\ v &\longmapsto -v, \end{aligned}$$

is the antipodal map, then we have $\tau_- = \tau_+ \circ \alpha$. Since α is the composition of $n-1$ reflections of $\mathbb{R}^{n-1}$, it is of degree $(-1)^{n-1}$, so that χ represents $s_n := \big(1 + (-1)^{n-1}\big)$ times the generator $[\tau_+]$ of $\pi_{n-2}(S^{n-2})$.

Since $s_{\text{even}} = 2$ and $s_{\text{odd}} = 0$, the group $H_{n-2}(V) = \pi_{n-2}(V) \cong \mathbb{Z}/s_n\mathbb{Z}$ is as claimed. □

Remark 3.2.4. Since $V_2(\mathbb{R}^n)$ is the set of pairs (v, w) with $v \in S^{n-1}$ and $w \perp v$, it can be seen as the set of unit vectors in the tangent bundle TS^{n-1}. This is a S^{n-2}-bundle associated to a principal $\mathrm{SO}(n-1)$-bundle, and it can be shown that the image of the element 1 of the fiber cohomology group $\mathbb{Z} = H^{n-2}(S^{n-2})$ in the base cohomology group $H^{n-1}(S^{n-1}) = \mathbb{Z}$ is the *Euler class* of this bundle (see Section 7.5); the fact that this number alternates between zero and two can be seen as a reflection of the fact that the Euler characteristics (Appendix A.2.3) of spheres obey the rule $\chi(S^n) = 1 + (-1)^n$.

Corollary 3.2.5 (Stiefel [Sti35]). *The nonzero integral cohomology groups of the real Stiefel manifold* $V = V_2(\mathbb{R}^n)$ *are*

$$H^0(V) \cong H^{2n-3}(V) \cong \mathbb{Z}, \qquad H^{n-2}(V) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases} \qquad H^{n-1}(V) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ \mathbb{Z}/2 & n \text{ odd}. \end{cases}$$

In particular, the differential $H^{n-2}(S^{n-2}) \xrightarrow{d} H^{n-1}(S^{n-1})$ *shown in Figure 3.2.2) is zero if n is even and multiplication by 2 if n is odd. The mod 2 cohomology ring of V is*

$$H^*(V; \mathbb{F}_2) \cong \Lambda[v_{n-2}, v_{n-1}]$$

Proof. If n is even, we have $\pi_{n-2}(V) = H_{n-2}(V)$ infinite cyclic from Lemma 3.2.1, so by universal coefficients, $H^{n-1}(V)$ is also free abelian, and it follows $d = 0$ and $H^{n-2}(V) \cong \mathbb{Z}$.

If n is odd, we have $\mathbb{Z}/2 \cong \pi_{n-2}(V) = H_{n-2}(V)$, so by universal coefficients, $H^{n-2}(V) = 0$ and $H^{n-1}(V)$ is the sum of $\mathbb{Z}/2$ and a free abelian group. But $H^{n-1}(V)$ is cyclic, since it is $\operatorname{coker} d$, so we have $H^{n-1}(V) \cong \mathbb{Z}/2$.

As for the modulo 2 case, we have $2 \equiv 0 \pmod 2$, so the map d is always zero and the SSS collapses. There is no extension problem simply by a dimension count. □

The main point of this argument, for us, is that the map d is trivial for n even and an isomorphism over $\mathbb{Z}[\frac{1}{2}]$ if n is odd. In the mod 2 case, these differentials are all zero, so we can induct up with spheres rather than $V_2(\mathbb{R}^n)$s, but we do have an extension problem because exterior algebras are not free CGAs in characteristic 2.

Corollary 3.2.6. *The mod 2 cohomology ring of $V = V_j(\mathbb{R}^n)$ has a simple system $v_{n-1}, \ldots, v_{n-j}$ of generators (see Definition A.2.4), where* $\deg v_i = i$*. That is,*

$$H^*(V; \mathbb{F}_2) = \Delta[v_{n-1}, v_{n-2}, \ldots, v_{n-j}].$$

Proof. We fix n and prove the result by induction on $j \in [1, n]$. For $j = 1$, the result is just $H^*(S^{n-1}) = \Lambda[v_{n-1}]$. Suppose by induction the result holds for $V_{j-1}(\mathbb{R}^n)$ and the Serre spectral sequence of $S^{n-(j-1)} \to V_{j-1}(\mathbb{R}^n) \to V_{j-2}(\mathbb{R}^n)$ collapses at E_2. Then the E_2 page of the Serre spectral sequence of $S^{n-j} \to V_j(\mathbb{R}^n) \longrightarrow V_{j-1}(\mathbb{R}^n)$ is (additively)

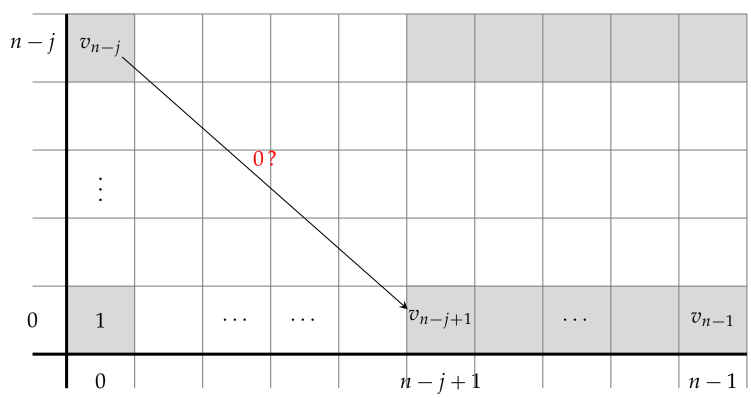
$$E_2 = \Delta[v_{n-1}, \ldots, v_{n-(j-1)}] \otimes \Delta[v_{n-j-1}],$$

so the induction will go through if and only if $E_2 = E_\infty$ in this spectral sequence as well. The only potentially nontrivial differential is $d_{n-(j-1)}$, which vanishes on the base $\Delta[v_{n-1}, \ldots, v_{n-(j-1)}]$ and so is determined by the map

$$H^{n-j-1}(S^{n-j-1}) \xrightarrow{d_{n-j+1}} H^{n-j}\big(V_j(\mathbb{R}^n)\big)$$

indicated in Figure 3.2.7.

Figure 3.2.7: The Serre spectral sequence of $S^{n-j} \to V_j(\mathbb{R}^n) \to V_{j-1}(\mathbb{R}^n)$ over $\mathbb{F}_2$



To see this map is zero, we identify it with the analogous differential in the Serre spectral sequence of $S^{n-j} \to V_2(\mathbb{R}^{n+2-j}) \to S^{n+1-j}$, which we already know to be zero by Corollary 3.2.5. To do that, consider the following commutative diagram:

$$\begin{array}{ccccc}
S^{n-j} & = & S^{n-j} & & \\
\downarrow & & \downarrow & & \\
V_2(\mathbb{R}^{n+2-j}) & \longrightarrow & V_j(\mathbb{R}^n) & \longrightarrow & V_{j-2}(\mathbb{R}^n) \\
\downarrow & & \downarrow & & \| \\
S^{n+1-j} & \longrightarrow & V_{j-1}(\mathbb{R}^n) & \to & V_{j-2}(\mathbb{R}^n).
\end{array}$$

Each row and column is a bundle, and the bundle projections are of the form "consider the first few vectors"; for example, the map $V_j(\mathbb{R}^n) \to V_{j-2}(\mathbb{R}^n)$ simply forgets the last two vectors of a

j-frame on $\mathbb{R}^n$, and the fiber over a $(j-2)$-frame is the set of 2-frames orthogonal to those $j-2$ vectors in $\mathbb{R}^n$, and so is a $V_2(\mathbb{R}^{n-j+2})$.

The map of columns induces a map (ψ_r) of spectral sequences from (E_r, d_r) to the spectral sequence $({}'E_r, {}'d_r)$ of the left column, which collapses at ${}'E_2$. The bottom row is the bundle whose Serre spectral sequence we inductively assumed collapses, so $\psi_{n+1-j}\colon H^{n+1-j}\big(V_{j-1}(\mathbb{R}^n)\big) \longrightarrow H^{n+1-j}(S^{n+1-j})$ is an isomorphism. The relation

$$0 = {}'d_{n+1-j}\psi_{n+1-j} = \psi_{n+1-j}d_{n+1-j}$$

then ensures $d_{n+1-j} = 0$ and we have collapse. □

Taking $j = n-1$ yields the result we really were after.

Corollary 3.2.8. *The mod 2 cohomology ring of the special orthogonal group* $\mathrm{SO}(n)$ *has a simple system* $v_1, \ldots, v_{n-1}$ *of generators:*

$$H^*\big(\mathrm{SO}(m); \mathbb{F}_2\big) = \Delta[v_1, v_2, \ldots, v_{n-1}],$$

where $\mathbb{F}_2\{v_{n-1}\}$ *is the image of* $H^{n-1}(S^{n-1}) \longrightarrow H^{n-1}\big(\mathrm{SO}(n)\big)$*.*

Remark 3.2.9. We used the induction $S^{n-j} \to V_j(\mathbb{R}^n) \to V_{j-1}(\mathbb{R}^n)$ to pick up the cohomology of the Stiefel manifolds along the way to that of $\mathrm{SO}(n)$. We could also have inducted the other way, using

$$\begin{array}{ccccccccccc}
\mathrm{SO}(2) & \longrightarrow & \mathrm{SO}(3) & \longrightarrow & \mathrm{SO}(4) & \longrightarrow & \cdots & \longrightarrow & \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}(n+1) \\
\downarrow \wr & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 & & S^2 & & S^3 & & \cdots & & S^{n-1} & & S^n,
\end{array}$$

in analogy with (3.1.2). Then the task is to show that the differential $H^{n-1}\big(\mathrm{SO}(n)\big) \longrightarrow H^n(S^n)$ is zero. We can still use the collapse of the Serre spectral sequence of $S^{n-1} \to V_2(\mathbb{R}^{n+1}) \to S^n$ to do this; the relevant bundle map is

$$\begin{array}{ccc}
\mathrm{SO}(n) & \longrightarrow & S^{n-1} \\
\downarrow & & \downarrow \\
\mathrm{SO}(n+1) & \longrightarrow & V_2(\mathbb{R}^{n+1}) \\
\downarrow & & \downarrow \\
S^n & = & S^n.
\end{array}$$

The induction is substantially subtler over $\mathbb{Z}$ or even over $k = \mathbb{Z}[\frac{1}{2}]$, because the differentials no longer must be trivial. We can use the real Stiefel manifolds $V_2(\mathbb{R}^n) \cong \mathrm{SO}(n)/\mathrm{SO}(n-2)$ as building blocks now, though, the same way we used spheres before:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{SO}(n-4) & \longrightarrow & \mathrm{SO}(n-2) & \longrightarrow & \mathrm{SO}(n) \\
& \downarrow & & \downarrow & & \downarrow \\
& V_2(\mathbb{R}^{n-4}) & & V_2(\mathbb{R}^{n-2}) & & V_2(\mathbb{R}^n).
\end{array} \tag{3.2.10}$$

Proposition 3.2.11. *Let $2n+1 \geqslant 3$ be an odd integer and $2j < 2n+1$ an even integer. Then taking coefficients in $k = \mathbb{Z}[\frac{1}{2}]$, we have*

$$\begin{aligned} H^*\big(\mathrm{SO}(2n+1)\big) &\cong \Lambda[z_3, z_7, \ldots, z_{4n-1}], \quad \deg z_{4i-1} = 4i-1. \\ H^*\big(V_{2j}(\mathbb{R}^{2n+1})\big) &\cong H^*\big(\mathrm{SO}(2n+1)\big) \big/\!\!\big/ H^*\big(\mathrm{SO}(2n-2j+1)\big) \cong \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-1}]. \end{aligned}$$

Proof. By Corollary 3.2.5, we have $H^*\big(V_2(\mathbb{R}^{2j+1})\big) = \Lambda[z_{4j-1}]$, so the objects in (3.2.10) have the same cohomology as those in (3.1.4) which yielded the same structure (over $\mathbb{Z}$) for $H^*\big(\mathrm{Sp}(n)\big)$. The result for $H^*\big(V_j(\mathbb{R}^n)\big)$ follows as in Proposition 3.1.8. □

To recover $V_{2j-1}(\mathbb{R}^{2n})$, consider the map of bundles

$$\begin{array}{ccccc} V_{2j-2}(\mathbb{R}^{2n-1}) & = & V_{2j-2}(\mathbb{R}^{2n-1}) & & \\ \downarrow & & \downarrow & & \\ V_{2j-1}(\mathbb{R}^{2n}) & \longrightarrow & V_{2j}(\mathbb{R}^{2n+1}) & \longrightarrow & S^{2n} \\ \downarrow & & \downarrow & & \| \\ S^{2n-1} & \longrightarrow & V_2(\mathbb{R}^{2n+1}) & \longrightarrow & S^{2n}. \end{array}$$

The Serre spectral sequence of the middle column collapses at E_2 by an elaboration of our calculation above.[2] Thus we can use the bundle lemma Theorem 2.4.1 to conclude

$$H^*\big(V_{2j-1}(\mathbb{R}^{2n-1})\big) \cong \Lambda[e_{2n-1}] \underset{\Lambda[z_{4n-1}]}{\otimes} \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-1}] = \Lambda[e_{2n-1}] \otimes \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-5}].$$

Taking $n = j$, we recover $H^*\big(\mathrm{SO}(2n)\big)$.

Proposition 3.2.12. *Let $2n \geqslant 2$ be an even integer and $2j-1 < 2n$ odd. Then taking coefficients in $k = \mathbb{Z}[\frac{1}{2}]$,*

$$H^*\big(V_{2j-1}(\mathbb{R}^{2n})\big) \cong \Lambda[e_{2n-1}] \otimes \Lambda[z_{4(n-j)+3}, \ldots, z_{4n-5}],$$

where $\deg z_i = i$ and $\deg e_{2n-1} = 2n-1$. In particular,

$$H^*\big(\mathrm{SO}(2n)\big) \cong \Lambda[e_{2n-1}] \otimes \Lambda[z_3, \ldots, z_{4n-5}].$$

We can state the result for $\mathrm{SO}(m)$ more uniformly as follows:

---

[2] The relevant bundle map is this:

$$\begin{array}{ccccc} \mathrm{SO}(2n-2j+1) & \rightarrowtail & \mathrm{SO}(2n-1) & \twoheadrightarrow & V_{2j-2}(\mathbb{R}^{2n-1}) \\ \| & & \downarrow & & \downarrow \\ \mathrm{SO}(2n-2j+1) & \rightarrowtail & \mathrm{SO}(2n+1) & \longrightarrow & V_{2j}(\mathbb{R}^{2n+1}). \end{array}$$

By Proposition 3.2.11, both rows yield tensor decompositions in cohomology and the fiber inclusion $\mathrm{SO}(2n-1) \longrightarrow \mathrm{SO}(2n+1)$ is surjective in cohomology with kernel (z_{4n-1}), which is in the image of $H^*V_{2j}(\mathbb{R}^{2n+1})$, so the same holds of the right-hand map we are interested in.

Corollary 3.2.13. *Over $k = \mathbb{Z}[\frac{1}{2}]$, the cohomology ring of* $\mathrm{SO}(m)$ *is*

$$H^*(\mathrm{SO}(m); \mathbb{Z}[\tfrac{1}{2}]) = \begin{cases} \Lambda[z_3, z_7, \ldots, z_{4n-5}] \otimes \Lambda[e_{2n-1}], & m = 2n, \\ \Lambda[z_3, z_7, \ldots, z_{4n-5}] \otimes \Lambda[z_{4n-1}], & m = 2n+1, \end{cases}$$

where $k \cdot e_{2n-1}$ is the image of $H^{2n-1}(S^{2n-1}) \longrightarrow H^{2n-1}\big(\mathrm{SO}(2n)\big)$.

To get the cases $V_\ell(\mathbb{R}^m)$ where $\ell \equiv m \pmod 2$, we can use the Serre spectral sequence of

$$S^{m-\ell} \longrightarrow V_\ell(\mathbb{R}^m) \longrightarrow V_{\ell-1}(\mathbb{R}^m).$$

as we did in Corollary 3.2.6. The E_2 page is $H^*\big(V_{\ell-1}(\mathbb{R}^m)\big) \otimes \Delta[s_{m-\ell}]$, and the only potentially nonzero differential, $d_{m-\ell+1}$, is determined by a map $d\colon H^{m-\ell}(S^{m-\ell}) \longrightarrow H^{m-\ell+1}\big(V_{\ell-1}(\mathbb{R}^m)\big)$. By the last two propositions, the ring $H^*\big(V_{\ell-1}(\mathbb{R}^m)\big)$ is an exterior algebra on generators of degree at least $2m - 2\ell + 3$ if m is odd, and at least $m - 1$ if m is even. In the former case, d is zero by lacunary considerations. In the latter, $\ell \geqslant 2$, since ℓ is of the same parity as m, so we have $m - \ell + 1 \leqslant m + 1$, with equality if and only if $\ell = 2$. Thus, if $\ell > 2$, then $d = 0$ by lacunary considerations, and if $\ell = 2$, then we showed $d = 0$ in Corollary 3.2.5. So no matter what, the sequence collapses at E_2, and then by Proposition A.4.4, we have

$$H^*\big(V_\ell(\mathbb{R}^m)\big) \cong H^*\big(V_{\ell-1}(\mathbb{R}^m)\big) \otimes \Delta[s_{m-\ell}].$$

To compile these cases into one statement, we introduce some notation. Let S be a free k-module or basis thereof and φ a proposition whose truth or falsehood is easily verifiable. We write

$$\Lambda\big[\{S : \varphi\}\big] = \begin{cases} \Lambda[S] & \text{if } \varphi \text{ is true,} \\ k & \text{otherwise.} \end{cases}$$

Then, gathering cases and doing some arithmetic on indices, we arrive at the following.

Proposition 3.2.14 ([BCM, Thm. 2.5])**.** *The cohomology of the real Stiefel manifold $V_\ell(\mathbb{R}^m)$ with coefficients in $k = \mathbb{Z}[\frac{1}{2}]$ is given by*

$$H^*\big(V_\ell(\mathbb{R}^m)\big) \cong \Lambda[z_{4j-1} : 2m - 2\ell + 1 \leqslant 4j - 1 \leqslant 2m - 3] \otimes \Lambda[e_{m-1} : m \text{ even}] \otimes \Delta[s_{m-\ell} : m - \ell \text{ even}].$$

Remark 3.2.15. The author found the useful notation for abbreviating case distinctions in Proposition 3.2.14 in the notes by Bruner, Catanzaro, and May [BCM].[3]

It is standard to discuss along with $\mathrm{SO}(n)$ its simply-connected double cover $\mathrm{Spin}(n)$.

Proposition 3.2.16. *The cohomology of* $\mathrm{Spin}(n)$ *for $n \geqslant 2$ satisfies*

$$H^*(\mathrm{Spin}(n); \mathbb{Z}[\tfrac{1}{2}]) \cong H^*(\mathrm{SO}(n); \mathbb{Z}[\tfrac{1}{2}]).$$

Proof. Since $\pi\colon \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is a connected double cover and 2 is invertible, the isomorphism follows immediately from Corollary B.2.2. ☐

[3] It seems uncommon to find another statement of this result without typos. Both the excellent books of Mimura and Toda [MT00, Thm. III.3.14, p. 121] and of Félix, Oprea, and Tanré [FOT08, Prop. 1.89, p. 84] have misprints in their statements of the result Proposition 3.2.14 where the (even, even) case is omitted and another case repeated twice with different results. For example, Mimura and Toda list two nonisomorphic rings for the case (odd, odd). For those keeping score, the misprint in [FOT08] is nonisomorphic to the misprint in [MT00]

Finally, we will relate without proof the multiplicative structure of $H^*\mathrm{SO}(n)$ and $H^*\mathrm{Spin}(n)$ with $\mathbb{F}_2$ coefficients. The standard proofs invoke Steenrod squares, which we decided not to assume as background.

Proposition 3.2.17. *The mod 2 cohomology of* $\mathrm{SO}(n)$ *for* $n \geqslant 2$ *is given by*

$$H^*\big(\mathrm{SO}(n);\mathbb{F}_2\big) = \mathbb{F}_2[v_1,\ldots,v_{n-1}]/\mathfrak{a},$$

where the ideal $\mathfrak{a}$ *is generated by the relations*

$$v_i^2 \equiv \begin{cases} v_{2i}, & 2i < n,\\ 0, & 2i \geqslant n.\end{cases}$$

Shedding excess generators, we can write

$$H^*\big(\mathrm{SO}(n);\mathbb{F}_2\big) = \mathbb{F}_2[v_1,v_3,\ldots,v_{\lfloor n/2\rfloor-1}]/\mathfrak{b},$$

where $\mathfrak{b}$ *is the truncation ideal* $(v_i^{\lceil n/i\rceil})$ *generated by the least powers of* v_i *of degree exceeding* $n-1$.

The mod 2 cohomology of $\mathrm{Spin}(n)$ *admits a simple system of generators containing an element* z *of degree* $2^{\lceil \log_2 n\rceil}-1$ *and generators* v_j *for each* $j \in [1, n-1]$ *which is not a power of* 2*:*

$$H^*\big(\mathrm{Spin}(n);\mathbb{F}_2\big) = \Delta\big[z_{2^{\lceil \log_2 n\rceil}-1}, v_j \ : \ 1 \leqslant j < n, \quad j \neq 2^r\big].$$

Hopf's theorem 1.0.4 allows one to be more specific about the ring structure of $H^*\big(\mathrm{Spin}(n);\mathbb{F}_2\big)$, but the description is disappointingly complicated. A simpler description can be obtained for the countable-dimensional group

$$\mathrm{Spin} := \varinjlim \mathrm{Spin}(n),$$

where the colimit is taken along the unique maps $\mathrm{Spin}(n) \rightarrowtail \mathrm{Spin}(n+1)$ lifting the composition $\mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1)$ of the covering map with the canonical inclusion, which exist because the spinor groups are simply-connected. As by construction the diagrams

$$\begin{array}{ccc} \mathrm{Spin}(n) & \rightarrowtail & \mathrm{Spin}(n+1) \\ \downarrow & \searrow & \downarrow \\ \mathrm{SO}(n) & \hookrightarrow & \mathrm{SO}(n+1) \end{array}$$

commute, Spin can be seen as a simply-connected double covering of $\mathrm{SO} := \bigcup \mathrm{SO}(n)$.

Theorem 3.2.18 ([BCM, Thm. 6.10, p. 55])**.** *The mod 2 cohomology ring of* Spin *is given by*

$$H^*(\mathrm{Spin};\mathbb{F}_2) = \mathbb{F}_2[v_{2n+1} : n \geqslant 1]$$

and that of SO *by*

$$H^*(\mathrm{SO};\mathbb{F}_2) = \mathbb{F}_2[v_{2n+1} : n \geqslant 0],$$

the map $H^*\mathrm{SO} \longrightarrow H^*\mathrm{Spin}$ *induced by* $\mathrm{Spin} \twoheadrightarrow \mathrm{SO}$ *being the obvious surjection.*

Historical remarks 3.2.19. The lemma 3.2.1 is due to Eduard Stiefel also the namesake of the Stiefel manifolds and the Stiefel–Whitney classes. A comprehensive account of this material, also including explicit computations for the cohomology of the exceptional groups, can be found in the much-recommended book of Mimura and Toda [MT00]. As an indication of the nontriviality of computing $H^*\mathrm{SO}(n)$, we point out that while the cohomology ring $H^*\big(\mathrm{SO}(n);k\big)$ for k any field follows immediately from what we have done in this section and extracting the *additive* structure of the integral cohomology is not hard afterward, the recovery of the *integral cohomology ring* from this data seems to only have been completed in 1989 [Pit91].

Chapter 4

Formality and polynomial differential forms

In this chapter we define and develop two concepts from rational homotopy theory to the extent we will need them. Formality will let us exchange a cochain algebra for its cohomology, in a manner of speaking, and the algebra of polynomial differential forms will give us a functorial commutative model for rational singular cohomology.

4.1. Formality

The real cohomology of a Lie group exhibits a remarkable property. Like the rational singular cohomology, it is an exterior Hopf algebra, but unlike rational cohomology, it admits a classical *commutative* model. Since G is among other things a smooth manifold, by de Rham's theorem, one can compute $H^*(G;\mathbb{R})$ as the cohomology of the *de Rham algebra* $\Omega^\bullet(G)$ of differential forms, an $\mathbb{R}$-CDGA. If $\vec{z}$ is an $\mathbb{R}$-basis for the primitive elements of $H^*(G;\mathbb{R}) \cong H^*(G;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$, then we can pick out one closed form $\omega_j \in \Omega^\bullet(G)$ representing each z_j, and because $\Omega^\bullet(G)$ is a commutative graded algebra, we have an exterior subalgebra $\Lambda[\vec{\omega}]$ of $\Omega^\bullet(G)$ representing $H^*(G;\mathbb{R})$. That is to say, we have can define an *algebra* section of the projection $Z^*(G) \longrightarrow H^*(G;\mathbb{R})$, or, put another way, we have found a quasi-isomorphism

$$\big(H^*(G;\mathbb{R}), 0\big) \longrightarrow \big(\Omega^\bullet(G), d\big)$$

to the de Rham algebra from *its own cohomology*, viewed as a CDGA with zero differential.

Of course, one can of course always find a *vector space* of representative forms, but the ability to make these to form a subring on the nose, rather than up to homotopy, is rather special. This behavior will be sufficiently useful to us that we formalize the situation.[1]

Definition 4.1.1. A differential graded k-algebra (A, d) is said to be ***formal*** if there exists a zig-zag of k-DGA quasi-isomorphisms

$$\begin{array}{ccccccccc} & & (B_1, d_1) & & & & \cdots & & (A, d) \\ & \swarrow & & \searrow & & \swarrow & & \searrow \quad \swarrow & \\ \big(H^*(A), 0\big) & & & & (B_2, d_2) & & & (B_n, d_n) & \end{array}$$

connecting $\big(H^*(A), 0\big)$ and (A, d). A simply-connected topological space X is said to be ***k-formal*** if there exists a formal k-DGA with cohomology $H^*(X;\mathbb{Q})$. A zig-zag of k-DGA quasi-isomorphisms from (A, d) to the singular cochain algebra $\big(C^*(X;k), \delta\big)$ is called a ***model*** of X.

[1] Pun unintended, but retained.

Example 4.1.2. We will find in Section 7.4 that for a compact, connected Lie group G, the cohomology $H^*(BG;\mathbb{Q})$ of its classifying space BG (see Chapter 5) is a symmetric algebra, hence a free CGA. In Section 4.2, we will produce a commutative model, A_{PL} for BG, and then, as for $H^*(G;\mathbb{R})$ and $\Omega^\bullet(G)$, it will follow by Proposition A.4.3 that after assigning generators, $H^*(BG;\mathbb{Q})$ lifts to back to a subalgebra of cocycles of $\big(A_{\mathrm{PL}}(BG)\big)$, inducing a quasi-isomorphism $\big(H^*(BG),0\big) \longrightarrow A_{\mathrm{PL}}(BG)$ and showing BG is formal.

Example 4.1.3. Élie Cartan demonstrated that symmetric spaces G/K are formal, in fact showing that the collection of *harmonic* forms on a symmetric space forms a subring of the differential forms $\Omega^\bullet(G/K)$ consisting of one element from each class in $H^*(G/K)$. We will produce a version of this proof in Proposition 8.5.2.

It is a remarkable fact about fields k of characteristic zero that they allow one to construct small models. Moreover one can easily piece such models together. We have already found such a model of G. The overall plan of the rest of this work is to find such a simple model for the classifying space BK of a connected Lie group, to be defined and constructed in Chapter 5, and use these models to construct a simple model of a homogeneous space G/K. However, BK will not be a manifold, but will almost always be infinite-dimensional, so the methods of differential topology will not directly apply. Instead, we will find a $\mathbb{Q}$-CDGA computing the rational singular cohomology of any topological space.

4.2. Polynomial differential forms

The obvious stumbling block to defining differential forms on an arbitrary topological space X is the absence of a smooth structure. There are at least two ways around this. The first historically, due to Leray and summarized in Historical remarks C.3.2, is to abstract the features of the de Rham algebra and prove that analogous objects exist over any sufficiently regular space.[2] This approach led Leray to sheaf theory and spectral sequences. The second approach is to replace X with a homotopy equivalent space that does admit forms, and it is that tack we take here.

We can at least define smooth forms on a single n-simplex

$$\Delta^n = \big\{(t_0,\ldots,t_n) \in \mathbb{R}^{n+1} : \sum t_j = 1\big\},$$

since this is just a manifold with corners. It is reasonable to say a form is continuous if it is smooth on the interior of Δ^n, its restriction to the interior of each face Δ^{n-1} is smooth, etc. Doing this yields a perfectly reasonable complex of forms. Of course the simplex is contractible, so the cohomology of this complex will be trivial.

If X is a polyhedron $\bigcup \Delta_\alpha$, meaning a union of simplices (glued whole face to whole face) then one can define a smooth form on X to be given by a collection of smooth forms ω_α on Δ_α such that whenever Δ_β is a face of Δ_α, then $\omega_\alpha|_{\Delta_\beta} = \omega_\beta$. This amounts to decomposing X into a union of simplices, defining forms on each, and asking these forms respect the gluings. We should make this more precise, so we will review simplices a bit.

4.2.1. Semisimplicial sets

A polyhedron X may be embedded, in the case of utmost extremity, as a piecewise affine subspace of a sufficiently high-dimensional vector space V: form the abstract vector space $V = \mathbb{R} \cdot X_0$ with

[2] Compact metrizable will do.

basis the vertex set X_0 of X and on an n-simplex σ^n in X with vertices $x_0, \ldots, x_n$, send the element with barycentric coordinates $\vec{t}$ in σ^n to the vector $\sum_{j=0}^{n} t_j x_j$ in V.

This canonical embedding is evidently monstrously inefficient, and one can usually get away with a much smaller vector space V. It is still the case that an individual embedded affine simplex $\sigma \approx \Delta^n$ in V is the convex hull of its $n+1$ vertices $v_0, \ldots, v_n$, so we may parameterize an embedded simplex as a tuple $[v_0, \ldots, v_n]$ without losing information about the embedding. This σ has $n+1$ faces, given by $\partial_j \sigma = [v_0, \ldots, \widehat{v}_j, \ldots, v_n]$ for $0 \leqslant j \leqslant n$, where the hat denotes omission,[3] and any subsimplex is described by a composition of these vertex omissions. A subsimplex is determined by which vertices are omitted, independent of what order they are forgotten in, so there are some relations among the omission operations ∂_j. These relations are all generated by the familiar relation $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$ responsible for the fact the boundary operator ∂ defining singular and simplicial homology satisfies $\partial^2 = 0$.

Viewing the polyhedron as a sort of construction kit snapping together pre-packaged parts, one sees that (up to piecewise linear homeomorphism) it is fully specified by a listing of its simplexes and the omission/inclusion operations between them. We extract this specification, writing K_n for the set of n-simplices of X.

Definition 4.2.1. A ***semisimplicial set*** $K_\bullet = (K_n)_{n \in \mathbb{N}}$ is family of sets K_n indexed by nonnegative integers, equipped with functions $\partial_j \colon K_n \longrightarrow K_{n-1}$ for $0 \leqslant j \leqslant n$ satisfying $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$.

This semisimplicial set is no longer a geometric object in any meaningful sense; it's closer to the truth to think of it as a set of labels and gluing instructions. To get X back out of $K_\bullet$, one follows the instructions, producing a distinct geometric simplex Δ^n for each $\sigma \in K_n$ and including the simplex corresponding to $\partial_j \sigma$ as its j^{th} face. In coordinates, the inclusion of Δ^{n-1} as the j^{th} face of $\Delta^n < \mathbb{R}^{n+1}$ is given by

$$\begin{aligned} i_j \colon \Delta^{n-1} &\longrightarrow \Delta^n, \qquad\qquad (0 \leqslant j \leqslant n) \\ \vec{t} &\longmapsto (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{n-1}) \end{aligned}$$

and explicitly, one recovers X from $K_\bullet$ as

$$X \approx \left. \bigcup_{n \in \mathbb{N}} K_n \times \Delta^n \middle/ (\partial_j \sigma, \vec{t}) \sim (\sigma, i_j \vec{t}) \right. . \tag{4.2.2}$$

Not every semisimplicial set $K_\bullet$ comes from a polygon to begin with—for example, there is nothing in the definition preventing us from having $\partial_i \sigma = \partial_j \tau = \partial_k v$ a common face of three distinct simplices—but this process produces a topological space even so.

Definition 4.2.3 (Milnor, Segal). Given a semisimplicial set $K_\bullet$, the result $\|K_\bullet\|$ of the process (4.2.2) is called the (*fat*) ***geometric realization*** of $K_\bullet$.

4.2.2. Forms on semisimplicial sets

Now that we have a more exact way of describing how simplices fit together, we are able to describe forms on polyhedra. A differential form on Δ^n should be a formal linear combination of terms

$$f_I \, dt^{i_1} \wedge \cdots \wedge dt^{i_q}$$

and each f_I is the restriction of a real-valued C^∞ function on a neighborhood of Δ^n in $\mathbb{R}^{n+1}$.

[3] The convention is due to Eilenberg and Mac Lane.

Definition 4.2.4. Write $C^\infty(\Delta^n) := \varinjlim_{U \supseteq \Delta^n} C^\infty(U)$. The $\mathbb{R}$-CDGA of ***smooth differential forms*** on Δ^n is

$$(A_{\mathrm{DR}})_n := C^\infty(\Delta^n) \underset{\mathbb{R}}{\otimes} \Lambda[dt^0, \ldots, dt^n] \big/ (dt^0 + \cdots + dt^n).$$

The differential d is the exterior derivative given on generators by

$$df = \sum \frac{\partial f}{\partial t^j} dt^j, \quad d(dt^j) = 0.$$

The restriction to the j^{th} face is defined on generators by

$$\begin{aligned} i_j^* \colon (A_{\mathrm{DR}})_n &\longrightarrow (A_{\mathrm{DR}})_{n-1}, \\ f &\longmapsto f \circ i_j, \\ dt^k &\longmapsto d(i_j^* t^k). \end{aligned}$$

In full detail, for $k < j$ we have $i_j^* dt^k = dt^k$, for $k > j$ we have $i_j^* dt^k = dt^{k-1}$, and $i_j^* dt^j = 0$. Note that the restrictions are DGA homomorphisms and $i_j^* i_k^* = i_{k-1}^* i_j^*$ for $j < k$, so $(A_{\mathrm{DR}})_\bullet$ is a semisimplicial set. We call such an object a ***semisimplicial* CDGA**.

Given a semisimplicial set $K_\bullet$, to define a smooth differential form consistently on $|K_\bullet|$, is to give an element $\omega_\sigma \in (A_{\mathrm{DR}})_n$ for each $\sigma \in K_n$ in such a way that $\omega_{\partial_j \sigma} = i_j^* \omega_\sigma$.

Definition 4.2.5. A ***semisimplicial map*** $\phi_\bullet \colon K_\bullet \longrightarrow L_\bullet$ between semisimplicial sets is a collection $(\phi_n \colon K_n \longrightarrow L_n)$ of functions satisfying $\partial_j \phi_n = \phi_n \partial_j$ for all $j \leqslant n$. We write the collection of such maps as $\mathrm{Hom}_{\mathrm{ss}}(K_\bullet, L_\bullet)$.

The ***algebra of smooth differential forms*** on a semisimplicial set is

$$\begin{aligned} A_{\mathrm{DR}}(K_\bullet) := \mathrm{Hom}_{\mathrm{ss}}\big(K_\bullet, (A_{\mathrm{DR}})_\bullet\big), \\ \sigma \mapsto \omega_\sigma. \end{aligned}$$

This inherits a "simplexwise" $\mathbb{R}$-CDGA structure via

$$(\phi + \psi)(\sigma) = \phi(\sigma) + \psi(\sigma), \quad (\phi \wedge \psi)(\sigma) = \phi(\sigma) \wedge \psi(\sigma), \quad (d\phi)(\sigma) = d\big(\phi(\sigma)\big).$$

Moreover this algebra is contravariantly functorial in that a semisimplicial map $\varkappa \colon K_\bullet \longrightarrow L_\bullet$ induces $\varkappa^* \colon A_{\mathrm{DR}}(L_\bullet) \longrightarrow A_{\mathrm{DR}}(K_\bullet)$ via precomposition, taking $\lambda : L_\bullet \longrightarrow (A_{\mathrm{DR}})_\bullet$ to $\lambda \circ \varkappa : K_\bullet \to L_\bullet \to (A_{\mathrm{DR}})_\bullet$.

The distinguished coordinates on a simplex make it possible to isolate a subalgebra of especial interest in $(A_{\mathrm{DR}})_\bullet$.

Definition 4.2.6. The semisimplicial $\mathbb{Q}$-CDGA of ***polynomial differential forms*** is the semisimplicial differential graded subalgebra of $(A_{\mathrm{DR}})_\bullet$ defined by

$$(A_{\mathrm{PL}})_n := \mathbb{Q}[t_0, \ldots, t_n] \underset{\mathbb{Q}}{\otimes} \Lambda_{\mathbb{Q}}[dt^0, \ldots, dt^n] \Big/ \big(1 - \sum t^j, \sum dt^j\big).$$

The polynomial differential forms on a semisimplicial complex are given by

$$A_{\mathrm{PL}}(K_\bullet) := \mathrm{Hom}_{\mathrm{ss}}\big(K_\bullet, (A_{\mathrm{PL}})_\bullet\big).$$

We claim, and will show that these forms compute cohomology in the standard sense.

Recall that the simplicial homology of $X = \|K_\bullet\|$ with coefficients in $\mathbb{Z}$ is given by taking the homology of the chain complex $C_n^\Delta(X)$ of finite formal sums $\sum \ell_\alpha \sigma_\alpha^n$ of n-simplices under the differential $\partial = \sum(-1)^j \partial_j$, and the simplicial cohomology with coefficients in an abelian group k is given by the dual complex $C^n_\Delta(X) := \mathrm{Hom}_{\mathbb{Z}}\big(C_n^\Delta(X), k\big) = \mathrm{Map}(K_n, k)$. This definition does not depend directly on the space X, and only on the sets of simplices, so it is an instance of the following definition.

Definition 4.2.7. Let $K_\bullet$ be a semisimplicial set and k an abelian group. The ***homology*** $H_*(K_\bullet, k)$ of $K_\bullet$ is the homology of the chain complex $\bigoplus k \cdot K_n$ of free k-modules equipped with the k-linear differential ∂ defined on a basis element $\sigma \in K_n$ by $\partial\sigma := \sum_{j=0}^n (-1)^j \partial_j \sigma$.

The ***cohomology*** $H^*(K_\bullet, k)$ of $K_\bullet$ is the cohomology of the cochain complex $\bigoplus \mathrm{Map}(K_n, k)$ of free k-modules equipped with the dual differential

$$\begin{aligned}\delta\colon \mathrm{Map}(K_n, k) &\longrightarrow \mathrm{Map}(K_{n+1}, k),\\ c &\longmapsto \Big(\sigma \mapsto \sum_{j=0}^{n+1} (-1)^j c(\partial_j \sigma)\Big).\end{aligned}$$

If k is a ring with unity, $\mathrm{Map}(K_\bullet, k)$ becomes a DGA under the cup product

$$\begin{aligned}\mathrm{Map}(K_m, k) \times \mathrm{Map}(K_n, k) &\overset{\smile}{\longrightarrow} \mathrm{Map}(K_{m+n}, k)\\ (c \smile c')(\sigma) &:= c(\partial_{m+1}^{\circ n} \sigma) \cdot c'(\partial_0^{\circ m} \sigma).\end{aligned}$$

The cup product induces a product on $H^*(K_\bullet, k)$ making it a CGA.

The cup product on the level of cochains is not commutative, so it is not immediately obvious if this cohomology relates in any way to those of our new algebras of forms. We can show isomorphisms on the level of semisimplicial sets, but for now we prefer to return to the level of spaces.

4.3. Comparison with singular cohomology

Singular cohomology can be seen as an instance of simplicial cohomology.

Definition 4.3.1. Given any topological space X, the ***total singular complex*** is the semisimplicial set $C_\bullet(X)$

$$C_n(X) := \mathrm{Top}(\Delta^n, X),$$

the set of singular simplices in X, with face maps given by restriction $\partial_j \sigma = \sigma i_j \colon \Delta^{n-1} \xrightarrow{i_j} \Delta^n \xrightarrow{\sigma} X$. The total singular complex is functorial in that a continuous map $X \longrightarrow Y$ induces a semisimplicial map $C_\bullet(X) \longrightarrow C_\bullet(Y)$ by precomposition.

Then singular homology and cohomology with constant coefficients are just the homology and cohomology of $C_\bullet(X)$ under Definition 4.2.7. Moreover, the total singular complex gives us a way to define A_{DR} and A_{PL} on an arbitrary space.

Definition 4.3.2. Given any topological space X, we define

$$A_{\mathrm{DR}}(X) := A_{\mathrm{DR}}\big(C_\bullet(X)\big) \quad \text{and} \quad A_{\mathrm{PL}}(X) := A_{\mathrm{PL}}\big(C_\bullet(X)\big).$$

These constructions are functorial in X because A_{DR}, A_{PL}, and $C_\bullet$ are.

[We need to compare these to singular cohomology. I am considering two tacks at present:

- Show they are cohomology theories and show integration induces an isomorphism on the cohomology groups of a point.
- Directly construct a zigzag of dga quasi-isomorphisms connecting them to $C^\bullet(X)$.

]

Historical remarks 4.3.3. Sullivan attributes the idea of forms on simplices to Whitney and Thom [Track down citations].

4.4. Simplicial sets

[We need to introduce simplicial sets and thin geometric realization for Section 5.4.]

Chapter 5

Classifying spaces

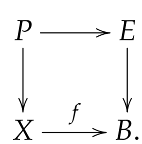
In this section, we carry out the construction of the ***universal principal*** *G**-bundle*** $EG \to BG$, which we use essentially as a tool to convert actions into closely related *free actions*. The existence of this bundle is more important than the details of its construction in almost everything that follows, but we may at some points use the fact that EG admits commuting right and left actions of G.

5.1. The weak contractibility of EG

The original purpose of the universal principal G-bundle $EG \to BG$ was to be a principal G-bundle such that all others $G \to E \to B$ arose as pullbacks. Moreover, it was seen that under these conditions, isomorphism classes of principal G-bundles over a given CW complex B correspond bijectively to homotopy classes of maps $B \longrightarrow BG$. Thus a map $B \longrightarrow BG$ of base spaces inducing E as a pullback of EG "classifies" the bundle $E \to B$, and so is called the ***classifying map*** of the bundle; and BG is called a ***classifying space*** for principal G-bundles.

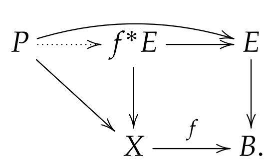
The fact that EG is weakly contractible—which is much of why we care about the universal bundle—turns out to be a consequence of that demand. In this subsection, we explain the relevance of this demand. It will simplify the argument to know that all maps of principal G-bundles are pullbacks.

Proposition 5.1.1. *Consider a principal G-bundle map*

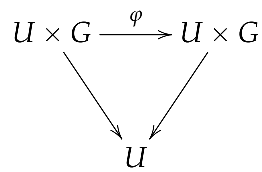


The pullback bundle $f^*E \to X$ is isomorphic to $P \to X$ as a principal G-bundle.

Proof. Recall from Appendix B.3 that the total space $f^*E = X$ is the pullback in Top of the diagram $X \to B \leftarrow E$. Since P also admits a map to such a diagram, there is a continuous map $P \to f^*E$ commutatively filling in



For any $x \in X$, by assumption, the maps of fibers $P|_x \to E|_{f(x)} \leftarrow (f^*E)|_x$ are G-equivariant homeomorphisms, so $P \to f^*E$ is a bijective G-map. To see its inverse is continuous, it is enough to restrict attention to an open $U \subseteq X$ trivializing both P and f^*E, so we need only show the inverse of a continuous G-bijection φ filling in the diagram



is continuous. By commutativity, we may write $\varphi(x,1) = \big(x, \psi(x)\big)$ for a continuous $\psi\colon U \to G$, so that $\varphi(x,g) = \big(x,\psi(x)g\big)$ by equivariance. Then $\varphi^{-1}(x,g) = \big(x,\psi(x)^{-1}g\big)$, and since ψ and $g \mapsto g^{-1}$ are continuous, so is φ^{-1}. □

Thus the $EG \to BG$ we seek needs to be a final object in the category of principal G-bundles. Recall that Top admits CW approximations, so that up to homotopy, we may assume the base space of our principal G-bundle $P \to X$ is a CW complex. Then X is built one level at a time from a discrete set X^0 of vertices by gluing disks D^{n+1}_α to the n-skeleton X^n along attaching maps $\varphi_\alpha\colon \partial D^{n+1}_\alpha \approx S^n \longrightarrow X^n$, so we can view P as being constructed inductively from principal G-bundles over these attached cells.

We require one intuitively plausible lemma, which we will not prove.

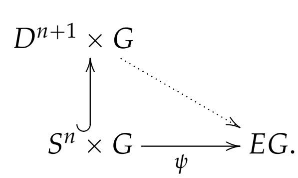
Lemma 5.1.2 ([Ste51, Cor. 11.6, p. 53]). *Let B be a contractible, paracompact Hausdorff space and $E \to B$ an F-bundle for some fiber F. Then E is isomorphic as an F-bundle to $B \times F$.*

By the lemma, principal G-bundles over disks are trivial, so $P|_{X^{n+1}}$ is the identification space of $P|_{X^n}$ with some bundles $D^{n+1}_\alpha \times G \to D^{n+1}_\alpha$, the identifications given by G-maps $S^n_\alpha \times G \longrightarrow P|_{X^n}$.

The task of constructing a G-map $P \longrightarrow EG$ can now be undertaken one cell at a time. To start, $P|_{X^0}$ is a disjoint union of copies of G, and any homeomorphic map of these to fibers of $EG \to BG$ will work. Suppose inductively that a G-map $P|_{X^n} \longrightarrow EG$ has been built, and we want to extend this to the space $P|_{X^n} \cup (D^{n+1} \times G)$, where $D^{n+1} \times G$ is attached by a G-map $S^n \times G \longrightarrow P|_{X^n}$. We can do anything we want over the *interior* of D^{n+1}, and we know what must happen over $P|_{X^n}$, so our only constraint is the composition of the preexisting G-map and the attaching map,

$$\psi\colon S^n \times G \longrightarrow P|_{X^n} \longrightarrow EG.$$

Thus the task is really to extend an arbitrary G-map $S^n \times G \longrightarrow EG$ over the interior of $D^{n+1} \times G$:



But a G-map $\widetilde{\psi}\colon D^{n+1} \times G \longrightarrow EG$ is uniquely determined by its restriction to the standard section $D^{n+1} \times \{1\}$ since $\widetilde{\psi}(x,g) = \widetilde{\psi}(x,1)g$, so it is necessary and sufficient to extend the restriction $S^n \longrightarrow EG$ to a map $D^{n+1} \longrightarrow EG$. If it is possible to do so, then restrictions of the latter map to concentric spheres of decreasing radius form a nullhomotopy of the map $S^n \longrightarrow EG$, so the condition finally turns out to be that $\pi_n(EG) = 0$.

Proposition 5.1.3. *A principal G-bundle $EG \longrightarrow BG$ is universal just if $\pi_*(EG) = 0$: for every principal G-bundle $G \to P \to B$, there is a G-bundle map*

$$\begin{array}{ccc} G & = & G \\ \downarrow & & \downarrow \\ P & \xrightarrow{\tilde{\chi}} & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{\chi} & BG \end{array}$$

realizing P as the pullback $\chi^* EG$.

Thus the collapse $EG \longrightarrow *$ of the total space is a weak homotopy equivalence, and so if EG is a CW complex, then it is actually contractible by Whitehead's theorem B.1.6.

[Show that BG is a classifying space.]

Now seems as good a time as any to derive a corollary we will use repeatedly later.

Corollary 5.1.4. *If G is a path-connected group, then BG is simply-connected.*

Proof. The long exact homotopy sequence Theorem B.1.4 of $G \to EG \to BG$ contains subsequences

$$0 = \pi_{n+1}(EG) \longrightarrow \pi_{n+1}(BG) \longrightarrow \pi_n(G) \longrightarrow \pi_n(EG) = 0,$$

yielding isomorphisms $\pi_{n+1}(BG) \cong \pi_n(G)$ for all n, and in particular for $n = 0$. □

We have not shown existence yet, but it is easy to show uniqueness in a strong sense, using a construction that will be useful again later. For G-spaces X, Y, there is a diagonal G-action on $X \times Y$,[1] which gives rise to the following ***mixing diagram***:

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X/G & \xleftarrow{\varpi_X} & \dfrac{X \times Y}{G} & \xrightarrow{\varpi_Y} & Y/G. \end{array} \tag{5.1.5}$$

Exercise 5.1.6. Show that if $X \longrightarrow X/G$ in (5.1.5) is a G-bundle (automatically principal), then ϖ_X is a bundle with fiber Y.

Proposition 5.1.7. *Given any two principal G-bundles $G \to E_j \to B_j$ with $\pi_* E_j = 0$ $(j \in \{1, 2\})$, there is a string of weak homotopy equivalences connecting B_1 with B_2.*

Proof (Borel [Bor53, Prop. 18.2]). Consider the mixing diagram (5.1.5) for $X = E_1$ and $Y = E_2$. The fiber of ϖ_{E_1} is E_2, which is weakly contractible, so from the long exact homotopy sequence of this bundle we conclude ϖ_{E_1} is a weak homotopy equivalence; and symmetrically for ϖ_{E_2}. □

Exercise 5.1.8. Show that the homotopy isomorphism $\beta_{12} := (\varpi_{E_1})_* (\varpi_{E_2}^{-1})_*$ is unique in the sense that if we have a third universal principal G-bundle $E_3 \to B_3$, then $\beta_{13} = \beta_{12} \circ \beta_{23}$. *Hint*: Consider the orbit-space of $E_1 \times E_2 \times E_3$.

[1] This is the product in the category of G-spaces.

Exercise 5.1.9. Prove a weak homotopy equivalence $B_1 \longrightarrow B_2$ directly using the universal property of $G \to E_2 \to B_2$.

Remark 5.1.10. The reader has probably seen the Eilenberg–MacLane space $K(\pi, 1)$ for π a non-topological group characterized up to homotopy as a CW complex with $\pi_* K(\pi, 1) = \pi_1 K(\pi, 1) = \pi$ the only nonzero homotopy group. This is the case of our BG with $G = \pi$ a discrete group. Proposition 5.1.7 and Exercise 5.1.9 show $K(\pi, 1)$ is unique up to homotopy.

5.2. An ad hoc construction of EG for G compact Lie

As we have seen in the previous section, the specification for EG is somewhat loose; it is really a G–homotopy type rather than any one single space. In this section we construct an avatar which will serve most of our needs.

Example 5.2.1. Embedding $\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$ as $\mathbb{C}^n \times \{0\}$, the direct union is the countable direct sum $\mathbb{C}^\infty = \bigoplus_{\mathbb{N}} \mathbb{C}$, which can be seen as the subspace of the countable direct product $\prod_{\mathbb{N}} \mathbb{C}$ such that all but finitely many coordinates are 0. Within $\mathbb{C}^\infty$ lies the ***unit ∞-sphere***

$$S^\infty := \{\vec{z} \in \mathbb{C}^\infty : \sum z_j^2 = 1\}.$$

Write $\mathbb{C}^\infty_\times := \mathbb{C}^\infty \backslash \{0\}$. The scalar multiplication of $\mathbb{C}$ on $\mathbb{C}^\infty$ restricts to a free action of $\mathbb{C}^\times$ on $\mathbb{C}^\infty_\times$ and of S^1 on S^∞, with the same orbit space

$$\mathbb{CP}^\infty := \mathbb{C}^\infty_\times / \mathbb{C}^\times \approx S^\infty / S^1,$$

called ***infinite complex projective space***. The fiber space $S^\infty \to \mathbb{CP}^\infty$ can be seen as the increasing union of restrictions $S^{2n-1} \to \mathbb{CP}^{n-1}$, where we conceive S^{2n-1} as $S^\infty \cap \mathbb{C}^n$. Each $\mathbb{CP}^n$ admits an open cover by contractible affines, so these restrictions are all principal S^1-bundles, and $S^\infty \to \mathbb{CP}^\infty$ is as well.

We claim this bundle satisfies the requirements to be $ES^1 \to BS^1$. Because S^∞ is the union of the unit spheres $S^{2n-1} \subsetneq \mathbb{C}^n$, by a compactness argument, any map $S^m \longrightarrow S^\infty$ must lie inside some sufficiently large S^n, and $\pi_m S^n = 0$ for $m < n$. Thus S^∞ is weakly contractible. There is a natural CW structure on S^∞ where two hemispheres D^n attach to each S^{n-1} to form S^n, so we know from Whitehead's theorem S^∞ is contractible, but in fact, it is possible to see so directly as well.

Proposition 5.2.2. *The unit ∞-sphere S^∞ is contractible.*

Proof. There is a homotopy taking the subspace $S' := S^\infty \cap \big(\{0\} \times \mathbb{C}^\infty\big) \approx S^\infty$ with first coordinate zero to the point $e_1 = (1, \vec{0})$, given by

$$h_t(\vec{z}) := (\sin t) e_1 + (\cos t) \vec{z};$$

this is just a renormalization of the straight-line homotopy. Now it will be enough to find a homotopy from S^∞ to S'. Write $s \colon \vec{z} \longmapsto (0, \vec{z})$ for the shift homeomorphism. One's first inclination is to take

$$f_t(\vec{z}) = (1-t)\vec{z} + t \cdot s(\vec{z}).$$

If we can show $f_t(S^\infty)$ avoids $\vec{0} \in \mathbb{C}^\infty$, then the renormalization $\hat{f}_t := f_t / |f_t|$ will suit our purposes. Now note any $\vec{z} \in \mathbb{C}^\infty$ has a last nonzero coordinate z_n, so the n^{th} and $(n+1)^{\text{st}}$ coordinates $\big((1-t)z_n, t z_n\big)$ of $f_t(\vec{z})$ will never simultaneously be zero, and the linear maps $f_t \in \operatorname{End}_{\mathbb{C}} \mathbb{C}^\infty$ are injective. Thus $\hat{f}_t$ is an isotopy. □

Example 5.2.3. Replacing $\mathbb{C}$ with the quaternions $\mathbb{H}$ (respectively, the reals $\mathbb{R}$) and S^1 with $\mathrm{Sp}(1) \approx S^3$ (resp., $\mathrm{O}(1) \approx S^0 \cong \mathbb{Z}/2$), one finds a universal $\mathrm{Sp}(1)$-bundle $E\mathrm{Sp}(1) \to B\mathrm{Sp}(1)$ is

$$S^3 \longrightarrow S^\infty \longrightarrow \mathbb{H}\mathrm{P}^\infty$$

and a universal $\mathrm{O}(1)$-bundle $E\mathrm{O}(1) \to B\mathrm{O}(1)$ is

$$S^0 \longrightarrow S^\infty \longrightarrow \mathbb{R}\mathrm{P}^\infty.$$

Any closed subgroup $K \leqslant G$ acts freely on EG by a restriction of the G-action, so one has a natural map $EG \longrightarrow EG/K$ with fiber K. It is intuitively plausible that this is also a fiber bundle, and this is actually the case in the event G is a Lie group: by Theorem B.4.4, $G \longrightarrow\!\!\!\!\!\rightarrow G/K$ is a principal K-bundle, and the local trivializations $\phi\colon (EG)|_U \xrightarrow{\approx} U \times G$ of $EG \to BG$ and $G|_V \xrightarrow{\approx} V \times K$ of $G \to G/K$ combine to yield local trivializations $\phi^{-1}(U \times G|_V) \longrightarrow U \times V \times K$ making $EG \to EG/K$ a principal K-bundle, so that EG can serve as EK and EG/K as BK.

To make use of this observation, we can use the classic result Theorem B.4.8, due to Peter and Weyl, that every compact Lie group has a faithful finite-dimensional unitary representation. Thus, if we can find $E\mathrm{U}(n)$, we will have bundles $EG \to BG$ for all compact Lie groups G. Here is one construction.

Example 5.2.4. The infinity-sphere S^∞ can be seen as the collection of orthonormal 1-frames in $\mathbb{C}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ as the space of 1-dimensional vector subspaces of $\mathbb{C}^\infty$. Analogously, one can form the infinite complex ***Stiefel manifolds*** $V_n(\mathbb{C}^\infty)$ of orthonormal n-frames in $\mathbb{C}^\infty$, which is to say, sequences $(v_1, \ldots, v_n)$ of n mutually orthogonal vectors of length one, topologized as a subspace of $\prod_n S^\infty$, and the infinite complex ***Grassmannian*** $G_n(\mathbb{C}^\infty)$ of n-planes in $\mathbb{C}^\infty$. Just as S^∞ projects onto $\mathbb{C}\mathrm{P}^\infty$, so does each $V_n(\mathbb{C}^\infty)$ project onto $G_n(\mathbb{C}^\infty)$ through the span map $(v_1, \ldots, v_n) \longmapsto \sum \mathbb{C}v_j$. The unitary group $\mathrm{U}(n)$ acts freely on $V_n(\mathbb{C}^\infty)$; if one considers an element of S^∞ as an infinite vertical vector, or a $\infty \times 1$ matrix, then an element of $V_n(\mathbb{C}^\infty)$ can be seen as an $\infty \times n$ matrix, and right multiplication by an $n \times n$ matrix in $\mathrm{U}(n)$ produces another $\infty \times n$ matrix spanning the same column space, so that the fiber of the span map $V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty)$ is homeomorphic to $\mathrm{U}(n)$. With a little work, it can be seen that $\mathrm{U}(n) \to V_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ is a fiber bundle.

Moreover, an analogue of the contraction of S^∞ in Example 5.2.1 shows $V_n(\mathbb{C}^\infty)$ to be contractible: the idea is to first conduct the isotopy $\hat{f}_t$ of S^∞ consecutively n times, taking S^∞ into $\{0\}^n \times S^\infty$ and hence $V_n(\mathbb{C}^\infty)$ into $V_n\big(\{0\}^n \times \mathbb{C}^\infty\big)$, and then use a renormalized straight-line homotopy generalizing h_t to take $V_n\big(\{0\}^n \times \mathbb{C}^\infty\big)$ to the identity matrix $I_n \in \mathbb{C}^{n\times n} \subsetneq \mathbb{C}^{\infty\times n}$, representing the standard basis of the subspace $\mathbb{C}^n < \mathbb{C}^\infty$. Write g_t for the resulting homotopy $V_n(\mathbb{C}^\infty) \times I \longrightarrow \mathbb{C}^{\infty\times n}$. In the same way that our first guess for S^∞ failed to have image strictly unit-length, this map g_t, while it preserves linear independence, does not preserve orthogonality. But if we postcompose to g_t the Gram–Schmidt orthonormalization procedure, which is a well-defined projection

$$\{n\text{-tuples of linearly independent vectors in } \mathbb{C}^\infty\} \longrightarrow V_n(\mathbb{C}^\infty),$$

we achieve the desired homotopy.

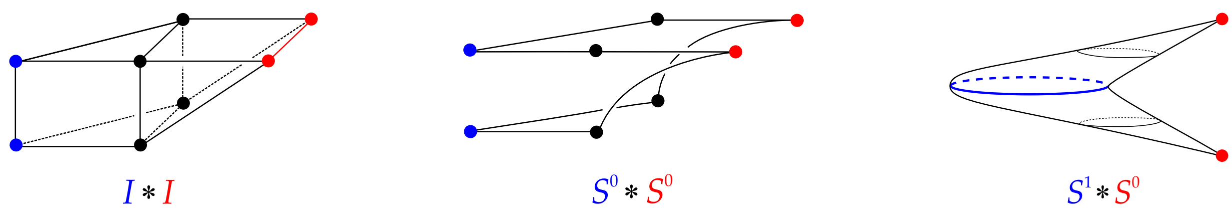
One analogously finds that $V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ and $V_n(\mathbb{H}^\infty) \to G_n(\mathbb{H}^\infty)$ respectively satisfy the hypotheses for $E\mathrm{O}(n) \to B\mathrm{O}(n)$ and $E\mathrm{Sp}(n) \to B\mathrm{Sp}(n)$. The double cover $V_n(\mathbb{R}^\infty)/\mathrm{SO}(n) =: \widetilde{G}_n(\mathbb{R}^\infty)$ of $G_n(\mathbb{R}^\infty)$, the ***oriented Grassmannian*** consisting of all *oriented* n-planes in $\mathbb{R}^\infty$, is a $B\mathrm{SO}(n)$.

5.3. Milnor's functorial construction of EG

These pleasing constructions do not generalize. In 1955, Milnor [Mil56] found a functorial construction of $EG \to BG$ that works for any topological group G, not even assumed Hausdorff.

To lay the groundwork, the ***join*** $X * Y$ of two topological spaces X and Y is the quotient of the product $X \times Y \times I$ with an interval by identifications $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and all $y, y' \in Y$. We may think of this as an $(X \times Y)$-bundle over I that has been collapsed to X over 0 and to Y over 1, and consider X and Y to be included as these particular end-subspaces.

Figure 5.3.1: Some low-dimensional joins



Examples 5.3.2. The join $I * I$ of two intervals is a 3-simplex Δ^3, the join $S^0 * S^0$ is a circle S^1, and the join $S^1 * S^0$ is a 2-sphere S^2.

It is not hard to see that generally $X * \mathrm{pt}$ is the cone CX on X and, as in the examples above, $X * S^0$ is the suspension SX of X, so the process of iteratively joining points generates the simplices Δ^n and that of iteratively joining copies of S^0 yields spheres S^n. One can also see $S^3 \approx S^1 * S^1$ geometrically. The unit sphere in $\mathbb{C}^2$ has a singular foliation by

$$T_r := \{(z \cos r, w \sin r) : z, w \in S^1\},$$

for $r \in [0, \pi/2]$, which are tori $S^1 \times S^1$ for $r \in (0, \pi/2)$ and circles for $r \in \{0, \pi/2\}$: the S^1 factor corresponding to the w-coordinate collapses at $r = 0$ and the S^1 corresponding to the z-coordinate collapses at $r = \pi/2$.

One important property of joins is that they are (bi)*functorial*: continuous maps $X \longrightarrow X'$ and $Y \longrightarrow Y'$ uniquely induce a map $X * Y \longrightarrow X' * Y'$ in a manner respecting composition of maps. Another key feature is that they are *more connected* than their factor spaces, as one already sees in the sphere examples above, in the following sense.

Definition 5.3.3. A nonempty space X is ***(-1)-connected***, and, for each $n \in \mathbb{N}$, is ***n-connected*** if $\pi_j(X) = 0$ for all $j \leqslant n$.

The relevant fact is that one can find an CW-replacement of an n-connected space such that after the basepoint, the next smallest cell is of dimension $n + 1$. (This is true but content-free if $n = -1$.)

Corollary 5.3.4. *If X is n-connected and Y is m-connected, then $X * Y$ is $(m + n + 2)$-connected.*

We decompose this into two lemmas.

Lemma 5.3.5. *Let X and Y be spaces homotopy equivalent to CW complexes. Then $X * Y$ is homotopy equivalent to the reduced suspension $\Sigma(X \wedge Y)$ of the smash product of X and Y.*

Proof. Fix basepoints $x_0 \in X$ and $y_0 \in Y$. The subspace $x_0 * Y \supseteq Y$ deformation retracts to x and the subspace $X * y_0 \supseteq X$ deformation retracts to y; their intersection $x_0 * y_0$ is also contractible, so their union A is as well. Thus the *reduced join* $(X * Y)/A$ is homotopy equivalent to $X * Y$. But A is comprised of elements $[x, y, t] \in X * Y$ with $x = x_0$ or $y = y_0$ or $t \in \{0, 1\}$ which are precisely the things one mods out of $X \times Y \times I$ to get $\Sigma(X \wedge Y)$. □

Recall that reduced suspension is equivalent to smashing with S^1. Then Corollary 5.3.4 follows by applying the following lemma twice to $X \wedge Y \wedge S^1$.

Lemma 5.3.6. *If X is m-connected and Y is n-connected, then $X \wedge Y$ is $(m + n + 1)$-connected.*

Proof. Replace X and Y with weakly homotopy equivalent CW complexes, such that the smallest cells besides the basepoints are of dimensions $m + 1$ and $n + 1$ respectively. Then $X \times Y$ decomposes into cells $\sigma \times \tau$ for σ and τ respectively cells in X and Y, and $X \vee Y$ is the union of cells $\{x_0\} \times \tau$ and cells $\sigma \times \{y_0\}$. The cells of the inherited CW structure on $X \wedge Y = X \times Y / X \vee Y$ are the 0-cell representing the collapsed $X \vee Y$ and the images of the other $\sigma \times \tau$, the minimum dimension of which is $(m + 1) + (n + 1)$. □

It follows that if X is n-connected, then the n-fold iterated join $\circledast^n X$ is $\big(n(m + 2) - 2\big)$-connected. Including $\circledast^n X$ as the second factor of $\circledast^{n+1} X = X * (\circledast^n X)$, we can form the direct limit

$$EX := \varinjlim \circledast^n X.$$

Because for all n we have $EX \approx (*^{n+1} X) * EX$, it follows that every $\pi_n(EX) = 0$. We will show in the next section that EX is actually contractible. Note that $E(-)$ is functorial: a continuous map $\psi \colon X \longrightarrow Y$ induces a continuous map $E\psi \colon EX \longrightarrow EY$.

Now let G be a topological group. To construct a G-action on EG, we first provide a different description of it. For any topological space X, write CX for the unreduced ***cone*** on X, the quotient of the product $X \times I$ obtained by pinching $X \times \{0\}$ to a point. Then $X * Y$ can be seen as the subspace of $CX \times CY$ consisting of elements $[x, t_1, y, t_2]$ such that $t_1 + t_2 = 1$ and X as the subspace where $t_2 = 0$. Similarly, the triple join $X * Y * Z$ can be seen as $\big\{[x, t_1, y, t_2, z, t_3] \in CX \times CY \times CZ : t_1 + t_2 + t_3 = 1\big\}$, and $X * Y$ as the subspace where $t_3 = 0$, and the infinite join EG can be seen as

$$\Big\{\big([g_j, t_j]\big)_{j \in \mathbb{N}} \in \prod_{\mathbb{N}} CG : \text{only finitely many } t_j \neq 0 \text{ and } \sum t_j = 1\Big\}. \tag{5.3.7}$$

Write these elements briefly as $e = \big[(g_j), \vec{t}\big]$. A free, continuous right action of G on EG is given by

$$\big[(g_j), \vec{t}\big] \cdot g := \big[(g_j g), \vec{t}\big].$$

Set $BG := EG/G$, with the quotient topology.

We still must show $p \colon EG \to BG \colon e \longmapsto eG$ is a fiber bundle. Much like projective space, EG admits an open cover by sets $U_j = t_j^{-1}(0, 1]$. On U_j, the g_j-coordinate is well-defined and continuous, so

$$\phi_j = (p, g_j) \colon U_j \longrightarrow p(U_j) \times G$$

is a continuous bijection. To see the inverse ϕ_j^{-1} is continuous, note that the continuous map $e \longmapsto e \cdot g_j^{-1}(e)$ determines the unique representative e' of eG such that $g_j^{-1}(e') = 1$, and since p is open, the restriction of p to this set of representatives is a homeomorphism p_j onto its image

$p(U_j)$, with inverse $eG \longmapsto eg_j^{-1}(e)$. Now we can write ϕ_j^{-1} as $(eG, g) \longmapsto p_j^{-1}(eG) \cdot g$, which is plainly continuous. Where defined, $\phi_i \circ \phi_j^{-1}$ is given by

$$(eG, g) \longmapsto \phi_i(p_j^{-1}(eG) \cdot g) = \Big(eG, g_i\big(p_j^{-1}(eG) \cdot g\big)\Big),$$

which is continuous, so the transition function on $U_i \cap U_j$, is also continuous; explicitly in terms of any representative e of eG, this transition function sends $g \longmapsto g_i(e)g_j(e)^{-1}g$. Thus $EG \to BG$ is a principal G-bundle.

The classifying space construction B is also functorial, because if $\psi \colon G \longrightarrow H$ is a continuous homomorphism, $E\psi$ is fiber-preserving and equivariant in a sense—

$$E\psi\big([\vec{g}_j, \vec{t}] \cdot g\big) = E\psi[\overrightarrow{g_j g}, \vec{t}] = \big[\overrightarrow{\psi(g_j g)}, \vec{t}\big] = \big[\overrightarrow{\psi(g_j)}, \vec{t}\big] \cdot \psi(g) = E\psi\big([\vec{g}_j, \vec{t}]\big) \cdot \psi(g)$$

—so that $E\psi$ descends to a well-defined continuous map $B\psi \colon BG \longrightarrow BH$.

Remark 5.3.8. The spaces EG can actually be seen to be contractible by an argument due to Dold.

Historical remarks 5.3.9. The notation for EG and BG descends from a proud historical precedent. The way to denote a bundle $F \to E \xrightarrow{\pi} B$ equipped with a local trivialization with transition functions taking values in $G \leqslant \mathrm{Homeo}(F)$, as late as the 1960s [Ste51, BH58, BH59, BH60], was a quintuple (E, B, F, p, G), with the last two entries often omitted. This arrowless notation requires one to remember which object lives in which position, but does have the benefit that if a bundle is named ξ, it has canonically associated with it an entourage of ready-named objects

$$(E_\xi, B_\xi, F_\xi, \pi_\xi, G_\xi) = \xi.$$

The standard name for the universal principal G-bundle under this convention is, naturally enough,

$$(E_G, B_G, G, \pi_G, G).$$

In subsequent decades, perhaps as the functorial nature of $E \colon G \mapsto EG$ and $B \colon G \mapsto BG$ is embraced, one can see the subscripts of E_G and B_G gradually move up until one has the $EG \longrightarrow BG$ of modern day.

5.4. Segal's functorial construction of EG

Although we only need one functorial construction of EG, there is another that is very attractive, uses ideas we have already seen, and whose generalizations had an important impact on later directions in algebraic topology.

The conditions $t_j \geqslant 0$ and $\sum t_j = 1$ in (5.3.7) describe, of course, a simplex, so writing $J = [j_0, \ldots, j_n]$ for a decreasing tuple of indices $j \in \mathbb{N}$ with $t_j \neq 0$, what we have done is represent each element of EG uniquely as a pair $(\vec{g}, \vec{t}) \in G^J \times \mathring{\Delta}^n$. To see how these pieces fit together, we consider elements $(\vec{g}, \vec{t}) \in G^J \times \Delta^n$. If $t_j = 0$, then the tuple is represented in EG by the same tuple with g_j omitted. If we write $\partial_\ell J = [j_0, \ldots, \widehat{j_\ell}, \ldots, j_n]$, and $\partial_\ell \colon G^J \longrightarrow G^{\partial_\ell J}$ for the coordinate projection omitting g_j, then this identification can be expressed as

$$(\partial_\ell \vec{g}, \vec{t}) \sim (\vec{g}, i_\ell \vec{t}) \qquad \text{for } \vec{g} \in G^J,\ \vec{t} \in \Delta^{n-1},$$

which is just the relation one has in defining the geometric realization (Definition 4.2.3). In fact, since projection ∂_ℓ are given by entry omission, it is clear the G^J fit into a semisimplicial set, namely $\mathscr{N}G$, where $(\mathscr{N}G)_n := \coprod_{|J|=n+1} G^J$ and ∂_ℓ is projection as above, and then it becomes clear that

$$EG = \|\mathscr{N}G\|.$$

This change of viewpoint actually makes it easier to see that EG is contractible. Let $\mathfrak{e}$ be the semisimplicial subset $\mathscr{N}G$ consisting of elements all of whose entries are $1 \in G$. There is a unique map ε of semisimplicial sets $\mathscr{N}G \longrightarrow \mathfrak{e}$ defined on the 0^{th} level by sending each element of $G^{[n]}$ to $1 \in G^{[n+1]}$. Let $\Delta[1]$ be the semisimplicial set with two 0-simplices (0) and (1) and n-simplices nonincreasing length-$(n+1)$ sequences of 0's and 1's.[2] The maps $\mathrm{id}_{\mathscr{N}G}$ and ε prescribe a map of simplicial sets $\mathscr{N}G \times \Delta[1] \longrightarrow \mathscr{N}G$ determined on the 0-level by

$$\begin{aligned} (g_\ell) \times (0) &\longmapsto (g_\ell), \\ (g_\ell) \times (1) &\longmapsto (1_{j+1}). \end{aligned}$$

Compatibility with the face maps means this prescription actually specifies the map completely; for example, for $q > p > m > \ell > j$,

$$\begin{aligned} (h_j, h_j) \times (1,0) &\longmapsto (1_{j+1}, h_j), \\ (g_\ell, h_j) \times (0,0) &\longmapsto (g_\ell, h_j), \\ (g_\ell, h_j) \times (1,1) &\longmapsto (1_{\ell+1}, 1_{j+1}), \\ (a_q, b_p, c_m, g_\ell, h_j) \times (1,1,1,0,0) &\longmapsto (1_{q+1}, 1_{p+1}, 1_{m+1}, g_\ell, h_j). \end{aligned}$$

Taking geometric realizations yields a map

$$\|\mathscr{N}G \times \Delta[1]\| \longrightarrow EG$$

which is the identity on the subcomplex $EG \times \{(0)\}$ and sends the subcomplex $EG \times \{(1)\}$ to $\|\mathfrak{e}\|$. But $\|\mathfrak{e}\| = \varinjlim \ast^n\{1\} = \Delta^\infty$ is an infinite-dimensional simplex, hence contractible.

Exercise 5.4.1. Write an explicit nullhomotopy of Δ^∞.

Theorem 5.4.2 (Dold)**.** *The Milnor model of EG is contractible.*

[SIMPLICIAL HOMOTOPY INDUCES HOMOTOPY IS [?, COR., P. 360]]

The semisimplicial set $\mathscr{N}G$ realizing to EG descends to a semisimplicial set $(\mathscr{N}G)/G$ with realization BG, whose levels are unions of G^J/G. An element $[\vec{g}]$ of G^J/G is represented equally well by $\vec{g}$ and $(g_j h)$ for any other $h \in G$, and it would be nice to have unique representatives. One observation to make is that the ratios $g_j g_\ell^{-1}$ are invariant under the substitution $\vec{g} \longmapsto \vec{g} \cdot h$, so an element of G^J is uniquely determined by its list of ratios $(g_{j_0} g_{j_1}^{-1}, \ldots, g_{j_{n-1}} g_{j_n}) \in G^n$. Let us see explicitly what the face operators do downstairs, for $J = [3,2,1,0]$:

$$(a,b,c) \longleftarrow (abc, bc, c, 1) \begin{cases} \overset{\partial_0}{\longmapsto} (bc, c, 1) \longmapsto (b,c), \\ \overset{\partial_1}{\longmapsto} (abc, c, 1) \longmapsto (ab, c), \\ \overset{\partial_2}{\longmapsto} (abc, bc, 1) \longmapsto (a, bc), \\ \overset{\partial_3}{\longmapsto} (abc, bc, c) \longmapsto (a,b). \end{cases} \tag{5.4.3}$$

[2] The idea is that $\alpha = (1,0)$ represents the nontrivial edge and every other simplex is degenerate, with image one of the endpoints or this edge. The geometric realization, as we have defined it, will only be homotopy-equivalent to I, but this is all right.

Thus ∂_0 and ∂_n respectively omit the first and last entry, as before, but the other ∂_j multiply two consecutive entries.

This generalizes substantially. We may consider a monoid G as a category in at least two different ways. One way is to construct the category $\widetilde{\mathscr{C}}_G$ whose objects are the elements of G and whose morphisms are given by the multiplication table: there is a unique morphism $\ell_g : h \longrightarrow gh$ for every $g, h \in H$. If G is a group, then for every pair of objects $h, x \in G$, there is a unique morphism $\ell_{xh^{-1}} : h \longrightarrow x$. In other words, the space of morphisms is $G \times G$. This category is clearly equivalent to the category $*$ with one object and one morphism, for the unique functor $\widetilde{\mathscr{C}}_G \longrightarrow *$ is surjective on objects and bijective on each hom-set. Another to consider G as a category is to construct the category $\mathscr{C}_G$ with one object $*$ such that the morphism set $\mathscr{C}_G(*,*)$ endowed with composition is just G. There is a natural functor $\pi_G : \widetilde{\mathscr{C}}_G \longrightarrow \mathscr{C}_G$ between these categories, taking every object to $*$ and each morphism ℓ_g to $g \in \mathscr{C}_G(*,*)$.

Associated to every category, and these in particular, is a semisimplicial set, as per Definition 4.2.1, whose levels are its strings of composable arrows.

Definition 5.4.4. Given a topological category $\mathscr{C}$, we write $\mathscr{C}_0$ for its class of objects and $\mathscr{C}_1$ for its class of morphisms.[3] The ***nerve*** $N\mathscr{C}$ of $\mathscr{C}$ is the simplicial space $N\mathscr{C}$ with levels

$$(N\mathscr{C})_0 = \mathscr{C}_0, \qquad (N\mathscr{C})_n = \big\{(f_{n-1}, \ldots, f_0) \in \mathscr{C}_1^n : \mathrm{source}(f_{j+1}) = \mathrm{target}(f_j)\big\}.$$

If we write down $\mathscr{C}$ as a graph, then elements of $(N\mathscr{C}_n)$ correspond to paths $\cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n-1}} \cdots \xleftarrow{f_2} \cdot \xleftarrow{f_1} \cdot$. In other words, $\vec{f}$ is an element of $\mathscr{C}_n$ if the composition $f_n f_{n-1} \cdots f_2 f_1$ is defined. The face maps are

$$\begin{aligned} \partial_0 \vec{f} &:= (f_n, \cdots, f_2) \\ \partial_j \vec{f} &:= (f_n, \cdots, f_{j+1} f_j, \cdots, f_1), \qquad 0 < j < n, \\ \partial_n \vec{f} &:= (f_{n-1}, \cdots, f_1). \end{aligned} \tag{5.4.5}$$

and the degeneracies are

$$s_j \vec{f} := (f_n, \cdots, f_{j+1}, \mathrm{id}_{\mathrm{target}(f_j)}, f_j, \cdots, f_1). \tag{5.4.6}$$

We write $B\mathscr{C} := |N\mathscr{C}|$ for the geometric realization of the nerve, and call it the ***classifying space*** of $\mathscr{C}$.

To make it clearer that the nerve is indeed a semisimplicial set, recall that we initially came by the relations $\partial_j \partial_i = \partial_i \partial_{j+1}$ for $i < j$ by analyzing what happened when we removed entries from an $(n+1)$-tuple. To put the nerve back in that framework, note that face map also removes one of $n+1$ things from $\vec{f}$, namely f_n, f_1, or one of the $n-1$ commas separating entries. Thus the only cases to be checked are $(i,j) = (n-1, n-1)$ and $(i,j) = (0,0)$.

Exercise 5.4.7. Check these cases.[4]

[3] In the cases we consider, these will just be sets.

[4] *In practice, one usually specifies a simplicial set X by describing the sets X_n of n-simplices and then defining the required face and degeneracy maps. Mercifully, the required relations are often obvious, and even if they are not, it is still advisable to assert that they are, after privately verifying that they do in fact hold.*

—Emily Riehl [Rie11].

A continuous functor between topological categories induces a continuous simplicial map of simplicial spaces and so a continuous map between classifying spaces. We have already seen an example of this.

Example 5.4.8. Our semisimplicial set $\mathscr{N}G$ realizing EG is the nerve of the topological category $\widetilde{\mathscr{C}}_{\mathbb{N}}G$ whose space of objects is $\mathbb{N} \times G$, and whose nonidentity morphisms are unique arrows $(\ell, g) \longleftarrow (j, h)$ for $\ell > j$ and any $g, h \in G$. If we agree to write these morphisms as (g_ℓ, h_j), we can write a pair of composable arrows $(m, c) \leftarrow (\ell, g) \leftarrow (j, h)$ as $(c_m, g_\ell, h_j) \in G^{[j,\ell,m]} \subsetneq (\mathscr{N}G)_2$ and so on. With this notational convention, omitting the first or last coordinate corresponds to dropping the first or last arrow and projecting out a middle coordinate corresponds to composition since nonempty hom-sets contain only one element.

Example 5.4.9. Our semisimplicial set $\mathscr{N}G/G$ realizing EG is the nerve of the topological category $\mathscr{C}_{\mathbb{N}}G$ whose space of objects is $\mathbb{N}$ with hom-sets $\mathrm{Hom}_{\mathscr{C}_{\mathbb{N}}G}(j, n) \cong G$ for $j < n$, the identity for $j = n$, and empty if $j > n$. If we think of these arrows as left multiplication by g, then the face operators in (5.4.3) exactly meet the specification set by (5.4.5).

We will show that $B\widetilde{\mathscr{C}}_G \longrightarrow B\mathscr{C}_G$ is a model for $EG \longrightarrow BG$ and in the process provide another proof that the Milnor model of EG is contractible.

Proposition 5.4.10. *The functor B preserves products.*

Proof. An object in a product $\mathscr{C} \times \mathscr{D}$ of categories is a pair (c, d) of objects of each and a morphism is a pair (f, g) of arrows. It follows $N(\mathscr{C} \times \mathscr{D}) = N\mathscr{C} \times N\mathscr{D}$ as a set and we set $\partial_j = (\partial_j, \partial_j)$ in $N(\mathscr{C} \times \mathscr{D})$; this is the product simplicial set. By [CREATE IN SIMPLICIAL SET SECTION AND CITE], then, we have

$$B(\mathscr{C} \times \mathscr{D}) = |N\mathscr{C} \times N\mathscr{D})| = |N\mathscr{C}| \times |N\mathscr{D}| = B\mathscr{C} \times B\mathscr{D}. \qquad \square$$

Proposition 5.4.11. *Let $F_0, F_1 \colon \mathscr{C} \longrightarrow \mathscr{D}$ be continuous functors between topological categories. A natural transformation $F_0 \longrightarrow F_1$ induces a homotopy $B\mathscr{C} \times I \longrightarrow B\mathscr{D}$ from BF_0 to BF_1.*

Proof. Let $\mathscr{C}_{\Delta^1}$ be the category with two objects $0, 1$ linked by one nonidentity arrow $0 \to 1$. Then the data of a natural transformation $\vartheta \colon F_0 \longrightarrow F_1$ is exactly that of a functor $H \colon \mathscr{C} \times \mathscr{C}_{\Delta^1} \longrightarrow \mathscr{D}$. Explicitly

$$\begin{aligned} H(X, j) &= F_j X, \\ H(f, \mathrm{id}_j) &= F_j f \quad \text{for } j \in \{0, 1\}, \\ H(\mathrm{id}_X, 0 \to 1) &= (\vartheta_X \colon F_0 X \to F_1 X). \end{aligned}$$

Taking classifying spaces, since $B\mathscr{C} \times I = B\mathscr{C} \times B\mathscr{C}_{\Delta^1} \approx B(\mathscr{C} \times \mathscr{C}_{\Delta^1})$ by Proposition 5.4.10, we see H induces a map $B\mathscr{C} \times I \longrightarrow B\mathscr{D}$ as claimed. $\square$

Proposition 5.4.12. *A adjunction between topological categories induces a homotopy equivalence of classifying spaces.*

Particularly, an equivalence $\mathscr{C} \equiv \mathscr{D}$ induces a homotopy equivalence $B\mathscr{C} \simeq B\mathscr{D}$.

Proof. An adjunction of topological categories is a pair of continuous functors $F \colon \mathscr{C} \longrightarrow \mathscr{D}$ and $G \colon \mathscr{D} \longrightarrow \mathscr{C}$ such that there are natural transformations $\eta \colon \mathrm{id}_{\mathscr{C}} \longrightarrow GF$ and $\varepsilon \colon FG \longrightarrow \mathrm{id}_{\mathscr{D}}$ satisfying universal properties that we don't actually need here. By Proposition 5.4.11, these induce homotopies from $\mathrm{id}_{B\mathscr{C}}$ to $BG \circ BF$ and from $BF \circ BG$ to $\mathrm{id}_{B\mathscr{D}}$, as was to be shown. $\square$

Theorem 5.4.13 (Segal)**.** *$B\widetilde{\mathscr{C}}_G$ is contractible.*

Proof. We have already seen that $\widetilde{\mathscr{C}}_G \longrightarrow *$ is an equivalence.[5] Thus, by Proposition 5.4.12, $B\widetilde{\mathscr{C}}_G \simeq B* \approx *$. ☐

It is not always the case that $B\widetilde{\mathscr{C}}_G \longrightarrow B\mathscr{C}_G$ is a bundle map, although it is if G is a Lie group, as we will always assume. But this model is still relevant.

Proposition 5.4.14. *There exists a homotopy equivalence $BG \longrightarrow B\mathscr{C}_G$.*

Proof. We define a continuous functor $\widetilde{\mathscr{C}}_{\mathbb{N}}G \longrightarrow \widetilde{\mathscr{C}}_G$ taking $\big((j,g) \to (\ell,h)\big) \longmapsto (g \mapsto h)$. This is not an equivalence[6] but is a continuous G-equivariant functor, so it induces a G-map $EG \longrightarrow B\widetilde{\mathscr{C}}_G$. (It is a homotopy equivalence simply because both spaces are contractible, but this does not imply it is a G–homotopy equivalence.)

[This is a hole which still remains to be filled.] ☐

Historical remarks 5.4.15. [To be written...]: [Commentary on the geometric cobar construction, Steenrod–Rothenberg, and group cohomology.]

5.5. The Borel construction

We have now constructed, for every topological group G, a universal principal G-bundle $G \to EG \to BG$ such that EG is weakly contractible. Given a left G-space X, we can construct the mixing diagram (5.1.5) of EG and X. The product space $EG \times X$, equipped with the diagonal action, is another G-space weakly homotopy equivalent to X, but the new action is free since

$$(e,x) = g\cdot(e,x) = (eg^{-1}, gx) \implies e = eg^{-1}$$

and the G-action on EG is free. The middle entry on the bottom of the diagram, the orbit space of this new, free action, serves as a sort of "homotopically correct" substitute for X/G when the action of G is not free, and a useful auxiliary even when it is.

Definition 5.5.1 (Borel [BBF+60, Def. IV.3.1, p. 52])**.** The orbit space

$$X_G := EG \underset{G}{\otimes} X \;=\; EG \times X \big/ (eg,x) \sim (e,gx),$$

of the diagonal action of G on $EG \times X$ is the ***homotopy quotient*** of X by G (or the *Borel construction*). We denote the elements of X_G by $e \otimes x$, since $eg \otimes x = e \otimes gx$.

The homotopy quotient is functorial, in that a continuous G-map $X \longrightarrow Y$ induces a continuous map $X_G \longrightarrow Y_G$ in a manner respecting composition. Every G-space X admits a G-map $X \longrightarrow *$ to a single point (equipped with the unique possible G-action), inducing, since $*_G = EG \underset{G}{\otimes} * \approx EG/G = BG$, a canonical map

$$\begin{aligned} X_G &\longrightarrow *_G \approx BG \\ e\otimes x &\longmapsto e \otimes * \leftrightarrow eG. \end{aligned}$$

The fiber of this map over eG is the set $\{e \otimes x : x \in X\} \approx X$.

[5] As an explicit pseudoinverse, one may take the map $* \longmapsto 1$; any point $g \in G$ will do.

[6] It is surjective on objects and faithful, but not full because if $g \neq h$, then $\operatorname{Hom}_{\widetilde{\mathscr{C}}_{\mathbb{N}}}\big((n,g),(n,h)\big)$ is empty, while $\operatorname{Hom}_{\widetilde{\mathscr{C}}_G}(g,h)$ is not.

Definition 5.5.2. The bundle $X \to X_G \to BG$ is the ***Borel fibration*** of the action of G on X.

Proposition 5.5.3. *Let G act freely on a CW complex X. Then the projection $X_G \longrightarrow X/G$ is a weak homotopy equivalence.*

Proof. The map $e \otimes x \longmapsto Gx$ from $X_G \to X/G$ has fiber $EG/\mathrm{Stab}(x)$ in general. If G acts freely on X, then all fibers are EG. Since EG is contractible, the long exact homotopy sequence of the bundle $EG \to X_G \to X/G$ shows the map is a weak homotopy equivalence. By Whitehead's theorem, it is a homotopy equivalence. □

To use this map as an auxiliary, we will want to be able to replace the map $X \longrightarrow X/G$ with $X \longrightarrow X_G$ and $X/G \longrightarrow BG$ with $X_G \longrightarrow BG$ as needed when the action on X is free. The first is natural: one has a triangle

$$\begin{array}{ccc} & & X_G \\ & \nearrow & \downarrow \\ X & \searrow & X/G, \end{array} \qquad\qquad \begin{array}{ccc} & & e \otimes x \\ & \nearrow & \downarrow \\ x & \searrow & Kx, \end{array}$$

which commutes on the nose. The map $\chi\colon X/G \longrightarrow BG$ in question, on the other hand, is the classifying map of $G \to X \to X/G$, which exists from the abstract considerations of Proposition 5.1.3, but which we do not typically have any concrete description of. It is not *a priori* clear it should have anything to do with the projection $X_G \longrightarrow BG$ of the Borel fibration. To see it does, quotient the G-map

$$\mathrm{id} \times \tilde{\chi}\colon EG \times X \longrightarrow EG \times EG$$

by the diagonal G-action. The projections to the either factor on both sides in the resulting homotopy quotient yield the following diagram (this is a map of bottom rows of mixing diagrams (5.1.5)).

$$\begin{array}{ccccc} BG & \xleftarrow{/X} & EG \underset{G}{\otimes} X & \xrightarrow{\sim} & X/G \\ \| & & \downarrow{\scriptstyle \mathrm{id} \otimes \tilde{\chi}} & & \downarrow{\scriptstyle \chi} \\ BG & \xleftarrow{\sim} & EG \underset{G}{\otimes} EG & \xrightarrow{\sim} & BG \end{array} \qquad\qquad \begin{array}{ccccc} eK & \longleftarrow & e \otimes x & \longmapsto & Kx \\ \| & & \downarrow & & \downarrow \\ eK & \longleftarrow & e \otimes \tilde{\chi}(x) & \longmapsto & \chi(Kx) \end{array}$$

Here the map $X_G \longrightarrow X/G$ and the maps along the bottom are weak homotopy equivalences because they are fibrations with fiber EG; this was the proof in Proposition 5.1.7 of the uniqueness of BG. It follows that we can indeed replace $\chi\colon X/G \longrightarrow BG$ with the projection $X_G \longrightarrow *_G = BG$ up to homotopy.

Proposition 5.5.4. *If $G \to X \to X/G$ is a principal bundle, then the weak homotopy equivalence $X_G \longrightarrow X/G$ identifies $X \longrightarrow X_G$ with $X \longrightarrow X/G$ and the classifying map $X/G \longrightarrow BG$ with the Borel fibration $X_G \longrightarrow BG$ up to homotopy.*

Remark 5.5.5. The singular cohomology $H^*(X_G;)$ of the homotopy quotient X_G is the (*Borel equivariant cohomology* $H^*_G(X;)$ of the action of G on X [BBF+60, IV.3.3, p. 53], a classical tool in the study of group actions and one of the topics of the thesis this book derives from. The equivariant cohomology of a point is $H^*_G(*) = H^*(BG)$. As this is the coefficient ring of Borel cohomology, we will abbreviate this ring by H^*_G later on.

Chapter 6

The cohomology of complete flag manifolds

The algebraic relation between a compact group and its maximal torus informs all discussion of invariant subalgebras going forward, and is epistemologically prior to much of our discussion of the cohomology of homogeneous spaces, being treated with *sui generis* methods that do not apply in the general case.

The quotient G/T of a compact, connected Lie group by its maximal torus T, called a ***complete flag manifold***, was among the first homogeneous spaces other than groups and symmetric spaces whose cohomology was understood. This material will be cited in Section 8.3.2.It is fundamental, and but for the discussion of the Serre spectral sequence in Theorem 2.2.2, could have gone earlier.

6.1. The cohomology of a flag manifold

The cornerstone result is the following.

Theorem 6.1.1. *Let G be a compact, connected Lie group and T a maximal torus in G. Then the cohomology of $H^*(G/T)$ is concentrated in even dimensions.*

[CITE BOTT–SAMELSON]

Proof sketch 1. Associated to G is a *complexified* Lie group $G^{\mathbb{C}}$ which is a complex manifold, and which contains a *Borel subgroup* B, a complex Lie group containing T and such that

$$G^{\mathbb{C}}/B \approx G/T.$$

Thus G/T admits a complex manifold structure and hence a CW structure with even-dimensional cells. This actually shows $H^*(G/T;\mathbb{Z})$ is free Abelian. □

We reproduce Borel's original 1950 proof. This argument was first published somewhat telegraphically in Leray's contribution [Ler51] to the 1950 Bruxelles *Colloque*, and is elaborated in Borel's thesis [Bor53]. It invokes two facts we shall not prove about invariant differential forms, which are these.

Proposition 6.1.1. *Suppose a compact, connected Lie group G acts on a manifold M. Then every cohomology class in $H^*(M;\mathbb{R})$ is represented by a G-invariant differential form ω. Such a form is determined uniquely by its value $\omega_x \in \Lambda T^*_x M$, an alternating multlinear form on the tangent space of one point x of M.*

Sketch of proof. Given a closed form ω, note that since G is path-connected, for any $g \in G$ the left translation ℓ_g^* on $\Omega^\bullet(M)$ induces an isomorphism on cohomology, so $\omega - \ell_g^*\omega$ is an exact form $d\tau_g$. Using an invariant probability measure μ on G, average $\omega - \ell_g^*\omega = d\tau_g$ and get $\omega - \int_G \ell_g^*\omega\, d\mu = d\int_G \tau_g\, d\mu$, showing ω is cohomologous to an invariant form.[1] Thus inclusion of invariant forms induces a surjection in de Rham cohomology. It is an injection because the composition $\Omega^\bullet(M)^G \hookrightarrow \Omega^\bullet(M) \xrightarrow{\int_G - d\mu} \Omega^\bullet(M)^G$ is the identity. □

Proposition 6.1.2. *Let G be a compact, connect Lie group and K a closed subgroup. The alternating multlinear form $\omega_{1K} \in \Lambda(\mathfrak{g}/\mathfrak{k})^\vee$ representing a G-invariant form $\omega \in \Omega^\bullet(G/K)$ is invariant under the action $\mathrm{Ad}^*|_K$ of K induced by the conjugation action on K on G.*

Proof. The adjoint action of G on $\mathfrak{g}$ is the derivative at $1 \in G$ of the conjugation action $x \longmapsto gxg^{-1}$. The action of K on G/K induced by conjugation is identical to the left action $k.gK = (kg)K$, since the right k^{-1} is absorbed by K, so $(\ell_k)^* = \mathrm{Ad}^*(k)$ on $\Omega^\bullet(G/K)$. Now

$$\mathrm{Ad}^*(k)\omega|_{1K} = \big(\mathrm{Ad}^*(k)\omega\big)_{1K} = (\ell_k^*\omega)_{1K} = \omega_{1K}. \qquad \square$$

Borel's proof of Theorem 6.1.1. By Theorem B.1.1, we may use $\mathbb{R}$ coefficients. Write $\ell = \operatorname{rk} G$ and $n = \dim G - \operatorname{rk} G$. We prove the result by a double induction on ℓ and n. If $\ell = 0$, then G is discrete, and we are done. Inductively suppose we have proven the result for all groups of rank $\ell - 1$. If $n = 0$, then $\operatorname{rk} G = \dim G$, so $G = T$ is a torus and we are done.

Now suppose inductively we have proven the result for ℓ and $n - 1$. Note that without loss of generality, by Theorem B.4.5, G can be taken to be of the form $A \times K$ with A a torus and K simply-connected. Since A is a factor of the maximal torus T of G, one has $G/T = K/(T \cap K)$, and $\operatorname{rk} K = \operatorname{rk} G - \operatorname{rk} A < \ell$ if $\operatorname{rk} A \neq 0$.

Otherwise $G = K$ is simply-connected. We claim there exists an element $x \in G$ such that $x \notin Z(G)$ and $x^2 \in Z(G)$. Indeed, $1 \in Z(G)$ lies in every maximal torus T. There is $y_1 \in T$ with $y_1^2 = 1$, and since a torus is divisible for all $m \geqslant 0$ there are y_m with $y_m^2 = y_{m-1}$. If these simultaneously lay in all tori, then $Z(G)$ would fail to be discrete, so there is some first m such that $y_m \notin Z(G)$ and we may take $x = y_{m-1}$. Let K be the identity component of the centralizer $Z_G(x)$ of x. Because x lies in the maximal torus T of G, we know $\operatorname{rk} K = \operatorname{rk} G$, and because $x \notin Z(G)$, the dimension $\dim Z_G(x) = \dim K$ is strictly less than $\dim G$. Thus $H^*(K/T)$ is evenly graded by the inductive assumption.

The tangent space $\mathfrak{g}/\mathfrak{k} = T_{1K}(G/K)$ to the identity coset $1K$ in G/K can be identified with an orthogonal complement $\mathfrak{k}^\perp$ to $\mathfrak{k}$ in $\mathfrak{g}$ in such a way that the isotropy action of K on $T_{1K}(G/K)$ corresponds to the adjoint action of K on $\mathfrak{k}^\perp$.

By Proposition 6.1.1, each de Rham cohomology class on G/K contains a left G-invariant element, which is then determined by its restriction to $T_{1K}(G/K) \cong \mathfrak{k}^\perp$. Such a restriction is, by Proposition 6.1.2, an alternating $\mathrm{Ad}^*(K)$-invariant multilinear form on $\mathfrak{k}^\perp$. Because x^2 is central, $\mathrm{Ad}(x) \in \mathrm{GL}(\mathfrak{g})$ is an involution; thus $\mathfrak{g}$ splits as the 1-eigenspace $\mathfrak{k}$ and an orthogonal (-1)-eigenspace, which must be $\mathfrak{k}^\perp$. Since $\mathrm{Ad}(x)$ acts as multiplication by -1 on $\mathfrak{k}^\perp$, a nonzero $\mathrm{Ad}^*(x)$-invariant alternating form on $\mathfrak{k}^\perp$ can only have even degree. As $x \in K$, it follows we must have $H^*(G/K)$ concentrated in even degree.

Now we can apply the Serre spectral sequence to $K/T \to G/T \to G/K$. Both $H^*(K/T)$ and $H^*(G/K)$ are evenly-graded, so by Theorem 2.2.2, so also is G/T. In fact, by Corollary 2.2.9, the

[1] Particularly, this is sketchy because we have not shown how to choose τ_g such that $g \longmapsto \tau_g$ is measurable.

spectral sequence collapses at E_2 and $H^*(G/T) \cong H^*(G/K) \otimes H^*(K/T)$ as an $H^*(K/T)$-module. □

Corollary 6.1.3. *Let G be a compact, connected Lie group and T a maximal torus in G. Then the Euler characteristic of $\chi(G/T)$ is positive.*

6.2. The acyclicity of $G/N_G(T)$

In this section we prove another result whose importance will not immediately be clear, but which recurs in Section 6.3.

Proposition 6.2.1. *Let G be a compact, connected Lie group, T a maximal torus in G, and $N = N_G(T)$ the normalizer. Then* $\dim G/N$ *is even and G/N is $\mathbb{Q}$-acyclic:*

$$H^*(G/N;\mathbb{Q}) = H^0(G/N;\mathbb{Q}) \cong \mathbb{Q}.$$

Proof [MT00, Thm. 3.14, p. 159]. The torus T acts on G/N on the left, fixing the identity coset $1N$ (since $T \leqslant N$); we claim this is the only such fixed point. Indeed, let $t \in T$ be a topological generator. If an element $gN \in G/N$ is fixed under multiplication by t, it is fixed under multiplication by all powers of t, and thus, by continuity, by all of T, so that $TgN = gN$, or $g^{-1}Tg \leqslant N$. Since T is a connected component of N and $1 = g^{-1}1g \in T$, it then follows $g^{-1}Tg = T$, or $g \in N$.

Because T fixes $1N$, there is an induced isotropy action of T by isometries on the tangent space $\mathfrak{g}/\mathfrak{n} = T_{1N}(G/N)$ to G/N at the identity coset $1N$, which can be identified with the orthogonal complement $\mathfrak{n}^\perp < \mathfrak{g}$. Because T acts by isometries on the vector space $\mathfrak{n}^\perp \cong \mathbb{R}^m$, it leaves invariant ε-spheres S^{m-1} about the origin. The exponential $\exp\colon \mathfrak{n}^\perp \longrightarrow G/N$ will map a sufficiently small sphere isometrically and T-equivariantly into G/N, and this T-invariant image sphere S^{m-1} divides G/N into a T-invariant disk D^m and a T-invariant complement M. Since T is path connected, the map ℓ_t is homotopic to the identity, so $\chi(\ell_t) = \chi(\mathrm{id})$ on both S^{m-1} and M. As only $1N \in G/N$ is fixed by multiplication by T, and this point lies in the interior of D^m, it follows ℓ_t acts without fixed points on S^{m-1} and M. By the Lefschetz fixed point theorem B.1.10, then,

$$\chi(M) = \chi(S^{m-1}) = 0.$$

It follows m is even. Note that by excision $H^*(G/N, M) \cong H^*(D^m, S^{m-1}) \cong \widetilde{H}^*(S^m)$, so that the relative Euler characteristic $\chi(G/N, M)$ is $(-1)^m = 1$. The long exact sequence of the pair $(G/N, M)$ then gives

$$\chi(G/N) = \chi(M) + \chi(G/N, M) = 0 + 1 = 1.$$

As $G/T \to G/N$ is a finite cover with fiber $W = N/T$ and $H^{\mathrm{odd}}(G/T) = 0$ by Theorem 6.1.1, it follows from Proposition B.2.1 that

$$H^{\mathrm{odd}}(G/N) \cong H^{\mathrm{odd}}(G/T)^W = 0.$$

Thus $h^\bullet(G/N) = \chi(G/N) = 1$, so it must be that $H^*(G/N) = H^0(G/N) \cong \mathbb{Q}$. □

We have the following useful corollary.

Corollary 6.2.2 (Weil [DIG UP CITATION]). *Let G be a compact, connected Lie group, T a maximal torus in G, and W the Weyl group of G. Then*

$$\chi(G/T) = |W|.$$

Proof. Since $G/T \longrightarrow G/N$ is a $|W|$-sheeted covering and $\chi(G/N) = 1$ by Proposition 6.2.1, it follows from Proposition B.2.5 that

$$\chi(G/T) = \chi(G/N) \cdot |W| = |W|. \qquad \square$$

This means in a homogeneous space G/K, one can for cohomological purposes replace K with the normalizer of its maximal torus.

Corollary 6.2.3. *Let G be a compact, connected Lie group, K a closed, connected subgroup of lesser rank, S a maximal torus of K, and $N = N_K(S)$ the normalizer of this torus in K. Then the natural projection $G/N \longrightarrow G/K$ induces a ring isomorphism*

$$H^*(G/K) \xrightarrow{\sim} H^*(G/N).$$

Proof. There is a fiber bundle $K/N \to G/N \to G/K$, whose fiber K/N is acyclic by Proposition 6.2.1, so $\pi_1(G/K)$ acts trivially on $H^*(K/N) = H^0(K/N) \cong \mathbb{Q}$, and the Serre spectral sequence of this bundle collapses on the E_2 page, yielding an $H^*(G/K)$-module isomorphism

$$\operatorname{gr}_\bullet H^*(G/N) = H^*(G/K) \otimes \mathbb{Q} \cong H^*(G/K).$$

Because the bigraded algebra $H^*(G/N)$ is concentrated in the bottom row, the associated graded construction leaves it unchanged, so this is a ring isomorphism. $\square$

There is also the following further result, later generalized by Chevalley.

Corollary 6.2.4 (Leray). *The ring $H^*(G/T)$ is isomorphic to the regular representation of the Weyl group W.*

Proof. One characterization of the regular representation $W \rightarrowtail \operatorname{Aut}\big(\mathbb{Q}[W]\big)$ of a group W is through the character $w \longmapsto \operatorname{tr} w|_{\mathbb{Q}[W]}$ of the representation: a representation V is W-isomorphic to the regular representation just if

$$\operatorname{tr} w|_V = \begin{cases} |W| & w = 1, \\ 0 & w \neq 1. \end{cases}$$

Consider the standard right action[2] of $W = N_G(T)$ on G/T given by $gT \cdot nT := gnT$. Since

$$gnT = gT \iff nT = g^{-1}gT = T \iff n \in T,$$

no element of W other than the identity has any fixed points. Now, this right action induces an representation of W in $H^*(G/T)$. For $w \neq 1$, since there are no w-fixed points, w has Lefschetz number $\chi(w) = 0$; but since $H^*(G/T)$ is evenly graded by Theorem 6.1.1, this means that $\operatorname{tr} w|_{H^*(G/T)} = 0$. On the other hand, $\chi(1) = \chi(G/T) = |W|$ by Corollary 6.2.2. $\square$

[2] N.B. The proof of this result in [MT00, Prop. VII.3.25, p. 399] is not quite right, as it tries to use the left multiplication action.

We also can show that the Euler characteristic of a generic compact homogeneous space is zero.

Corollary 6.2.5. *Let G be a compact, connected Lie group and K a closed, connected subgroup of lesser rank. Then $\chi(G/K) = 0$.*

Proof. Let S be a maximal torus of K and T be a maximal torus of G containing S. Then we have a fiber bundle $T/S \to G/S \to G/T$. Since the base is simply-connected, it follows from Proposition 2.3.6 that

$$\chi(G/S) = \chi(G/T)\chi(T/S) = \chi(G/T) \cdot 0,$$

this last since a torus T/S is a product of circles and $\chi(S^1) = 1 - 1 = 0$. Let $N = N_K(S)$ be the normalizer in K of its maximal torus S. Since $N \to S$ is a covering with fiber W_K, so also is $G/S \to G/N$, so by Proposition B.2.5,

$$\chi(G/N) = \chi(G/S)/|W_K| = 0.$$

Now by Corollary 6.2.3 we have $\chi(G/K) = \chi(G/N) = 0$. □

Historical remarks 6.2.6. The Euler characteristic dichotomy that $\chi(G/K) > 0$ or $= 0$ depending as $\operatorname{rk} G = \operatorname{rk} K$ or $\operatorname{rk} G > \operatorname{rk} K$ is due to Hopf and Samelson [HS40, p. 248].

6.3. Weyl-invariants and the restricted action a maximal torus

In Appendix B.4, we pointed that the maximal torus of a compact, connected Lie group and its Weyl group carry much of its algebraic structure. In this section, we show something analogous holds for the orbit space X/K of a free action and the orbit space X/S of the restricted action by that group's maximal torus S. To do so, we use Theorem 6.1.1 and the result of Section 7.2, which we will prove later.

To start, we state a natural enhancement of the motivating observation Proposition 5.5.3 about free homotopy quotients.

Lemma 6.3.1. *Let K be a group, S a subgroup, and X and Y free K-spaces admitting a K-equivariant map $X \longrightarrow Y$. Then these diagrams commute:*

$$\begin{array}{ccc} X_S & \longrightarrow & X_K \\ \downarrow\wr & & \downarrow\wr \\ X/S & \longrightarrow & X/K, \end{array} \qquad \begin{array}{ccc} X_K & \xrightarrow{\simeq} & X/K \\ \downarrow & & \downarrow \\ Y_K & \xrightarrow{\simeq} & Y/K; \end{array}$$

so up to homotopy, $X_K \longrightarrow Y_K$ is equivalent to $X/K \longrightarrow Y/K$ and $X_S \longrightarrow X_K$ to $X/S \longrightarrow X/K$.

In this statement, the horizontal maps in the first square can be realized as the "further quotient" maps $e \otimes x \longmapsto e \otimes x \colon EK \otimes_S X \twoheadrightarrow ES \otimes_K X$ and $xS \longmapsto xK \colon X/S \twoheadrightarrow X/K$.

Definition 6.3.2. In the rest of this section, we let K be a compact, connected Lie group, S a maximal torus, $N = N_K(S)$ the normalizer of S in K, and $W = N/S$ the Weyl group of K.

Write K-Top for the category of topological spaces with continuous K-actions and K-equivariant continuous maps, K-Free for the full subcategory of free K-actions, $\mathbb{Q}$-CGA for the category of (homomorphisms between) graded commutative $\mathbb{Q}$-algebras, and H_S^*-CGA for subcategory of graded commutative H_S^*-algebras.

Observation 6.3.3. Suppose K acts on the right on a space X. Then W acts on the right on the orbit space X/S by $xS \cdot nS = xnS$, and so on the cohomology $H^*(X/S)$. Given a K-equivariant map $X \longrightarrow Y$, the induced map $X/S \longrightarrow Y/S$ is W-equivariant, so the map $H^*(X/S) \longleftarrow H^*(Y/S)$ is as well.

Lemma 6.3.4. *Suppose a finite group W acts on spaces X and Y and there is a W-equivariant continuous map $X \longrightarrow Y$ inducing a surjection $H^*(X) \underset{\varphi}{\twoheadleftarrow} H^*(Y)$. Then the map $H^*(X)^W \longleftarrow H^*(Y)^W$ is also surjective.*

Proof. The restriction to elements $b \in H^*(Y)^W$ has image in $H^*(X)^W$ by W-equivariance: if $w \cdot b = b$ for all $w \in W$, then $w \cdot \varphi(b) = \varphi(w \cdot b) = \varphi(b)$ is invariant as well.

To see the restriction is surjective, let $a \in H^*(X)^W$. Then it has a preimage $b \in H^*(Y)$, not a priori W-invariant. However, the W-average $\bar{b} = \frac{1}{|W|}\sum_{w \in W} w \cdot b$ certainly is, and by equivariance, $\varphi(\bar{b}) = \bar{a}$. Since a was assumed invariant, this average is just a again. □

Lemma 6.3.5 (Leray, 1950)**.** *There is a natural isomorphism*

$$H^*(X/K) \xrightarrow{\sim} H^*(X/S)^W$$

of functors $(K\text{-Free})^{\mathrm{op}} \longrightarrow \mathbb{Q}\text{-CGA}$.

Proof. The quotient map $X/S \longrightarrow X/K$ factors as

$$X/S \longrightarrow X/N \longrightarrow X/K.$$

The factor $X/S \longrightarrow X/N$ is a regular covering with fiber W, which induces by Proposition B.2.1 an isomorphism $H^*(X/N) \xrightarrow{\sim} H^*(X/S)^W$. The fiber of the factor $X/N \to X/K$ is K/N, and $H^*(K/N) \cong H^*(K/S)$ by Corollary 6.2.3.

Naturality follows because the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X/S & \longrightarrow & X/N & \longrightarrow & X/K \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y/S & \longrightarrow & Y/N & \longrightarrow & Y/K \end{array}$$

commutes and because, by Observation 6.3.3, the map $X/S \longrightarrow Y/S$ is W-equivariant. □

This lemma makes available a natural phrasing of an important, well-known result [Hsi75, Prop. III.1, p. 31].

Corollary 6.3.6. *Let K be a compact, connected Lie group with maximal torus S. Then there is a natural isomorphism of functors* $(K\text{-Free})^{\mathrm{op}} \longrightarrow H_S^*\text{-CGA}$ *on spaces X with free K-action taking*

$$H^*(BS) \underset{H^*(BK)}{\otimes} H^*(X/K) \xrightarrow{\sim} H^*(X/S).$$

Proof. We use Lemma 6.3.1 to replace $X/S \longrightarrow X/K$ with $X_S \longrightarrow X_K$ for the rest of the proof. Note that $\xi_0 \colon BS \longrightarrow BK$ is a (K/S)-bundle. Because $H^*(K/S)$ is evenly-graded by Theorem 6.1.1 and H_S^* is evenly-graded by the result of Section 7.2, the E_2 page of the spectral sequence associated to ξ_0 is concentrated in even rows and columns, meaning it collapses by Corollary 2.2.9 and so the fiber inclusion $K/S \hookrightarrow BS$ is surjective on cohomology by Corollary 2.2.12.

Recall from the beginning of Section 2.4 the category F-Bun$/\xi_0$ of bundles over ξ_0. The construction $(-)_{S\hookrightarrow K}\colon X \longmapsto (X_S \to X_K)$ is a functor K-Top $\longrightarrow F$-Bun$/\xi_0$: that is, there is a map of K/S-bundles

$$\begin{array}{ccc} X_S & \longrightarrow & BS \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi_0} \\ X_K & \longrightarrow & BK. \end{array}$$

Here the map $X_S \longrightarrow BS$ is the projection of the Borel fibration and likewise for $X_K \longrightarrow BK$, Now the natural isomorphism follows by Theorem 2.4.1. □

Corollary 6.3.7. *Let K be a compact, connected Lie group with maximal torus S and Weyl group W. Then $H^*(BK) \cong H^*(BS)^W$.*

Proof. Take $X = *$ in Corollary 6.3.6. □

We will use this result to find explicit generators of $H^*(BK)$ in many examples to come.

Remarks 6.3.8. (*a*) The results Lemma 6.3.5 and Corollary 6.3.6 are classical and very well known, except that the naturality of these isomorphisms is never stated. This minor detail was actually critical to the author's dissertation results working.

(*b*) Lemma 6.3.5 can fail if there exist elements of $H^*(X/S;)$ annihilated by scalar multiplication by $|W|$. For example, consider the action of $G = \{\pm 1\} \subsetneq \mathbb{R}^\times$ by scalar multiplication on $X = S^\infty \subsetneq \mathbb{R}^\infty$. Then $X/G \approx \mathbb{RP}^\infty$, and the maximal torus T is trivial, so $W_G = G$, and $X/T = X = S^\infty$ again. With $\mathbb{Z}$ coefficients, one finds

$$H^*(X/G;\mathbb{Z}) \cong \mathbb{Z}[c_1]/(2c_1), \qquad \deg c_1 = 2,$$
$$H^*(X/T;\mathbb{Z})^{W_G} = H^0(S^\infty;\mathbb{Z})^G = \mathbb{Z}.$$

Similarly, with $\mathbb{F}_2$ coefficients,

$$H^*(X/G;\mathbb{F}_2) \cong \mathbb{F}_2[w_1], \qquad \deg w_1 = 1,$$
$$H^*(X/T;\mathbb{F}_2)^{W_G} = H^0(S^\infty;\mathbb{F}_2)^G = \mathbb{F}_2.$$

Historical remarks 6.3.9. Leray had proved a version of Lemma 6.3.5 for classical G [Ler49b] already in 1949, and proved the general version in his *Colloque* paper [Ler51, Thm. 2.2]. The author is indebted to Borel's summary of Leray's topological output [Bor98] for guiding him to these references. [DIG UP WEIL CR REFERENCE (CHECK DIEUDONNÉ).]

Chapter 7

The cohomology of classifying spaces

The Serre spectral sequence of $G \to EG \to BG$ allows us to compute the cohomology of the classifying spaces BG. This computation, due to Borel, can be seen (ahistorically) as a motivation for the definition of the Koszul complex, and through it, the definition of Lie algebra cohomology. Later we will use the result of this spectral sequence calculation, and the Koszul complex, to compute the cohomology of G/K.

7.1. The Serre spectral sequence of $S^1 \to ES^1 \to BS^1$

The ideological mainspring of all the spectral sequence calculations we will do in the rest of this document is a sequence that is only two pages long, the Serre sequence of the universal principal circle bundle $S^1 \to ES^1 \to BS^1$.[1] We use our knowledge of $H^*(S^1)$ and $H^*(ES^1)$ to work out $H^*(BS^1)$.

Proposition 7.1.1. *The cohomology of* $BS^1 = \mathbb{C}\mathrm{P}^\infty$ *is given by*

$$H^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[u], \quad \deg u = 2.$$

Proof. By Proposition 2.2.3, $\pi_1 BS^1$ acts trivially on $H^*(S^1)$, so we can use untwisted coefficients in Theorem 2.2.2.[2] Thus we can write

$$E_2^{p,q} = H^p\big(BS^1; H^q(S^1;\mathbb{Z})\big)$$

As the total space ES^1 is contractible, its cohomology ring $H^*(ES^1)$ is that of a point, a lone $\mathbb{Z}$ in dimension zero, and the associated graded ring E_∞ again $\mathbb{Z}$ because the filtration is trivial.

The cohomology $H^*(S^1)$ is an exterior algebra $\Lambda[z_1]$, where $z_1 \in H^1(S^1)$ is the fundamental class, so in particular it is a graded free abelian group, and

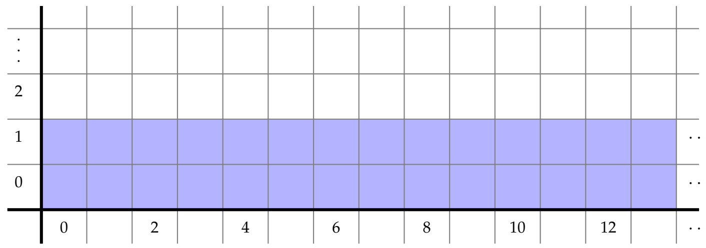
$$E_2^{p,q} \cong H^p(BS^1) \otimes H^1(S^1).$$

Since the second factor is nonzero only for $q \in \{0,1\}$, the entire sequence is concentrated in these two rows.

[1] We earlier, in Section 5.2, identified $S^\infty \to \mathbb{C}\mathrm{P}^\infty$ as a model, but the calculation does not actually require this "geometric" datum.

[2] In fact, from the homotopy long exact sequence of $S^1 \to ES^1 \to BS^1$, it follows that $\pi_2 BS^1 \cong \mathbb{Z}$ is its only nonzero homotopy group, so $\mathbb{C}\mathrm{P}^\infty \simeq BS^1$ is an Eilenberg–Mac Lane space $K(\mathbb{Z},2)$. In particular, BS^1 is in particular simply-connected.

Figure 7.1.2: The potentially nonzero region in the Serre spectral sequence of $S^1 \to ES^1 \to BS^1$



Thus $d = d_2$ is the only differential between nonzero rows, so $E_3 = E_\infty = \mathbb{Z}$ and d *must kill everything else* in E_2. Because the rows $E_2^{\bullet,q} = 0$ except for $q \in \{0,1\}$ and d decreases q by 1, the complex (E_2, d) breaks, for each $p \in \mathbb{Z}$, into short complexes

$$0 \to E_2^{p,1} \longrightarrow E_2^{p+2,0} \to 0.$$

Because the SSS is concentrated in the first quadrant, all groups in the short complex are definitionally zero for $p < -2$. For $p = -2$, we have the very short complex

$$0 \to E_2^{0,0} \to 0,$$

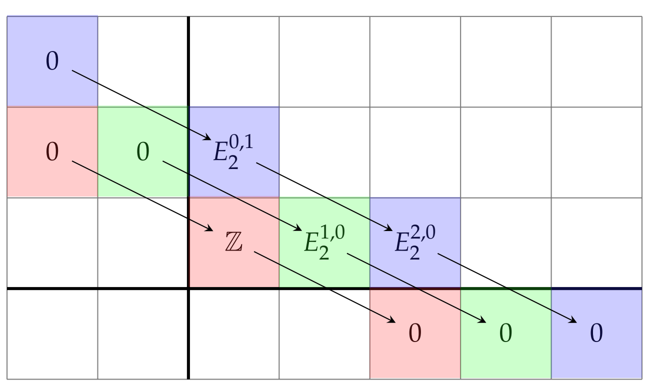
red in Figure 7.1.3, witnessing the apotheosis of $E_2^{0,0} \cong \mathbb{Z}$ to $H^0(ES^1) = E_\infty$. This in fact happens for any SSS where the fiber and base are path-connected, and *must* happen, since $H^0 = \mathbb{Z}$ for all three spaces.

For $p = -1$, we have the very short sequence

$$0 \to E_2^{1,0} \to 0,$$

green in Figure 7.1.3. The middle object must zero because otherwise it would survive to $E_3 = E_\infty$, which would mean $H^1(ES^1) \neq 0$. (Then again, we already knew this because BS^1 is simply-connected and H^0 is always free abelian, so that the universal coefficient theorem B.1.1 yields $H^1(BS^1) \cong H_1(BS^1) \cong \pi_1(BS^1)^{\mathrm{ab}} = 0$.)

Figure 7.1.3: The first few subcomplexes of E_2 in the Serre spectral sequence of $S^1 \to ES^1 \to BS^1$



For $p \geqslant 0$, the total degrees $p+1$ and $p+2$ are positive, so that both groups in the short complex must die in E_3. The only way this can happen is if the d linking them is both injective and surjective, so an isomorphism: that is,

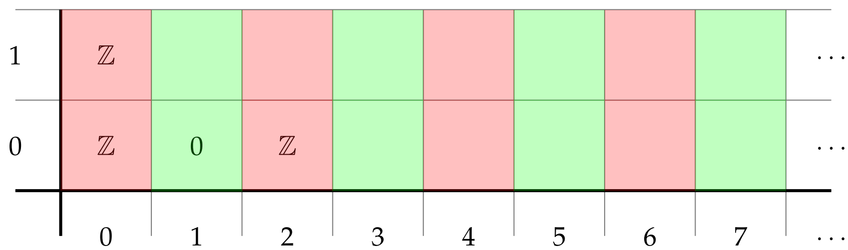
$$E_2^{p,1} \cong E_2^{p+2,0} \quad \text{for all } p \geqslant 0.$$

The first occurrence of this, for $p = 0$, is blue in Figure 7.1.3. On the other hand, the simple fact that $H^0(S^1) \cong \mathbb{Z} \cong H^1(S^1)$ as abstract groups implies, on tensoring with $H^p(BS^1)$, that likewise

$$E_2^{p,0} \cong E_2^{p,1}.$$

Assembling these isomorphisms, all groups in even columns $p = 0, 2, 4, \ldots$ (red in Figure 7.1.4), and all groups in odd columns (green) are isomorphic. The base cases $E_2^{0,0} = H^0(ES^1) = \mathbb{Z}$ and $E_2^{1,0} = \pi_1 BS^1 = 0$ then determine all the other entries: zero in odd columns and $\mathbb{Z}$ in even.

Figure 7.1.4: The partitioning by isomorphism class of groups $E_2^{p,q}$ in the Serre spectral sequence of $S^1 \to ES^1 \to BS^1$



Reading off the bottom row $E_2^{\bullet,0} \cong H^*(BS^1) \otimes H^0(S^1) \cong H^*(BS^1)$, we find the cohomology groups of $BS^1 = \mathbb{C}\mathrm{P}^\infty$ are

$$H^n(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

Recall that the differential $d = d_2$ was an antiderivation restricting to an isomorphism $H^1(S^1) \xrightarrow{\sim} H^2(BS^2)$. If we write $u = dz \in H^2(BS^2)$ for the image of the fundamental class of S^1, then since $du = 0$, applying the product rule yields

$$d(u^{k+1}z) = (k+1)\underbrace{du}_{0} \cdot u^k z \;+\; u^{k+1} \cdot \underbrace{dz}_{u} = u^{k+2}$$

for $k \geqslant 0$. Since this d is an isomorphism $E_2^{2k,1} \xrightarrow{\sim} E_2^{2k+2,0}$ and z and u are nonzero, it follows by induction that u^k generates $H^{2k}(\mathbb{C}\mathrm{P}^\infty)$ for all k. □

We could more easily have found the graded group structure of $H^*(\mathbb{C}\mathrm{P}^\infty)$ through cellular cohomology after pushing down the increasing union $S^\infty = S^1 \cup S^3 \cup S^5 \cup \cdots$ to a strictly even-dimensional CW structure $\mathbb{C}\mathrm{P}^\infty = e^0 \cup e^2 \cup e^4 \cup \cdots$, but the spectral sequence also makes computing the ring structure almost trivial.

For later reference, note that, topology aside, the calculation we just made is a manifestation of the following algebraic fact. Define B to be the graded ring $\mathbb{Z}[u]$, where $\deg u = 2$, and assign it the trivial differential. Let A be the graded ring $B \otimes \Lambda[z]$, where $\deg z = 1$. Make A a $\mathbb{Z}$-CDGA extending $(B, 0)$ by assigning as differential the unique antiderivation d that vanishes on 0 and satisfies

$$dz = u.$$

Then (A, d) is acyclic: $H^0(A) = \mathbb{Z}$ and $H^n(A) = 0$ for $n > 0$. The reason we were able to deduce $H^*(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}[u]$ is that $\mathbb{Z}[u]$ is the unique B that makes an $A = B \otimes \Lambda[z]$ constructed as above acyclic.

7.2. The Serre spectral sequence of $T \to ET \to BT$

The circle is the one-dimensional case of the *torus* $T^n = \prod^n S^1$. By the Künneth theorem, one has

$$H^*(T^n) \cong \bigotimes^n H^*(S^1) = \bigotimes^n \Lambda[z] = \Lambda[z_1, \ldots, z_n] = \Lambda H^1(T^n),$$

where z_j is the fundamental class of the j^{th} factor circle and $H^1(T^n) = \mathbb{Z}\{z_1, \ldots, z_n\}$ is the primitive subspace as discussed in Proposition 1.0.9.

To understand $H^*(BT)$, there are at least two options. The first is an analysis analogous to, but more intricate than, that in the last section: one sees easily $d_2 \colon H^1(T) \longrightarrow H^2(BT)$ must be an isomorphism and then puts more work into showing that means d_2 is injective on the entire first column $E_2^{0,\bullet} \cong H^*(T)$ and that $E_3 = E_\infty = \mathbb{Z}$. The second invokes the functoriality of the universal principal bundle construction $G \longmapsto (G \to EG \to BG)$ to make the problem trivial. As the functors E and B preserve products, one has the bundle isomorphism

$$\begin{array}{ccc} T & \xrightarrow{\sim} & \prod S^1 \\ \downarrow & & \downarrow \\ ET & \xrightarrow{\sim} & \prod ES^1 \\ \downarrow & & \downarrow \\ BT & \xrightarrow{\sim} & \prod BS^1, \end{array}$$

so that $BT = \prod^n \mathbb{C}\mathrm{P}^\infty$ and $H^*(BT) = \bigotimes \mathbb{Z}[u_j] \cong \mathbb{Z}[u_1, \ldots, u_n]$.

The bundle isomorphism in fact induces a Künneth isomorphism of SSSs, so that

$$E_2 = \bigotimes_{j=1}^{n} \big(S[u_j] \otimes \Lambda[z_j]\big) \cong S[\vec{u}] \otimes \Lambda[\vec{z}],$$

with differential d_2 the unique antiderivation taking $z_j \mapsto u_j$ for each j (and hence annihilating $S[\vec{u}]$). Thus

$$\Big(S[\vec{u}] \otimes \Lambda[\vec{z}], \quad z_j \mapsto u_j\Big)$$

is another example of an acyclic CDGA. We will investigate the natural algebraic generalization of this phenomenon in the next section.

7.3. The Koszul complex

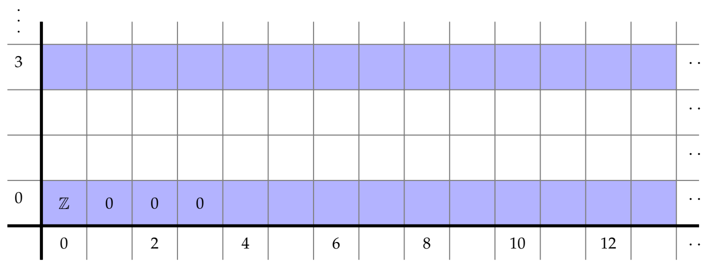
In the spectral sequences of universal bundles $T \to ET \to BT$, the cohomology $H^*(T)$ of the fiber is an exterior algebra and the cohomology H_T^* of the base is a polynomial algebra on the same number of generators, and the algebra generators of fiber and base cancel one another in a one-to-one fashion in the spectral sequence.

For another such example, consider the Lie group $\mathrm{Sp}(1)$. Recall that his group can be seen as the multiplicative subgroup of quaternions of norm 1, and hence is a homeomorphic 3-sphere, and that one can take $E\mathrm{Sp}(1) = \bigcup S^{4n-1} = S^\infty$ and $B\mathrm{Sp}(1) = \mathbb{H}\mathrm{P}^\infty$. Now, E_2 page of the Serre spectral sequence of the universal bundle

$$\mathrm{Sp}(1) \longrightarrow E\mathrm{Sp}(1) \longrightarrow B\mathrm{Sp}(1)$$

is thus $E_2^{\bullet,0} \otimes E_2^{0,\bullet} \cong H^*\big(B\mathrm{Sp}(1)\big) \otimes H^*(S^3)$. As with the spectral sequence of $S^1 \to ES^1 \to BS^1$, then, there are only two nonzero rows, now the 0^{th} and the 3^{rd}, so the only nontrivial differential can be d_4.

Figure 7.3.1: The potentially nonzero region in the Serre spectral sequence of $S^3 \to ES^3 \to BS^3$



Because $E_\infty = E_5 = \mathbb{Z}$ is trivial, all the differentials d_4 in and out of the other shaded boxes must be diffeomorphisms. Since the $E_2 = E_4$ page is a tensor product, the two entries in each column must be the same, so as in the S^1 case, one has isomorphisms

$$H^p(BS^3) = E_2^{p,0} \cong E_2^{p+4,0} \cong H^{p+4}(BS^3)$$

for each p. We know $H^0(BS^3) = \mathbb{Z}$ and since there are no nonzero differentials to or from the next three boxes, these are zero. If we write z for a generator of $H^3(S^3)$ and $q = d_4 z \in H^4(BS^3)$, then as in the S^1 case, we find $d_4(zq^n) = q^{n+1}$, so that finally $H^*(BS^3) = \mathbb{Z}[q]$. This example is very closely analogous to the S^1 example: in particular, the $E_2 = E_4$ page was of the form

$$\mathbb{Z}[q] \otimes \Lambda[z], \quad |z| = 3, \quad |q| = 4,$$

and there was only one nonzero differential, d_4

This example and the torus examples share the property of being tensor products of a very simple kind of spectral sequence, and we claim that for all compact, connected Lie groups G, the spectral sequence of $G \to EG \to BG$ is such a tensor product. To facilitate future reference, we axiomatize this situation.

Definition 7.3.2. Let $\Lambda[v]$ be the exterior k-algebra on one element v of odd degree ℓ and $S[dv]$ the symmetric algebra on one element dv of degree $\ell + 1$. Then

$$K[v] = S[dv] \otimes \Lambda[v]$$

is a k-algebra. The exterior factor Λz naturally has a grading defined by $|1| = 0$ and $|v| = \ell$, and $S[dv]$ inherits the natural grading $\big|(dv)^n\big| = n(\ell + 1)$, so $K[v]$ is bigraded by

$$K[v]_{p,q} := S[dv]_p \otimes \Lambda[v]_q$$

and singly graded by total degree:

$$K[v]_n := \bigoplus_{n=p+q} K[v]_{p,q} = \bigoplus_{n=p+q} S[dv]_p \otimes \Lambda[v]_q.$$

The map $d\colon v \longmapsto dv$ uniquely defines a derivation of (total) degree one on $K[v]$ since $d(dv) = 0$, explicitly given by

$$d\big(v \cdot (dv)^n\big) = (dv)^{n+1}.$$

It should be clear from our discussions of the Serre spectral sequences of $S^1 \to ES^1 \to BS^1$ and $\mathrm{Sp}(1) \to E\mathrm{Sp}(1) \to B\mathrm{Sp}(1)$ that the cohomology of $K[v]$ with respect to d is trivial. More complicated examples arise from tensoring these primordial contractible complexes.

Definition 7.3.3. Let $V = \bigoplus_{j>0} V_{2j-1}$ be a positively- and oddly-graded free graded k-module. The grading on V induces a grading on ΛV making it a free CGA. Let $\Sigma V = V_{\bullet-1}$ be the ***suspension***, the even regrading of V achieved by moving each graded level up one degree: symbolically, $(\Sigma V)_j := V_{j-1}$.[3] There is a naturally induced grading on the symmetric algebra $S\Sigma V$, making it a free CGA.

Let $KV := S\Sigma V \otimes \Lambda V$. As an algebra, this is just the tensor product of the $K[v_\alpha]$ for any basis v_α of V. Because $S^1[\Sigma V] \oplus \Lambda^1[V]$ generates KV as a k-algebra, to characterize a derivation on KV, it is enough to describe it on this submodule. The natural derivation is that which restricts on $\Lambda^1 V$ to the defining isomorphism

$$d = \Sigma\colon \Lambda^1[V] \xrightarrow{\sim} V \xrightarrow{\sim} S^1[\Sigma V]$$

of ungraded free k-modules; consequently, $dS^1[\Sigma V] = 0$ and hence $d(S\Sigma V) = 0$. It is called the ***Koszul differential*** and the complex (KV, d) is the ***Koszul algebra*** of V. As d is just the sum of the differentials on the $K[v_\alpha]$, so $(KV, d) \cong \bigotimes_\alpha (K[v_\alpha], d_\alpha)$ is a tensor product of the elementary Koszul CDGAs. It admits a natural bigrading

$$(KV)_{p,q} := (S\Sigma V)_p \otimes (\Lambda V)_q.$$

additively extending the gradings on V and ΣV. In addition to the associated single grading $KV_n := \bigoplus_{n=p+q} KV_{p,q}$, there is also another useful grading, setting

$$K^{-n}[V] := \bigoplus_{j=0}^{n} S^j[\Sigma V] \otimes \Lambda^{n-j}[V],$$

the submodule of KV spanned by products of $n \geqslant 0$ generators. This grading of KV, called the ***multiplicative grading***, induces a grading of the cohomology of KV such that $H^{-n}(KV)$ is the image of the cocycles in $K^{-n}[V]$.

From the Serre spectral sequence of $T \to ET \to BT$, we expect this cohomology to be trivial.

Proposition 7.3.4 (Koszul). *Let V be a free k-module and KV the Koszul complex. If k is of characteristic zero and contains $\mathbb{Q}$, or if V is of finite rank, then KV is acyclic.*

[3] The notation is meant to suggest the suspension ΣX of a topological space X, which satisfies $H^{n+1}(\Sigma X) \cong H^n(X)$.

First proof [Car51, Thm. 1]. Assume that $\mathbb{Q} \leqslant k$, so that all naturals $n \geqslant 1$ are invertible in k. The inverse isomorphism $h = d^{-1}\colon S^1[\Sigma V] \xrightarrow{\sim} \Lambda^1[V]$ extends uniquely, just as d does, to an antiderivation of KV of degree -1. We claim it is a chain homotopy of (KV, d).

The composition dh is the projection $K^{-1}[V] \to S^1[V]$ and hd the projection $K^{-1}[V] \to \Lambda^1[V]$, so $hd + dh = \mathrm{id}$ on $K^{-1}[V]$. Inductively assume that also $L = dh + hd = n\,\mathrm{id}$ on $K^{-n}[V]$ and write a decomposable (e.g., basis) element of $K^{-(n+1)}[V]$ as ab, for $a \in K^{-1}[V]$ and $b \in K^{-n}[V]$. Then by the product rule, the base case, and the inductive assumption,

$$L(ab) = (La)b + aL(b) = ab + nab = (n+1)ab,$$

concluding the induction. For any n-cocycle a we then have $na = (hd + dh)a = dha$, so each d-cocycle is a coboundary for $n \geqslant 1$. Thus $H^*(KV) = H^0(KV) \cong k$. □

This argument same argument incidentally also shows the h-cohomology of KV is trivial.

Second proof. Assume V is of finite rank over k. Find a k-basis v_j of V, so that $V = \bigoplus kv_j$ and $\Sigma V = \bigoplus kdv_j$. Then we have algebra isomorphisms

$$KV = S\Sigma V \otimes \Lambda V \cong S\left[\bigoplus kdv_j\right] \otimes \Lambda\left[\bigoplus kv_j\right] \cong \bigotimes \left(S[dv_j] \otimes \Lambda[v_j]\right) = \bigotimes K[v_j],$$

and this also holds on the level of CDGAs, as discussed in Definition 7.3.3. As everything in sight is a free k-module, the simplest version of the algebraic Künneth formula Corollary A.3.3 holds, and

$$H^*_d(KV) \cong \bigotimes_j H^*_{d_j}\big(K[v_j]\big) \cong k^{\otimes \operatorname{rk}_k V} \cong k. \qquad \square$$

As the Koszul algebra will be our chosen CDGA model for a universal bundle $G \to EG \to BG$, we will introduce a notation for its filtration spectral sequence.

Definition 7.3.5. Let V be a positively- and oddly-graded free graded k-module. Filter its corresponding Koszul algebra (KV, d) by the p-grading induced by the factor $S\Sigma V$. We denote by $EV_\bullet$ the associated filtration spectral sequence. Explicitly, for $V = kv$ one-dimensional, we have

$$\big(E[v]_r, d_r\big) = \big(K[v], 0\big) \text{ for } r \leqslant |v|, \qquad E[v]_{|v|+1} = \big(K[v], d\big), \qquad E[v]_{|v|+2} = E[v]_\infty \cong k,$$

and if (v_j) is a homogeneous basis for V, then $EV_\bullet = \bigotimes E[v_j]_\bullet$ on every page.

The Koszul complex, which makes its first appearance in thesis work of Koszul dealing with the Lie algebra cohomology which had been recently defined by Chevalley and Eilenberg, was soon discovered to have uses in commutative algebra. Here is a more general definition.

Definition 7.3.6. Let A be a unital commutative ring over k. Given a sequence $\vec{a} = (a_j)_{j \in J}$ of elements of A, we can form the k-algebra $\Lambda[z_j]_{j\in J} = \bigotimes_{j \in J} \Lambda[z_j]$ and the tensor algebra

$$K_A\vec{a} := \Lambda[z_j]_{j \in J} \underset{k}{\otimes} A.$$

Viewing A as a CGA graded in degree zero, we can make $K_A\vec{a}$ a CDGA by extending the k-linear map $\bigoplus_{j\in J} k \cdot z_j \longrightarrow A$ given by $z_j \mapsto a_j$ to an antiderivation d and assigning the $|z_j| = -1$. Then $\deg d = 1$ and

$$K_A^{-n}\vec{a} = \Lambda^n[z_j]_{j\in J} \otimes A.$$

We call this grading the ***resolution grading***. The k-CDGA $(K_A\vec{a}, d)$ is the ***Koszul complex*** associated to the sequence $\vec{a}$.

Given an A-module M, the tensor product module

$$K_A(\vec{a}, M) := K_A\vec{a} \underset{A}{\otimes} M = \big(\Lambda[z_j] \underset{k}{\otimes} A\big) \underset{A}{\otimes} M \cong \Lambda^p[z_j] \underset{k}{\otimes} M,$$

inherits a differential, vanishing on M, given by

$$d(z_j \otimes 1) = 1 \otimes a_j \qquad (j \in J),$$

and the resulting chain complex is again called a ***Koszul complex***.

Koszul complexes $K_A(\vec{a}, M)$ being defined by sequence of ring elements, their potential acyclicity is related to properties of this sequence.

Definition 7.3.7. Let A be a unital commutative ring over k. A finite or countable sequence (a_j) of elements of A is called a ***regular sequence*** if for each n, the image of a_n is not a zero-divisor in the quotient ring $A/(a_1, \ldots, a_{n-1})$. Given an A-module M, the same sequence is called ***M-regular*** (or an *M-sequence*) if each a_n annihilates no nonzero elements of the quotient module $M/(a_1, \ldots, a_{n-1})M$. An ideal $\mathfrak{a} \trianglelefteq A$ is called a ***regular ideal*** if it can be generated by a regular sequence.

Regular sequences do not normally remain regular under permutation, but do if all elements lie in the Jacobson radical of A, and in particular if A is a local ring and the elements a_j are non-units [Eis95, Cor. 17.2, p. 426].

Proposition 7.3.8. *Let A be a connected* CGA *and a_j elements of the augmentation ideal $\widetilde{A}$; then the sequence (a_j) is regular just if each permutation is.*

Since we really care only about cohomology rings, order in a regular sequence shall never be an issue for us. The connection between Koszul complexes and regular sequences is the following.

Proposition 7.3.9 ([Ser00, IV.A.2, Prop. 3, p. 54])**.** *Given a Noetherian commutative ring A, a sequence $\vec{a}$ of elements of the Jacobson radical of A, and a finitely-generated A-module M, the following conditions are equivalent:*

1. $H^{-n}\big(K_A(\vec{a}, M)\big) = 0$ *for* $n \geqslant 1$;
2. $H^{-1}\big(K_A(\vec{a}, M)\big) = 0$;
3. *the sequence $\vec{a}$ is M-regular.*

The last relevant fact about Koszul complexes is that they compute Tor.

Proposition 7.3.10. *Let $A = S[\vec{a}]$ be a free commutative k-*CGA *generated by a sequence $\vec{a}$ of elements of even degree, and let B be an A-*CGA*. Then the Koszul complex $K_A(\vec{a}, B)$ associated to $\vec{a}$ computes* Tor, *in that*

$$H^{-p}(K_A\vec{a} \underset{A}{\otimes} B) \cong \mathrm{Tor}_p^A(k, B), \qquad p \geqslant 0.$$

Proof. The base ring k is an A-algebra in a natural way via $A \twoheadrightarrow A/\widetilde{A} \xrightarrow{\sim} k$. Since the generators are independent, by Proposition 7.3.9, the Koszul complex $(K_A\vec{a}, d)$ is acyclic, with

$$H^*(K_A\vec{a}) = H^0(K_A\vec{a}) \cong k[\vec{a}]/(\vec{a}) \cong k.$$

It follows that $K_A^\bullet\vec{a}$, with the resolution grading from Definition 7.3.6, is an A-module resolution of k, so that the $-p^{\text{th}}$ cohomology of the sequence

$$\cdots \longrightarrow K_A^{-2}\vec{a} \underset{A}{\otimes} B \longrightarrow K_A^{-1}\vec{a} \underset{A}{\otimes} B \longrightarrow K_A^{0}\vec{a} \underset{A}{\otimes} B \longrightarrow 0$$

computes $\operatorname{Tor}_p^A(k, B)$. □

Note that in fact $\operatorname{Tor}_\bullet^A(k, B)$ is a bigraded CGA. The product descends from the product on $\Lambda[z_j] \otimes_k B$, and the second component of the grading from the grading $\bigoplus B^q$ on B. We set

$$\operatorname{Tor}_A^{-p,q}(k, B) = \operatorname{Tor}_p^A(k, B^q) = H^p\big(\Lambda[z_j] \underset{k}{\otimes} B^q\big).$$

Historical remarks 7.3.11. Regular sequences were introduced by Serre in 1955 as *E-sequences* [Bor67, p. 93], and this terminology apparently hung on for quite a while [Bau68, Def. 3.4]. Smith [Smi67, p. 79] uses *ESP-sequence* and calls a graded ideal generated by such a sequence a *Borel ideal*.

7.4. The Serre spectral sequence of $G \to EG \to BG$

> "...the behavior of this spectral sequence ...is a bit like an Elizabethan drama, full of action, in which the business of each character is to kill at least one other character, so that at the end of the play one has the stage strewn with corpses and only one actor left alive (namely the one who has to speak the last few lines)."[4] —*J. F. Adams*

7.4.1. Statements

We have found $H^*(BT)$ for all tori and by Corollary 6.3.7, we know that $H^*(BG;\mathbb{Q})$ can be viewed as the Weyl-invariant subring $H^*(BT;\mathbb{Q})^W$, so theoretically, we understand $H^*(BG)$ now. In practice, and especially if one wants to understand the torsion—something we will eventually punt on—there is more work to be done.

In the torus computation, the algebra generators $H^1(T) = PH^*(T)$ of $H^*(T)$ (the primitives, as defined in Definition 1.0.8) and $H^2(BT) \cong QH^*(BT)$ of $H^*(BT)$ (the indecomposables, as defined in Definition 1.0.8) were linked bijectively by nontrivial differentials and were annihilated, and the algebraic repercussions of this bijection sufficed to force $E_\infty = \mathbb{Z}$. To work with merely generators greatly simplifies any computation, so one might hope that such a pattern holds as well for nonabelian groups. The proof of this result is due to Borel in his thesis [Bor53]. Our moderately modernized version is based somewhat unfaithfully on the treatments contained in Mimura and Toda [MT00, p. 379–80] and Hatcher [Hat, Thm. 1.34].

The ultimate goal is the following, to be borne in mind as we regress further and further into the algebraic abstraction required for its proof in the next subsection. The transgression in the Serre spectral sequence is described in Proposition 2.2.21 and will return again in the proof of Theorem 8.1.5.

[4] This memorable analogy is repurposed from a famous description of the Adams spectral sequence [Ada76].

Theorem 7.4.1 (Borel [Bor53, Théorème 19.1]). *Let G be a compact, connected Lie group and let k be a ring (such as a field of characteristic zero) such that $H^*(G;k) \cong \Lambda PG$ is an exterior algebra on odd-degree generators (by Proposition 1.0.9, these are the primitives). Then $H^*(BG;k) \cong k[\tau PG]$ is a polynomial ring on generators $\tau PG \cong \Sigma PG$ of degree one greater, given by a choice of transgression on PG.*

In all this, it is to be remembered that the transgression on $H^p(G)$ is really a only a map from a submodule of $E_{p+1}^{0,p} \leqslant E_2^{0,p} \cong H^p(G)$ to $E_{p+1}^{p+1,0}$, which is a quotient of $E_2^{p+1,0} = H^{p+1}(BG)$, so that when we lift this maps to E_2, what we get is for each p a relation $\tau \subseteq H^p(G) \times H^{p+1}(BG)$, rather than a map, and what it retains of the homomorphism d_{p+1} is *additivity*: if (z_j, y_j) are finitely many elements of τ, then so also is $(\sum z_j, \sum y_j)$. Despite the imprecision, it is useful notationally and psychologically to write τ as a map in the event that the precise lift to E_2 is irrelevant, and we engage in this abuse already in the statement of Theorem 7.4.1 above.

That said, an precise rephrasing of Borel's result can be obtained as follows. Writing $Q(BG)$ for the space of indecomposables (defined in Definition 1.0.8), and noting that we have a well-defined isomorphism $\Lambda PG \cong H^*(G)$ and an isomorphism $H^*(BG) \cong S\big[Q(BG)\big]$ only defined up to some arbitrary lifting, the transgression in the spectral sequence of $G \to EG \to BG$ nevertheless descends to a sequence of well-defined isomorphisms

$$P^pG \xrightarrow{\sim} Q^{p+1}(BG)$$

summing to the isomorphism[5]

$$\tau\colon PG \xrightarrow{\sim} Q(BG).$$

Setting $V = PG$ and constructing the Koszul complex KV, this τ uniquely extends uniquely to the Koszul differential. Because $H^*(BG)$ is free on $Q(BG)$, on the level of CGAs, we recover

$$\widetilde{E}_2 = H^*(BG) \otimes H^*(G) = KV$$

and can consider τ as an antiderivation $\widetilde{E}_2 \longrightarrow \widetilde{E}_2$, sometimes called a ***choice of transgression***, which we will use extensively in Chapter 8. By construction, it satisfies the following proposition.

Proposition 7.4.2. *A choice of transgression τ lifts the edge homomorphisms $\tilde{d}_r$ in the sense that for each $r \geqslant 0$, the following diagram commutes:*

$$\begin{array}{ccc} H^*(G) & \xrightarrow{\tau} & H^*(BG) \\ \uparrow & & \downarrow \\ \widetilde{E}_r^{0,r-1} & \xrightarrow[\tilde{d}_r]{\sim} & \widetilde{E}_r^{r,0}. \end{array}$$

As the differences produced by starting with a different choice of transgression turn out to be immaterial, we will at times identify $Q(BG)$ with a graded subspace of $H^*(BG)$. We also need one corollary about the original, unlifted transgression to prove Cartan's theorem later in Theorem 8.1.5 and Theorem 8.1.14.

[5] I owe this description to Paul Baum's thesis [Bau62, p. 3.3].

Corollary 7.4.3 (Borel). *Let $G \to E \xrightarrow{\pi} B$ be a principal G-bundle classified by $\chi\colon B \longrightarrow BG$. Write τ for the transgression of the universal bundle $G \to EG \to BG$. In the spectral sequence of π, each primitive $z \in PH^*(G)$ transgresses to $\chi^*\tau z$.*

Proof. This follows from the existence of the bundle map from $G \to E \to B$ to $G \to EG \to BG$, which induces a spectral sequence map as in Theorem 2.2.2 intertwining the edge homomorphisms. □

7.4.2. Two proofs

We provide two proofs of Borel's key Theorem 7.4.1 on classifying spaces. The first is an immediate application of the following algebraic result to the Serre spectral sequence of the universal bundle $G \to EG \to BG$. It invokes the notion of transgression discussed in Section 2.8.

Theorem 7.4.4 (Borel [Bor53, Thm. 13.1].). *Let k be a commutative ring and P an oddly-graded free k-module. Suppose (E_r, d_r) is a spectral sequence of bigraded k-algebras such that*

- *E_2 admits a tensor decomposition $E_2^{\bullet,0} \otimes E_2^{0,\bullet}$ with $E_2^{0,\bullet} \cong \Lambda P$ the exterior algebra on P and*
- *the final page $E_\infty = E_\infty^{0,0} \cong k$ is trivial.*

Then P admits a homogeneous basis of transgressive elements and $E_2^{\bullet,0} \cong k[\tau P]$ is the symmetric algebra on these transgressions.

This in turn is the $n = \infty$ case of the following more general theorem involving simple systems of generators as discussed in Definition A.2.4.

Theorem 7.4.5 (Borel transgression theorem). *Let k be a commutative ring and (E_r, d_r) is a spectral sequence of bigraded k-DGAs with*

$$E_2 \cong E_2^{\bullet,0} \otimes E_2^{0,\bullet} =: B^\bullet \otimes F^\bullet$$

a tensor product of connected k-DGAs up to total degree $n+2$. Suppose that

- *$E_\infty^{\leqslant n+2} := \bigoplus_{p+q \leqslant n+2} E_\infty^{p,q} = E_\infty^{0,0} \cong k$, and that*
- *there exists a free k-module $P < F^{\leqslant n}$, oddly graded,*
  such that the induced map $\Delta P \longrightarrow F^\bullet$ is $\begin{cases} \text{bijective in degrees} & \leqslant n, \\ \text{injective in degree} & = n+1. \end{cases}$

Then

- *P admits a transgressive basis, and*
- *writing $Q = \tau P < B^\bullet$, the induced map $SQ \longrightarrow B^\bullet$ is $\begin{cases} \text{bijective in degrees} & \leqslant n+1, \\ \text{injective in degree} & = n+2. \end{cases}$*

We need the fiddly degree bounds because the proof itself is inductive. We would actually not need to induct if we knew in advance the exterior generators transgress, and the proof is substantially easier in that special case, so we will prove it first. The essential idea is the same in both cases. We already know an acyclic algebra of the form $\Delta P \otimes S\Sigma P$, namely the Koszul complex $KP = \Lambda P \otimes S\Sigma P$ of Section 7.3, and the strategy behind the proof of both results will be to use our knowledge of the transgressions to construct a map of spectral sequences $EP_\bullet \longrightarrow E_\bullet$ that shows $E_2 \cong KP$ as a bigraded $S\Sigma P$-module, at least in a prescribed range of degrees. This can be seen as a natural generalization of our analysis of the Serre spectral sequence of a universal torus bundle $T \to ET \to BT$. Recall that we constructed the Koszul algebra KP in analogy with the E_2 CDGA of that spectral sequence; now we reverse the process.

Theorem 7.4.6 (Borel "little" transgression theorem)**.** *Let k be a commutative ring. Suppose (E_r, d_r) is a spectral sequence of bigraded k-algebras such that*

- *E_2 admits a tensor decomposition $E_2^{\bullet,0} \otimes E_2^{0,\bullet}$,*
- *the k-algebra $E_2^{0,\bullet} \cong \Delta(z_\alpha)$ is free as a k-module and admits a simple system of generators z_α,*
- *these z_α transgress in the spectral sequence, and*
- *the final page $E_\infty = E_\infty^{0,0} \cong k$ is trivial.*

Then $E_2^{\bullet,0} \cong k[\tau z_\alpha]$ is the symmetric algebra on the transgressions of the z_α.

As Zeeman noted, this result will apply to the case $\Delta P = H^*(G)$ to yield the structure theorem 7.4.1 for $H^*(BG)$ as soon we know the odd-degree generators P in that spectral sequence transgress. Thus there is the following easier proof.

Alternate proof of Theorem 7.4.1. Considering the *homology* Serre spectral sequence of the universal bundle $G \to EG \to BG$, Remark 1.0.13 shows the homological primitives $PH_*(G) < H_*(G)$ are all in the image of the transgression. Because $H_*(G) \cong H^*(G)$ and $H_*(BG) \cong H^*(BG)$ on the level of graded vector spaces and the homological and cohomological transgressions are dual (Remark 2.2.23), this means all elements of PG transgress in the cohomological Serre spectral sequence. Thus, by Theorem 7.4.6, we have $H^*(BG) \cong k[\tau P]$. □

Here is the promised proof of the little transgression theorem.

Proof of Theorem 7.4.6 ([Zee58][McC01, Thm. 3.27, p. 85])*.* Select a homogeneous k-basis v_α of P and for each v_α lift the transgression $d_{|v_\alpha|+1}v_\alpha$ to an element τv_α of $E_2^{|v_\alpha|+1,0}$. We construct a map of spectral sequences $\lambda_\bullet \colon EP_\bullet \longrightarrow E_\bullet$, where the source is the filtration spectral sequence of KP defined in Definition 7.3.5, by

$$\begin{aligned} \lambda_2 \colon EP_2 &\longrightarrow E_2, \\ 1 \otimes v_\alpha &\longmapsto 1 \otimes v_\alpha, \\ dv_\alpha \otimes 1 &\longmapsto \tau v_\alpha \otimes 1. \end{aligned}$$

and $\lambda_{r+1} = H^*(\lambda_r)$. To see this is a cochain map, one need only check on generators of each page EP_r, which are (represented by) $1 \otimes v_\alpha$ and $dv_\alpha \otimes 1$ for $v_\alpha \in P^{\geqslant r-1}$. There is nothing to see for the symmetric generators $dv_\alpha \otimes 1$ as all differentials vanish on $S\Sigma P = EP_2^{\bullet,0}$ and $E_2^{\bullet,0}$ and their descendants. As for exterior generators, d_r vanishes by construction on generators (descending) from the complement in P of the graded component P^{r-1}, and writing $[x]_r$ an element on the r^{th} page represented by x on the second to be maximally careful, one has $d_r[1 \otimes v_\alpha]_r = [dv_\alpha \otimes 1]_r$ for $v_\alpha \in P^{r-1}$ by construction, so

$$\lambda_r d_r[1 \otimes v_\alpha]_r = \lambda_r[dv_\alpha \otimes 1]_r = \big[\lambda_2(dv_\alpha \otimes 1)\big]_r = [\tau v_\alpha \otimes 1]_r = d_r[1 \otimes v_\alpha]_r = d_r\lambda_r[1 \otimes v_\alpha]_r.$$

Because $S\Sigma P$ is a free k-CGA in any characteristic and we extended λ multiplicatively from a map on the generators ΣP, the row restriction $\lambda_2^{\bullet,0}\colon S\Sigma P \longrightarrow E_2^{\bullet,0}$ is a ring homomorphism. The column restriction $\lambda_2^{0,\bullet}\colon \Lambda P \longrightarrow \Delta P$ is a linear isomorphism, because both ΔP and ΛP admit a k-basis of ordered monomials in the v_α. (If the characteristic of k is not 2, then $\Delta P = \Lambda P$, so this column map is a ring isomorphism, but it need not be in characteristic 2, because then it is not required that $v_\alpha^2 = 0$ in ΔP.) The limiting map $\lambda_\infty\colon EP_\infty \longrightarrow E_\infty$ is by construction the identity map on k, so the Zeeman–Moore comparison theorem 2.7.1 applies to tell us the ring map $\lambda_2^{\bullet,0}\colon S\Sigma P \longrightarrow E_2^{\bullet,0}$ is a linear isomorphism. □

Our proof of the big transgression theorem is an adaptation of the proof of Mimura and Toda [MT00, p. 379–80].[6]

Proof of Theorem 7.4.5. The proof is an induction on n. In the $n = 0$ case, we assume $E_\infty^{\leqslant 2} = k$ and $P = 0$. The conclusion that a basis of P transgresses is vacuously true, and since $B^1 = E_2^{1,0} = E_\infty^{1,0} = 0$, we do indeed have $SQ = k \longrightarrow B^\bullet$ bijective in degrees 0 and 1 and injective in degree 2.

For the induction step, note from Definition A.2.4 of a simple system of generators that on the level of graded k-modules we have $\Delta P \cong \Lambda P$, and that from the view of differentials in this spectral sequence, the two are indistinguishable. Thus, when $P < F^\bullet$ transgresses to $Q < B^\bullet$, we can define a map $EP_\bullet \longrightarrow E_\bullet$ from the filtration spectral sequence of the Koszul algebra KP, as defined in Definition 7.3.5, sending $\Lambda P \longrightarrow \Delta P$ and $S\Sigma P \longrightarrow SQ$. Then we use the Zeeman–Moore comparison theorem 2.7.1 on this map:

- the hypothesis on $\Delta P \longrightarrow F^\bullet$ in the present theorem is the condition $(F)_n$,
- the hypothesis $E_\infty^{\leqslant n+2} \cong k$ implies the condition $(E)_{n+1}$, and
- the conclusion about $SQ \longrightarrow B^\bullet$ is the condition $(B)_{n+1}$.

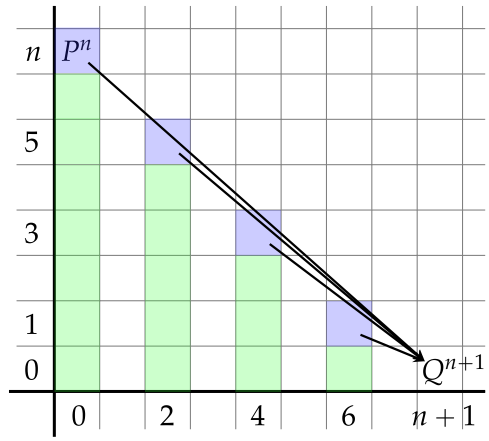
Now we assume the theorem holds for $n-1$ odd and must show it for n even. In this case, there is nothing to do: by hypotheses generators P are of odd degree $\leqslant n$, so P admits a transgressive basis by the induction hypothesis, and we apply the implication $(F)_n \,\&\, (E)_{n+1} \implies (B)_{n+1}$ of Theorem 2.7.1 to $EP_\bullet \longrightarrow E_\bullet$ to conclude.

Now we assume the theorem holds for $n-1$ even and must show it for n odd. The hypothesis is that $E_\infty^{\leqslant n+2} = k$ and there is a graded subspace $P < F^\bullet$ generated by elements of odd degree $\leqslant n$ such that $\Delta P \longrightarrow F^\bullet$ is an isomorphism in degrees $\leqslant n$ and an injection in degree $n+1$. Write $P_n = P \cap F^n$ and $P_{<n} < P$ for its complementary subspace. Then $\Delta P_{<n} \longrightarrow F^\bullet$ is an isomorphism in degrees $\leqslant n-1$ and an injection in degree n, so by the inductive hypothesis, $P_{<n}$ admits a transgressive basis, and map $SQ_{<n+1} \longrightarrow B^\bullet$ induced by inclusion of the transgressions $Q_{<n+1} < B^\bullet$ is a bijection in degree $\leqslant n$ and an injection in degree $n+1$. It follows we may pick out a basis of a complementary subspace Q_{n+1} to the image in B^{n+1}, and then setting $Q = Q_{<n+1} \oplus Q_{n+1}$, we have $SQ \longrightarrow B^\bullet$ bijective in degrees $\leqslant n+1$ by construction and injective in degree $n+2$ odd because Q is evenly graded.

It remains to show that d_{n+1} is a bijection $P_n \xrightarrow{\sim} Q_{n+1}$. First we show that Q_{n+1} lies in the image of the transgression. We know that $E_{n+2}^{n+1,0} = E_\infty^{n+1,0} = 0$, so $Q_{n+1} < E_{n+1}^{n+1,0}$ must be the image of some differential d_r. The potential differentials have source in bidegree $(n+1-r, r-1)$, and we must show it is only possible that $r = n+1$; see Figure 7.4.7. Now consider the spectral

[6] Which we believe is incomplete.

Figure 7.4.7: The differentials to $E_\bullet^{n+1,0}$ originate in the region receiving only differentials induced by those of $E[P^{<n}]_\bullet$



sequence map $E[P_{<n+1}]_\bullet \longrightarrow E_\bullet$. We know it is an isomorphism onto the rectangle $E_2^{\leqslant n, \leqslant n-1}$. None of the entries of bidegree $(n+1-r, r-1)$ receive differentials from outside this rectangle, so their elements correspond bijectively to elements of $E[P_{<n+1}]_r$, which is the Koszul subalgebra generated by P^j for $r-1 \leqslant j \leqslant n-2$. Thus, if $q \in Q_{n+1}$ were in the image of one of these differentials, it would lie in $SQ_{<n+1}$, contrary to assumption. It follows there is $P' \leqslant P_n$ such that $\tau P' \cong Q_{n+1}$. Now we may construct a map from the Koszul spectral sequence $E[P_{<n+1} \oplus P']_\bullet$ to $E_\bullet$, and applying the implication $(B)_{n+1} \,\&\, (E)_n \implies (F)_n$ of Theorem 2.7.1, we conclude that the map on $E_2^{\bullet,0}$ is a bijection $\Lambda[P_{<n+1} \oplus P'] \xrightarrow{\sim} \Delta P$ in degrees $\leqslant n$. It follows $P' = P_n$, which we already knows transgresses to Q_{n+1}, concluding the proof. □

Historical remarks 7.4.8. Coming at a later point in history affords us many luxuries Borel did not have when he was proving Theorem 7.4.6 and Theorem 7.4.5. For one, the Zeeman–Moore theorem was not available to him, so he did not construct a comparison map, but explicitly, inductively, and through careful bookkeeping ruled out the possibility of $H^*(BG)$ being anything other than a polynomial ring, keeping track at the same time of what elements of ΛP transgressed and ultimately determining them to be only the primitives P themselves.

More historically remarkably, in determining $H^*(BG)$ Borel did not have access to *BG itself*. In 1952, it was only known in general that *n-universal* principal bundles $E(n,G) \longrightarrow B(n,G)$ existed for each $n \in \mathbb{N}$ with $\pi_i E(n,G) = 0$ for $i \leqslant n$. Borel's $H^*(BG)$ is actually defined as the inverse limit of the rings $H^*\big(B(n,G)\big)$, known cohomology rings of already-existing manifolds. Resultingly, for numerous topological applications in which we cavalierly deploy BG, Borel must instead invoke $H^*\big(B(n,G)\big)$ for n sufficiently large. This approximation technique is still used in algebraic geometry, where each $B(n,G)$ can be considered an algebraic variety but BG cannot.

We imagine the alternate proof of Theorem 7.4.1 following Theorem 7.4.6 was known, but have no reference.

7.4.3. Complements

The rest of this section is devoted to related results we will not have need of in the sequel. For example, there is also a dual result whose proof falls out of what we have already done.

Corollary 7.4.9 (Borel [Bor54, Thm. 6.1, p. 297])**.** *Let k be a commutative ring and $Q \cong k\{y_\alpha\}$ an evenly-graded free k-module. Suppose (E_r, d_r) is a spectral sequence of bigraded k-algebras such that*

- *E_2 admits a tensor decomposition $SQ \otimes E_2^{0,\bullet}$, with $E_2 \cong SQ$ the symmetric algebra on Q and*
- *the final page $E_\infty = E_\infty^{0,0} \cong k$ is trivial.*

Then $E_2^{0,\bullet} \cong \Delta[v_\alpha]$ admits a simple system of transgressive generators v_α such that $\tau v_\alpha = y_\alpha$.

Proof. Since $E_\infty \cong k$, each y_α must eventually be annihilated by some differential. There can be no generators of degree 1 since all differentials out of this box are zero. The generators Q_2 of degree 2 can only be annihilated by elements $E_2^{0,1}$. Since d_2 is a differential, it follows $E_3^{\bullet,0} \cong SQ/(Q_2) \cong S[Q_{\geqslant 3}]$ is the subalgebra on generators of degree three or more. Assume inductively that Q_r survives in E_r. Since it doesn't survive to E_∞, it must be annihilated by the elements $E_r^{0,2}$, which are hence all transgressive. Inductively continuing this way, each element of Q must be killed by a transgressive element of $E_2^{0,\bullet}$.

Now define a cochain map $\lambda : SQ \otimes \Lambda[v_\alpha] \longrightarrow E_2$ as in the proof of Theorem 7.4.5. Applying the Zeeman–Moore comparison theorem 2.7.1 again, one sees the restriction $\Lambda[v_\alpha] \longrightarrow \Delta[v_\alpha] \leqslant E_2^{0,\bullet}$ must a linear isomorphism, so that $E_2^{0,\bullet} = \Delta[v_\alpha]$. □

Combining the two, one has the following.

Corollary 7.4.10. *Let k be a commutative ring. Suppose (E_r, d_r) is a spectral sequence of bigraded k-algebras such that*

- *E_2 admits a tensor decomposition $E_2^{\bullet,0} \otimes E_2^{0,\bullet}$,*
- *the final page $E_\infty = E_\infty^{0,0} \cong k$ is trivial.*

Then the following are equivalent:

- *The k-algebra $E_2^{0,\bullet} \cong \Delta[z_\alpha]$ admits a simple system of transgressive generators z_α.*
- *The k-algebra $E_2^{\bullet,0} \cong S[y_\alpha]$ is a symmetric algebra on generators y_α.*

If the statements hold, the z_α and y_α are related by $\tau z_\alpha = y_\alpha$.

Remarks 7.4.11. (a) In fact, there is a strengthening requiring only that the triangle $\bigoplus_{p+q \leqslant n} E_\infty^{p,q} \cong k$ is trivial, the map from $\Delta[z_\alpha]$ is a bijection in degrees $\leqslant n-1$ and an injection in degree n and that the map from $S[y_\alpha]$ is a bijection in degrees $\leqslant n$ and an injection in degree $n+1$, as in the proof of Theorem 7.4.5.

(b) The full strength version of Corollary 7.4.10 reflects a sort of duality between the category of modules over a symmetric algebra and that over an exterior algebra, called *Koszul duality*.

To round out this subsection we include without proof some other finite-characteristic results and conclude with some historical remarks. As regards the applicability of the little transgression theorem 7.4.6 in characteristic $\neq 2$, it is not universal. Borel found the following example. Recall that the cohomology rings $H^*(\mathrm{Spin}(n); \mathbb{F}_2)$ do admit simple systems of generators (Proposition 3.2.17).

Theorem 7.4.12. *Consider a simple system of generators for $H^*(\mathrm{Spin}(n); \mathbb{F}_2)$. These are all transgressive if and only if $n \leqslant 9$. Accordingly, $H^*(B\mathrm{Spin}(n); \mathbb{F}_2)$ is a polynomial ring if and only if $n \leqslant 9$.*

Specifically, as described in Proposition 3.2.17, one has

$$H^*\big(\mathrm{Spin}(10);\mathbb{F}_2\big) \cong \Delta[v_3, v_5, v_6, v_7, v_9, z_{15}],$$

but there is an element u of degree 15, congruent to z_{15} modulo decomposables, which has $d_{10}(u \otimes 1) = d_{10}(v_9 \otimes 1) \cdot (1 \otimes v_6)$. The nontransgression of this u is related to the failure of the homology ring $H_*\big(\mathrm{Spin}(10);\mathbb{F}_2\big)$ to be an exterior algebra, as described in Example A.2.8.

Nevertheless, in the universal bundle for the limiting group $\mathrm{Spin} = \varinjlim \mathrm{Spin}(n)$, all generators transgress again. One has then the following corollary of Theorem 3.2.18.

Theorem 7.4.13 ([BCM, Thm. 6.10, p. 55])**.** *The mod 2 cohomology ring of $B\mathrm{Spin}$ is given by*

$$H^*(B\mathrm{Spin};\mathbb{F}_2) = \mathbb{F}_2[w_j : j \neq 2^\ell + 1]$$

and that of $B\mathrm{SO}$ by

$$H^*(B\mathrm{SO};\mathbb{F}_2) = \mathbb{F}_2[w_j],$$

the map $H^*B\mathrm{SO} \twoheadrightarrow H^*B\mathrm{Spin}$ induced by $\mathrm{Spin} \twoheadrightarrow \mathrm{SO}$ being the obvious surjection. The transgressions are given, for j odd, by

$$\tau(v_j^{2^\ell}) = w_{2^\ell j+1}.$$

Borel also found a complement in characteristic not equal to 2, showing even-dimensional spheres (other than S^0) can't show up as factors in the fiber of a bundle with contractible total space.

Theorem 7.4.14. *Let k be a ring of characteristic not equal to 2. Suppose (E_r, d_r) is a spectral sequence of bigraded k-algebras such that*

- *E_2 admits a tensor decomposition $E_2^{\bullet,0} \otimes E_2^{0,\bullet}$ such that $E_2^{0,\bullet} = \Delta[p_\alpha]$ for a simple system (p_α) of generators with $p_\alpha^2 = 0$ and*
- *the final page $E_\infty = E_\infty^{0,0} \cong k$ is trivial.*

Then all of the p_α are of odd degree.

Explicitly, in this case, we have in the hypothesis that

$$E_2^{0,\bullet} = \Delta[p_\alpha] = \Lambda\big[p_j : |p_\alpha| \text{ odd}\big] \otimes \bigotimes_{|p_\alpha| \text{even}} S[p_\alpha]/(p_\alpha^2)$$

and in the conclusion that $\Delta[p_\alpha] = \Lambda P$ for P oddly graded. Clearly, then, if one wanted to generalize the "simple system of generators" to even-degree generators in characteristic $p > 2$, asking that they be nilsquare would not be the way to go. Postnikov would find the proper strategy in 1966 to generalize Theorem 7.4.6 to odd characteristic.

Definition 7.4.15 ([Pos66, p. 36])**.** Let p be an odd prime. A graded commutative $\mathbb{F}_p$-algebra F is said to admit a ***p-simple system of generators*** $(z_\alpha, y_\beta)_{\alpha \in \mathrm{A}, \beta \in \mathrm{B}}$, where the z_α are of odd degree and the y_β even, if F is spanned as an $\mathbb{F}_p$–vector space by the basis of ordered monomials

$$z_{\alpha_1} \cdots z_{\alpha_m} y_{\beta_1}^{\ell_1} \cdots y_{\beta_n}^{\ell_n},$$

where the indices α_i and β_j are strictly increasing and the exponents $\ell_j \leqslant p - 1$.

Theorem 7.4.16 ([Pos66, p. 36]). *Let p be an odd prime. Suppose (E_r, d_r) is a spectral sequence of bigraded $\mathbb{F}_p$-algebras such that*

- *E_2 admits a tensor decomposition $E_2^{\bullet,0} \otimes E_2^{0,\bullet}$ such that $E_2^{0,\bullet}$ admits a p-simple system (z_α, y_β) of transgressive generators, the z_α being of odd degree and the y_β even,*
- *the final page $E_\infty = E_\infty^{0,0} \cong \mathbb{F}_p$ is trivial.*

Then $E_2^{\bullet,0}$ is the free commutative algebra on the transgressions

$$x_\alpha := \tau z_\alpha, \qquad v_\beta := \tau y_\beta, \qquad u_\beta := \tau(v_\beta \otimes y_\beta^{p-1}).$$

Explicitly, $E_2^{\bullet,0} \cong \mathbb{F}_p[x_\alpha] \otimes \mathbb{F}_p[u_\beta] \otimes \Lambda[v_\beta]$

Sketch of proof. Assume first the elements $\tau y_\beta \otimes y_\beta^{p-1}$ transgress. Then we will be able to find the proper bigraded comparison complex admitting a chain map to (E_2, τ), where τ is a choice of transgression in E_2, and the proof will proceed exactly as the proof of Theorem 7.4.6.

For the odd generators z_α, one retains the bigraded Koszul spectral sequence $E[z_\alpha]$ as before, but for each even generator y_β one introduces a tensor-factor (not a DGA)

$$\big(\Lambda[\bar{v}_\beta] \otimes \mathbb{F}_p[\bar{u}_\beta]\big) \otimes \mathbb{F}[\bar{y}_\beta] \big/ (\bar{y}_\beta^{p-1}), \quad dy_\beta = \bar{v}_\beta, \quad d(\bar{v}_\beta \otimes \bar{y}_\beta^{p-1}) = \bar{u}_\beta$$

bigraded with the expected degrees with v_β and u_β in the bottom row and y_β in the left column. Collecting all of these, the assignment $z_\alpha \mapsto z_\alpha$, $\bar{y}_\beta \mapsto y_\beta$, $\bar{v}_\beta \mapsto v_\beta$, $\bar{u}_\beta \mapsto u_\beta$ is by definition a chain map, and restricts to a ring map $\mathbb{F}_p[\tau z_\alpha, \bar{u}_\beta] \otimes \Lambda[\bar{v}_\beta] \longrightarrow E_2^{\bullet,0}$ because on this subdomain it is defined by unique extension from free CGA generators.

That the $\tau y_\beta \otimes y_\beta^{p-1}$ must transgress is an induction like that in Theorem 7.4.5. To prove it for $\max_\beta |y_\beta| = n+1$, inductively assume it for degrees $\leqslant n$ and as well that $E_2^{\bullet,0}$ agrees up to degree $p(n+1)+1$ with the free CGA on

$$x_\alpha, \qquad v_\beta \quad \big(\text{for } |y_\beta| \leqslant n+1\big), \qquad u_\beta \quad \big(\text{for } |y_\beta| \leqslant n-1\big).$$

It follows from this assumption and the eventual triviality of $E_\infty \cong \mathbb{F}_p$ that on the page $E_{(n+1)(p-1)+1}$, the rectangle $\big[0, (n+1)p+1\big] \times \big[0, (n+1)(p-1)-1\big]$ is trivial, the cancellations being due solely to the elements in the induction hypothesis. This means that the differentials of the $\tau y_\beta \otimes y_\beta^{p-1}$ for $|y_\beta| = n+1$ which land in this rectangle must be trivial, and so the map $\tau \colon E_{(n+1)(p-1)+1}^{n+1,(n+1)(p-1)} \longrightarrow E_{(n+1)(p-1)+1}^{(n+1)p+2,0}$ must be an isomorphism, showing all the new $\tau y_\beta \otimes y_\beta^{p-1}$ also transgress. □

[

Remark 7.4.17.

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7.5. Characteristic classes

Borel's Theorem 7.4.1, the mod 2 addendum Section 7.4.2.(a), and knowledge of the cohomology rings of classical groups from Chapter 3 make instantly available a great deal of information about classifying spaces.

Corollary 7.5.1. *Let $k = \mathbb{Z}[1/2]$. The cohomology rings of the classifying spaces of the classical groups are*

$$\begin{aligned}
H^*\big(BO(n);\mathbb{F}_2\big) &\cong \mathbb{F}_2[w_1,\ldots,w_n], && \deg w_j = j,\\
H^*\big(BSO(n);\mathbb{F}_2\big) &\cong \mathbb{F}_2[w_2,\ldots,w_n], && \deg w_j = j,\\
H^*\big(BU(n);\mathbb{Z}\big) &\cong \mathbb{Z}[c_1,\ldots,c_n], && \deg c_j = 2j,\\
H^*\big(BSU(n);\mathbb{Z}\big) &\cong \mathbb{Z}[c_2,\ldots,c_n], && \deg c_j = 2j,\\
H^*\big(BSp(n);\mathbb{Z}\big) &\cong \mathbb{Z}[q_1,\ldots,q_n], && \deg q_j = 4j,\\
H^*\big(BSO(2n+1);k\big) &\cong k[p_1,\ldots,p_{n-1},p_n], && \deg p_j = 4j,\\
H^*\big(BSO(2n);k\big) &\cong k[p_1,\ldots,p_{n-1},e], && \deg p_j = 4j,\ \deg e = 2n.
\end{aligned}$$

Definition 7.5.2. The w_j in the preceding corollary are the ***Stiefel–Whitney classes***, the c_j the ***Chern classes***, the q_j the ***symplectic Pontrjagin classes***, the p_j the ***Pontrjagin classes***, and e the ***Euler class***.

Remark 7.5.3. For $G \in \{U, Sp, SO\}$, the inclusions $G(n) \hookrightarrow G(n+1)$ preserve c_j, q_j, p_j respectively for $j \leqslant n$ and annihilate $c_{n+1}, q_{n+1}, p_{n+1}$, with the exception that $H^*\big(BSO(2n+1)\big) \longrightarrow H^*\big(BSO(2n)\big)$ takes $p_n \longmapsto e^2$.

The Pontrjagin classes and Euler class as described above are actually *integral* in that they are in the image of the canonical map $H^*\big(BSO(m);\mathbb{Z}\big) \longrightarrow H^*\big(BSO(m);\mathbb{Z}[1/2]\big)$. These classes carry certain well-known relations. For example, the inclusion $U(n) \rightarrowtail SO(2n)$ induces a map $H^{2n}\big(BSO(2n);\mathbb{Z}\big) \longrightarrow H^{2n}\big(BU(n);\mathbb{Z}\big)$ carrying $e \longmapsto c_n$, and mod-2 coefficient reduction $H^n\big(BSO(n);\mathbb{Z}\big) \longrightarrow H^n\big(BSO(n);\mathbb{F}_2\big)$ takes $e \longmapsto w_n$.

All of these rings can also be calculated independently with $\mathbb{Q}$ coefficients from the result Corollary 6.3.7 that $H^*(BG) \cong H^*(BT)^W$ and an understanding of the Weyl group action on BT. For example, the existence of the Euler class can be seen as a result of the fact that $W_{SO(2n+1)} = \{\pm 1\}^n \rtimes S_n$ and $W_{SO(2n)}$ is the subgroup $S\{\pm 1\}^n \rtimes S_n$, where $S\{\pm 1\}^n < \{\pm 1\}^n$ is the index-two subgroup whose elements contain an even number of -1 entries. The product $e = t_1 \cdots t_n \in \mathbb{Z}[t_1,\ldots,t_n]$ is invariant under $S\{\pm 1\}^n$ but not under all of $\{\pm 1\}^n$, and as a result does not occur in $H^*\big(BSO(2n+1)\big)$; its square $p_n = t_1^2 \cdots t_n^2$ is however invariant under the larger group's action.

The cohomology classes of Definition 7.5.2, elements of a cohomology ring BG only known after 1955 to globally exist, are abstract manifestations of objects associated to vector bundles which were defined in the 1930s and early 1940s by their namesakes.[7]

Definition 7.5.4. Let $E \longrightarrow B$ be a principal G-bundle and $\chi\colon B \longrightarrow BG$ a classifying map. Given $c \in H^*(BG)$, its pullback $\chi^*(c) \in H^*(B)$ is written $c^*(E)$ and called a ***characteristic class*** of $E \longrightarrow B$.

These characteristic classes are functorial invariants of principal bundles: because the universal bundle is terminal, a map of bundles induces a homotopy-commutative triangle of maps of base spaces.

Proposition 7.5.5. *Let $E \to B$ be a principal G-bundle, let $f\colon B' \to B$ be a continuous map, and let $c \in H^*(BG)$. Then the pullback bundle f^*E satisfies*

$$c(f^*E) = f^*c(E) \in H^*(B).$$

[7] With the obvious exception of the Euler class.

Given a vector bundle $F \to V \xrightarrow{\xi} B$ with transition functions in a linear group G, there is an associated principal G-bundle $G \to P \to B$ as described in Appendix B.3.1, and one can associate to $V \to B$ the characteristic classes of $P \to B$,

$$c(V) := c(P),$$

calling them the ***characteristic classes of the vector bundle*** $V \to B$. For example

- if $\xi\colon V \to B$ is a quaternionic vector bundle it defines symplectic classes $q_j(\xi) \in H^{4j}(B;\mathbb{Z})$,
- if ξ is a complex vector bundle one has Chern classes $c_j(\xi) \in H^{2j}(B;\mathbb{Z})$,
- if ξ is a real vector bundle one has Pontrjagin classes $p_j(\xi) \in H^{4j}(B;\mathbb{Z})$ and Stiefel–Whitney classes $w_j(\xi) \in H^j(B;\mathbb{F}_2)$, and
- if ξ is an *orientable* vector bundle with fiber $F = \mathbb{R}^n$, it has an Euler class $e(\xi) \in H^n(B;\mathbb{Z})$, and the first Stiefel–Whitney class w_1 can be shown to vanish.

[Tie this in to the earlier discussion in the context of the Gysin sequence]

A smooth manifold M determines a tangent bundle $TM \to M$, which thus defines a characteristic class

$$c(M) := c(TM) \in H^*(M)$$

for each characteristic class c of the tangent bundle. For example, we can equip TM with a Riemannian or Hermitian metric to reduce its structure group to $\mathrm{O}(n)$ or $\mathrm{U}(n)$, so all smooth manifolds carry Pontjagin and Stiefel–Whitney classes, orientable smooth manifolds carry an Euler class $e(M) \in H^{\mathrm{top}}(M)$, and almost complex manifolds carry Chern classes.

These classes turn out to be well-defined invariants of the topological manifold underlying M in that they are independent of the chosen metrics and smooth or almost complex structures. To see at least that the metrics are irrelevant, one way to proceed is to note that the Gram–Schmidt construction can be seen as a product decomposition [BT82, Ex. 6.5(a)]

$$\mathrm{SL}(n,\mathbb{R}) = \mathrm{SO}(n)\cdot F,$$

where F is the contractible space of positive-definite symmetric matrices. If we consider $E\mathrm{SO}(n)$ to be $E\mathrm{SL}(n,\mathbb{R})$, which is valid, as discussed in Section 5.2, since $\mathrm{SO}(n)$ and $\mathrm{SL}(n,\mathbb{R})$ are Lie groups, the former closed in the latter, then taking quotients yields the bundle

$$F \longrightarrow B\mathrm{SL}(n,\mathbb{R}) \longrightarrow B\mathrm{SO}(n),$$

with fiber F contractible, so that $B\mathrm{SL}(n,\mathbb{R}) \simeq B\mathrm{SO}(n)$. Similar homotopy equivalences hold for other classifying spaces of linear groups, so one can dispense with the metrics at the negligible cost of viewing the characteristic classes instead as arising in $B\mathrm{GL}(n;\mathbb{F})$ or $B\mathrm{SL}(n;\mathbb{F})$ for $\mathbb{F} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$.

Assume now M is compact and oriented. A characteristic class c in $H^{\mathrm{top}}(M;\mathbb{Z}) \cong \mathbb{Z}$ is then some integer multiple $n\cdot[M]^*$ of the cohomological fundamental class $[M]^*$; alternately, evaluation of c against the homological fundamental class $[M]$ yields an integer n. These integers are called ***characteristic numbers*** of the manifold, and the data given by characteristic numbers for a real manifold can be seen as the composition

$$H^n\big(B\mathrm{SO}(n);\mathbb{Z}\big) \xrightarrow{\chi^*} H^n(M;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z},$$

where $\chi\colon M \longrightarrow B\mathrm{SO}(n)$ is the characteristic map of the associated principal $\mathrm{SO}(n)$-bundle.

The ***Pontrjagin numbers*** are the images under this composition of the degree-n level of the subring $\mathbb{Z}[p_1, \ldots, p_n]$, and the Euler characteristic can be seen as the image of e:

Theorem 7.5.6. *Let M be a smooth, compact, oriented n-manifold. Then the Euler class $e \in H^n(M;\mathbb{Z})$ and cohomological fundamental class $[M]^* \in H^n(M;\mathbb{Z})$ and the Euler characteristic $\chi(M) \in \mathbb{Z}$ satisfy the relation*

$$e = \chi(M) \cdot [M]^*.$$

This is the reason behind the nomenclature *Euler class*. This equivalence also yields an outlandishly complicated way of seeing the Euler characteristic of an odd-dimensional closed manifold is zero.

[CONNECT THESE EULER CLASSES AND STIEFEL–WHITNEY CLASSES WITH THOSE INTRODUCED IN SECTION 2.3.1.]

7.6. Maps of classifying spaces

The machine for computing $H^*(G/K)$ depends critically on an understanding of the map

$$\rho^* = (Bi)^*\colon H^*(BG) \longrightarrow H^*(BK)$$

induced by the inclusion $i\colon K \hookrightarrow G$; this understanding (in what by now should be starting to seem like a familiar theme) is also due to Borel [Bor53, §28].

7.6.1. Maps of classifying spaces of tori

To start, let $i\colon S \hookrightarrow T$ be an inclusion of tori. By using a functorial construction of the universal bundle as in Section 5.3, or else by taking $ES = ET$ and representing $BS \longrightarrow BT$ as the "further quotient" map $ET/S \twoheadrightarrow ET/T$, we have a bundle map

$$\begin{array}{ccc} S & \xhookrightarrow{\ i\ } & T \\ \downarrow & & \downarrow \\ ES & \xrightarrow{\ \simeq\ } & ET \\ \downarrow & & \downarrow \\ BS & \xrightarrow{\ Bi\ } & BT, \end{array}$$

which induces a map $\big(\psi_r\colon (\widetilde{E}_r, \tilde{d}_r) \longrightarrow (E_r, d_r)\big)$ between the spectral sequences of the bundles. Because these sequences both collapse on the third page, ψ_r is just an isomorphism $H^0(ET) \xrightarrow{\sim} H^0(ES) = \mathbb{Z}$ for $r \geqslant 3$, so we may as well drop page subscripts and consider the lone interesting map $\psi = \psi_2$, which by Theorem 2.2.2 is

$$\psi = (Bi)^* \otimes i^*\colon H^*(BT) \otimes H^*(T) \longrightarrow H^*(BS) \otimes H^*(S).$$

Because, by the definition of a chain map, we have $d\psi = \psi\tilde{d}$, and, as we have just seen, $d\colon H^1(S) \longrightarrow H^2(BS)$ and $\tilde{d}\colon H^1(T) \longrightarrow H^2(BT)$ are group isomorphisms, we have the commutative square

$$\begin{array}{ccc} H^1(S) & \xleftarrow{\ i^*\ } & H^1(T) \\ \downarrow \wr & & \downarrow \wr \\ H^2(BS) & \xleftarrow{(Bi)^*} & H^2(BT). \end{array} \tag{7.6.1}$$

Thus $i^*\colon H^1(T) \longrightarrow H^1(S)$ is conjugate through the transgression isomorphisms to $(Bi)^*\colon H^2(BT) \longrightarrow H^2(BS)$. Since $H^2(BT)$ generates $H^*(BT)$ as an algebra, and $(Bi)^*$ is a ring homomorphism, this means $(Bi)^*$ is determined uniquely by i^*. This i^*, in turn, is described by i in a transparent way. It is dual to the map $i_*\colon H_1S \longrightarrow H_1T$, or equivalently to the map $\pi_1(i)$.

In a case we will explore completely later, S will just be a circle, which we will identify with the standard complex unit circle $S^1 < \mathbb{C}^\times$. Similarly identify T with $(S^1)^n$. Then $i\colon S \hookrightarrow T$ can be written as

$$\begin{aligned} i\colon S^1 &\longrightarrow (S^1)^n, \\ z &\longmapsto (z^{a_1}, \dots, z^{a_n}), \end{aligned}$$

where the exponent vector $\vec{a} \in \mathbb{Z}^n$ is a list of integers with greatest common divisor 1, so that i is injective.[8] If $x_j \in \pi_1(T) = H_1(T)$ is the fundamental class of the j^{th} factor circle and $y \in H_1(S)$ the fundamental class of S, then

$$i_*\colon y \longmapsto \sum a_j x_j.$$

Let (x_j^*) be the dual basis for $H^1(T)$ and $y^* \in H^1(S)$ the cohomological fundamental class. Then the dual map $i^*\colon H^1(T) \twoheadrightarrow H^1(S)$ in cohomology takes $x_j^* \longmapsto a_j y^*$ since

$$(i^* x_j^*)y = x_j^*(i_* y) = x_j^*\left(\sum a_\ell x_\ell\right) = a_j.$$

Put another way, the matrix of i^* is the transpose of the matrix of i_*. Write $s = d_2 y^* \in H^2(BS)$ and $u_j = d_2 x_j^* \in H^2(BT)$ so that $H^*(BS) = \mathbb{Z}[s]$ and $H^*(BT) = \mathbb{Z}[\vec{u}]$. Then, the square above implies that $(Bi)^*(u_j) = a_j s$, so that if $p(\vec{u}) \in \mathbb{Z}[\vec{u}]$ is any homogeneous polynomial,

$$(Bi)^* p(\vec{u}) = p(a_1 s, \dots a_n s) = p(a_1, \dots, a_n) s^{\deg p}.$$

7.6.2. Maps of classifying spaces of connected Lie groups

Let $K \hookrightarrow G$ be an inclusion of compact, connected Lie groups. If S is a maximal torus of K, then there exists a maximal torus T of G containing S. Through the functoriality of the classifying space functor B and cohomology, this square of inclusions gives rise to two further commutative squares:

$$\begin{array}{ccc} S & \xhookrightarrow{\ i\ } & T \\ \downarrow & & \downarrow \\ K & \hookrightarrow & G \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} BS & \xrightarrow{Bi} & BT \\ \downarrow & & \downarrow \\ BK & \xrightarrow[\rho]{} & BG \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} H^*(BS) & \xleftarrow{(Bi)^*} & H^*(BT) \\ \uparrow & & \uparrow \\ H^*(BK) & \xleftarrow[\rho^*]{} & H^*(BG). \end{array}$$

[8] This vector $\vec{a}$ is only well-defined up to the choice of identifications $S \cong S^1$ and $T \cong (S^1)^n$, but will suffice for our later applications.

The vertical maps in the last square are inclusions by Corollary 6.3.7. Thus ρ^* can be computed as the composition

$$H^*(BG) \xrightarrow{\sim} H^*(BT)^{W_G} \xrightarrow{(Bi)^*} H^*(BS);$$

it follows from the commutativity of the square that the image lies in $H^*(BS)^{W_K} \cong H^*(BK)$.

Example 7.6.2. Let $G = \mathrm{U}(4)$ and $K = \mathrm{Sp}(2)$, identified as a subgroup of G through the injective ring map $\mathbb{H} \rightarrowtail \mathbb{C}^{2\times 2}$ given by $\alpha + j\beta \longmapsto \left[\begin{smallmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{smallmatrix}\right]$. A standard maximal torus for G is given by the subgroup $T = \mathrm{U}(1)^4$ of diagonal unitary matrices, which meets K in the subgroup

$$S = \left\{ \mathrm{diag}(z, \bar z, w, \bar w) \in \mathrm{U}(1)^4 : z, w \in S^1 \right\}.$$

With respect to the expected basis of $H_1(T)$ and the fundamental classes of the factor circles $w = 1$ and $z = 1$ of S, and the dual bases in H^1, the maps $H_1(S) \longrightarrow H_1(T)$ and $H^1(S) \longleftarrow H^1(T)$ are given respectively by

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

By the commutative square (7.6.1), the second matrix also represents $H^2(BS) \longleftarrow H^2(BT)$ with respect to the transgressed bases t_1, t_2, t_3, t_4 of $H^2(BT)$ and s_1, s_2 of $H^2(BS)$.

The Weyl group of $\mathrm{U}(4)$ is the symmetric group S_4 on four letters acting on T and hence BT by permutation of the four coordinates. It follows that when $H^*(\mathrm{U}(4))$ is conceived as the invariant subring $H^*(BT)^{S_4}$ of $H^*(BT)$, it is generated by the elementary symmetric polynomials $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ in t_1, t_2, t_3, t_4, lying in respective degrees $2, 4, 6, 8$. These are the first four Chern classes c_j.

The Weyl group of $\mathrm{Sp}(2)$ is the group $\{\pm 1\}^2 \rtimes S_2$, acting on $H^1(S)$ and hence on $H^2(BS) = \mathbb{Q}\{s_1, s_2\}$ by negating and/or switching the two coordinates. It follows the invariant subring $H^*(B\mathrm{Sp}(2)) \cong H^*(BS)^{W_{\mathrm{Sp}(2)}}$ is generated by $q_1 = s_1^2 + s_2^2$ and $q_2 = (s_1 s_2)^2$. These are the first two symplectic Pontrjagin classes. The generators c_j exhibit the following properties under $H^*(BT)^{S^4} \hookrightarrow H^*(BT) \longrightarrow H^*(BS)$:

$$\begin{aligned}
c_1 &= t_1 + t_2 + t_3 + t_4 \longmapsto (s_1 - s_1) + (s_2 - s_2) = 0, \\
c_2 &= \frac{1}{2}\big(\sigma_1^2 - \sigma_1(t_1^2, t_2^2, t_3^2, t_4^2)\big) \longmapsto \frac{1}{2}\big(0 - (s_1^2 + s_1^2 + s_2^2 + s_2^2)\big) = -(s_1^2 + s_2^2) = -q_1, \\
c_3 &= (t_1 + t_2)t_3 t_4 + t_1 t_2 (t_3 + t_4) \longmapsto (0 \cdot -s_2^2) + (-s_1^2 \cdot 0) = 0, \\
c_4 &= t_1 t_2 t_3 t_4 \longmapsto s_1^2 s_2^2 = q_2.
\end{aligned}$$

That is, $H^*(B\mathrm{U}(4)) \longrightarrow H^*(B\mathrm{Sp}(2))$ is surjective, a fact we will later be able to see as a consequence of the surjectivity of $H^*(\mathrm{U}(4)) \longrightarrow H^*(\mathrm{Sp}(2))$.

Example 7.6.3. Let $G = \mathrm{Sp}(2)$ and $K = S = \mathrm{SO}(2)$, identified as a subgroup of G through the standard inclusion $\mathbb{R} \hookrightarrow \mathbb{H}$. One maximal torus T of $\mathrm{Sp}(2)$ containing S is that generated by S and the block-diagonal subgroup $S' = \mathrm{U}(1) \oplus [1]$. As $|S \cap S'| = 1$, the standard isomorphisms $S^1 \longrightarrow S$ and $S^1 \longrightarrow S'$ determine a basis of $\pi_1(T) = H_1(T)$. With respect to this basis, the map $H_1(S) \longrightarrow H_1(T)$ is given by the matrix $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ so $H^1(S) \longleftarrow H^1(T)$ is given, with respect to the dual basis, by the transpose $\begin{bmatrix}1 & 0\end{bmatrix}$. By (7.6.1) again the second matrix also represents $H^2(BS) \longleftarrow H^2(BT)$ with respect to the transgressed bases t_1, t_2 of $H^2(BT)$ and t_1 of $H^2(BS)$. Generators for $H\big(B\mathrm{Sp}(2)\big)$ are q_1, q_2 as in Example 7.6.2, and we have

$$q_1 = t_1^2 + t_2^2 \longmapsto t_1^2,$$
$$q_2 = t_1^2 t_2^2 \longmapsto t_1^2 \cdot 0 = 0.$$

Example 7.6.4. Let $G = \mathrm{U}(2)$ and S, S', T as in the previous example. The map $H^*(BT) \longrightarrow H^*(BS)$ is again given by the map $\mathbb{Z}[t_1, t_2] \longrightarrow \mathbb{Z}[t_1]$, preserving t_1 and killing t_2. Generators for $H\big(B\mathrm{Sp}(2)\big)$ are c_1, c_2 as in Example 7.6.2, and we have

$$d_1 = t_1 + t_2 \longmapsto t_1,$$
$$q_2 = t_1 t_2 \longmapsto t_1 \cdot 0 = 0.$$

Example 7.6.3 illustrates a general result about the map ρ^* in the event G is semisimple and S a circle, which we will later use in determining the rings $H^*(G/S^1)$.

Lemma 7.6.5. *Let K be a semisimple Lie group containing a circle S. The image of $H_K^* \longrightarrow H_S^* \cong \mathbb{Q}[s]$ contains $s^2 \in H_S^4$.*

Proof. Let T be a maximal torus of K containing S, so that $H_K^* \longrightarrow H_S^*$ factors as $H_T^W \hookrightarrow H_T^* \to H_S^*$, where W is the Weyl group of K. Identifiying $H_T^* = \mathbb{Q}[u_1, \ldots, u_n]$ and $H_S^* = \mathbb{Q}[s]$, the exponents a_j of the inclusion $S^1 \hookrightarrow T \cong (S^1)^n$ give the matrix $[a_1 \ \cdots \ a_n]$ of $H_1(S) = \pi_1(S) \longrightarrow \pi_1(T) = H_1(T)$, and so the transpose is the matrix of $H^1(T) \longrightarrow H^1(S)$, which we can identify with $H_T^2 \longrightarrow H_S^1$. Thus $u_j \longmapsto a_j s$ and $H_T^4 \longrightarrow H_S^4$, takes a homogeneous quadratic polynomial $q(\vec{u})$ in the generators u_j to $q(\vec{a})s^2$.

To show $H^4(BK;\mathbb{Q}) \longrightarrow H^4(BS;\mathbb{Q})$ is surjective is equivalent to showing $H^4(BK;\mathbb{R}) \longrightarrow H^4(BS;\mathbb{R})$ is surjective, and elements of $H^4(BT;\mathbb{R})$ can be seen as quadratic forms on the vector space $H_2(BT;\mathbb{R}) \cong H_1(T;\mathbb{R}) \cong \pi_1 T \otimes \mathbb{R} \cong \mathfrak{t}$. Under this identification, the restriction $H^4(BT;\mathbb{R}) \longrightarrow H^4(BS;\mathbb{R})$ corresponds to restriction of a quadratic form q on $\mathfrak{t}$ to $\mathfrak{s}$. Thus, showing the map $H^4(BT;\mathbb{R})^W \longrightarrow H^4(BS;\mathbb{R})$ is surjective regardless of the choice of the circle S is equivalent to showing that for each tangent line $\mathfrak{s}$ in $\mathfrak{t}$, there is W-invariant quadratic form q not vanishing on $\mathfrak{s}$. In particular, it would more than suffice to find a W-invariant q such that $q(v) \neq 0$ for all $v \neq 0$. But the Killing form $B(-,-)\colon \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathbb{R}$ is an Ad-invariant, negative definite bilinear form by Proposition B.4.13, so its restriction to $\mathfrak{t} \times \mathfrak{t}$ is W-invariant, and its restriction to the diagonal is a W-invariant, quadratic form q on $\mathfrak{t}$ strictly negative on $\mathfrak{t}\backslash\{0\}$. □

Historical remarks 7.6.6. The choice of notation ρ^* for this important map follows historical precedent dating back to the heroic era of large tuples described in Historical remarks 5.3.9. Borel and later Hirzebruch canonically assigned the name $\rho(U, G)$ to the map $BU \longrightarrow BG$ induced by an inclusion $U \longleftrightarrow G$ and $\rho^*(U, G)$ to the resulting map $H^*(BG) \longrightarrow H^*(BU)$ in cohomology.

Chapter 8

The cohomology of homogeneous spaces

In this chapter we finally arrive at our stated goal, to compute the cohomology of a homogenous space G/K in terms of the transitively acting group G and the isotropy subgroup K.

Moreover, the Serre spectral sequence of $G \to EG \to BG$ induces a machine, invented by Borel in his thesis, computing the cohomology of homogeneous spaces G/K. The machine is, in slightly disguised form, the Cartan algebra computing the Borel K-equivariant cohomology of G. This Cartan algebra was one of the motivating examples behind the definition of minimal models, which developed into a central tool of rational homotopy theory in the late 1960s. We use one tool from rational homotopy theory, the algebra of polynomial differential forms, to update Borel's 1953 proof that the Cartan algebra computes the cohomology of a homogeneous space.

The innovation of this chapter is that we are able to present the Cartan algebra and its application in algebraic terms with essentially no use of the Lie algebra of G, of the Lie derivative, or of connections, and without developing rational homotopy theory. Though many sources cover this material in more or less detail [Cen51, And62, Ras69, GHV76, Oni94], all of them rely on Lie-algebraic methods. Rational homotopy theoretic proofs of Cartan's theorem can be found in texts [FHT01, FOT08], as an application of a much more of a general theory we for lack of space do not develop here. In fact, Cartan's theorem was an early instance of and an inspiration for such methods [Hes99].

Now seems like a good time to formalize the setup.

Definition. Let G be a compact, connected Lie group, and K a closed, connected subgroup. In this situation we call (G, K) a ***compact, connected pair*** of Lie groups.

Our discussion will really be about properties of such pairs. Associated to a compact pair (G, K) are three fiber bundles. The first, $K \to G \to G/K$, follows from Theorem B.4.4. The second is the Borel fibration $G \to G_K \to BK$, which is a principal G-bundle. The third is the fibration $G/K \to BK \to BG$, where the projection $\rho \simeq Bi\colon BK \longrightarrow BG$ can be seen as the "further quotient" map $EG/K \twoheadrightarrow EG/G$. Substituting the homotopy quotient G_K for G/K when convenient, we can then see that each three consecutive terms of the sequence

$$K \xrightarrow{i} G \xrightarrow{j} G/K \xrightarrow{\chi} BK \xrightarrow{\rho} BG \tag{8.0.1}$$

form a bundle up to homotopy. Here $\chi\colon gK \longmapsto e_0gK$ is the classifying map of $K \to G \to G/K$ and also the fiber of $BK \longrightarrow BG$ over e_0G, and we are able to substitute G_K in for G/K without changing j or χ up to homotopy by Proposition 5.5.4. This section is devoted to a general discussion of the implications of this fiber sequence in the resulting cohomology sequence

$$H^*(K) \xleftarrow{i^*} H^*(G) \xleftarrow{j^*} H^*(G/K) \xleftarrow{\chi^*} H^*_K \xleftarrow{\rho^*} H^*_G. \tag{8.0.2}$$

It is a curious historical coincidence that the study of the cohomology of homogeneous spaces seems to break into three basic periods, the first studying the Leray spectral sequence of the first three terms, the second studying the Leray–Serre spectral of the second three terms, and last studying the Eilenberg–Moore spectral sequence of the last three terms. It is the second period characterization that we employ in what follows, but these maps will all have some relevance for us.

Remark 8.0.3. We always assume our groups are compact and connected in what follows. Connectedness is essential, but what we say also goes for noncompact Lie groups. [Include this argument.]

8.1. The Borel–Cartan machine

We begin by introducing the device that will carry out our computations.

8.1.1. The fiber sequence

The five terms of (8.0.1), up to homotopy, form the labeled subdiagram in the following diagram of bundle maps, where the columns are bundles.

$$\begin{array}{ccccc}
K & \overset{i}{\hookrightarrow} & G & = & G \\
\downarrow & & \downarrow j & & \downarrow \\
EK & \longrightarrow & G_K & \longrightarrow & EG \\
\downarrow & & \downarrow \chi & & \downarrow \\
BK & = & BK & \xrightarrow{\rho} & BG
\end{array} \tag{8.1.1}$$

Here the middle row should be seen as

$$EK \underset{K}{\otimes} K \longhookrightarrow EG \underset{K}{\otimes} G \longrightarrow\!\!\!\!\rightarrow EG \underset{G}{\otimes} G,$$

the outer terms being homeomorphic to $EG \simeq EK$, and the fiber inclusions from the preceding row given by $g \longmapsto e_0 \otimes g$. The first and last columns are universal bundles and the second column is the Borel fibration. It is clear that $j \circ i$ and $\rho \circ \chi$ are nullhomotopic because they factor through EG. The classifying map $\rho\colon BK \longrightarrow BG$ is explicitly given by $\rho = Bi\colon eK \longmapsto eG$.

The Borel approach ([Bor53, §22]) to understanding the cohomology of $H^*(G/K)$ depends on the G-bundle map between the second two bundles,

$$\begin{array}{ccc}
G & = & G \\
j \downarrow & & \downarrow \\
G_K & \longrightarrow & EG \\
\chi \downarrow & & \downarrow \\
BK & \xrightarrow[\rho]{} & BG.
\end{array} \tag{8.1.2}$$

This bundle map induces a map from the spectral sequence $(\widetilde{E}_r, \tilde{d}_r)$ of the universal bundle, which we now completely understand, to the spectral sequence (E_r, d_r) of the Borel fibration, which we do not. As $G_K \simeq G/K$, the latter sequence converges to $H^*(G/K)$. We write

$$(\psi_r)\colon (E_r, d_r) \longleftarrow (\widetilde{E}_r, \tilde{d}_r)$$

for this map of spectral sequences. Recall from Section 2.6 that these maps $\psi_r \colon \widetilde{E}_r \longrightarrow E_r$ are DGA homomorphisms, meaning $d_r \circ \psi_r = \psi_r \circ \tilde{d}_r$, and each descends from that on the previous page, meaning $\psi_{r+1} = H^*(\psi_r)$. The map $\psi_2 \colon E_2 \longleftarrow \widetilde{E}_2$ between second pages is

$$\rho^* \otimes \mathrm{id}\colon H^*(BK) \otimes H^*(G) \longleftarrow H^*(BG) \otimes H^*(G),$$

where $\mathrm{id}_{H^*(G)}$ is the isomorphism $\widetilde{E}_2^{0,\bullet} \xrightarrow{\sim} E_2^{0,\bullet}$ of the leftmost columns and $\rho^* = (Bi)^* \colon H^*(BK) \longleftarrow H^*(BG)$ is the map $\widetilde{E}_2^{\bullet,0} \to E_2^{\bullet,0}$ of bottom rows.

It is a consequence of the following lemma that the map ρ^* at least largely determines $H^*(G/K)$.

Proposition 8.1.3. *Let G be a compact, connected Lie group whose primitive subspace $PG < H^*(G)$ is concentrated in degree $\leqslant q-1$. Then if $G \to E \to B$ is a principal G-bundle, its SSS collapses at E_{q+1}.*

Proof. Recall that the spectral sequence $(\widetilde{E}_r, \tilde{d}_r)$ of the universal G-bundle collapses at $\widetilde{E}_{q+1} = \widetilde{E}_\infty = \mathbb{Q}$. Because $G \to E \to B$ is principal, it admits a bundle map to the universal bundle, as in (8.1.2) inducing a spectral sequence map $(\psi_r)\colon (\widetilde{E}_r, \tilde{d}_r) \longrightarrow (E_r, d_r)$, which is a cochain map, meaning $d_r \psi_r = \psi_r \tilde{d}_r$. Thus the edge maps $d_r \colon E_r^{0,r-1} \longrightarrow E_r^{r,0}$ all vanish for $r > q$. Now, the d_r also vanish on the bottom row $E_r^{\bullet,0}$ by lacunary considerations, and are antiderivations with respect to an algebra structure on E_r descending from that of $E_2 = H^*(B) \otimes H^*(G)$, so they vanish entirely for $r > q$. □

In particular, since the edge homomorphisms of the universal bundle spectral sequence $(\widetilde{E}_r, \tilde{d}_r)$ are determined entirely composition by an isomorphism $\tau\colon PG \xrightarrow{\sim} Q(BG)$ restricting the transgression, it follows much of the structure of (E_r, d_r) is determined by the composition $\rho^* \circ \tau$. In fact, in the next subsection we will show that this composition *itself yields* a differential d on E_2, the *Cartan differential*, such that $H^*(E_2, d) \cong H^*(G/K)$ and (E_r, d_r) is the filtration spectral sequence associated to the filtered DGA (E_2, d), equipped with the horizontal filtration induced from H_K^*.

[ADD PROOF OF SAMELSON'S 1941 RESULT ABOUT TRANSITIVE ACTIONS ON SPHERES.]

8.1.2. Chevalley's and Cartan's theorems

In this subsection, we prove Cartan's theorem that the complex described above actually determines $H^*(G/K)$ completely. To do so, we will have to briefly invoke a cochain-level description of the situation, and rather than use singular cochains, we compute cohomology with A_{PL}. We only need two features: it is a CDGA and the filtration spectral sequence induced by the filtration $\big(A_{\mathrm{PL}}(X, X^{p-1})\big)$ of a bundle $F \to X \to B$ agrees with the cochain Serre spectral sequence after E_2.

Temporarily taking a step back from homogeneous spaces, consider the universal bundle $G \to EG \to BG$. Lifting indecomposables, which is possible by Proposition A.4.3 since $H^*(BG)$ is a free CGA, the transgression yields a map

$$P(G) \xrightarrow[\tau]{\sim} Q(BG) \hookrightarrow H^*(BG),$$

Since $H^*(BG)$ is also a free CGA, there exists a CGA section $i^*\colon H^*(BG) \longrightarrow A_{\mathrm{PL}}(BG)$, so we can lift τ to $i^*\tau\colon PH^*(G) \longrightarrow A_{\mathrm{PL}}(BG)$.

Now consider a principal G-bundle $G \to E \xrightarrow{\pi} B$. This bundle is classified by some map $\chi\colon B \longrightarrow BG$, inducing a ring map $\chi^*\colon A_{\mathrm{PL}}(BG) \longrightarrow A_{\mathrm{PL}}(B)$, and we can form the composition

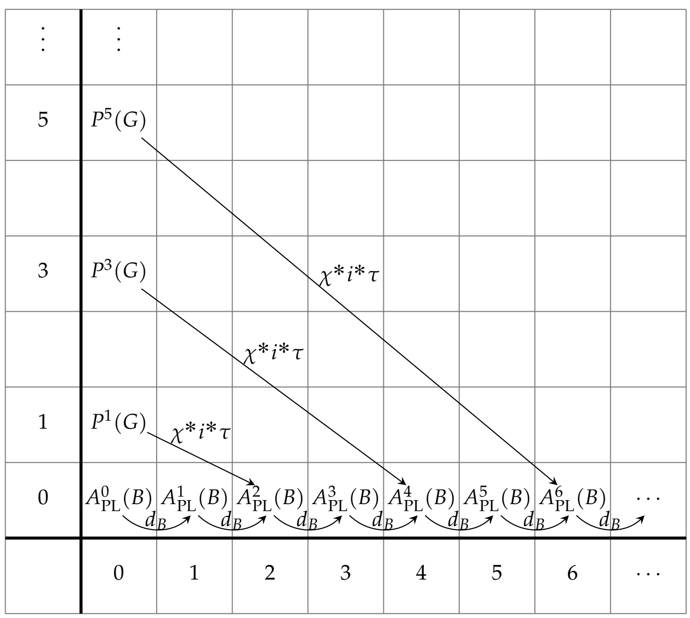
$$\chi^* i^* \tau\colon P(G) \longrightarrow H^*(BG) \longrightarrow A_{\mathrm{PL}}(BG) \longrightarrow A_{\mathrm{PL}}(B)$$

Because $H^*(G) = \Lambda P(G)$ is a free CGA, we can extend this lifted transgression uniquely to an antiderivation on

$$C := A_{\mathrm{PL}}(B) \otimes H^*(G)$$

which we will again call $\chi^* i^* \tau$ and which vanishes on $A_{\mathrm{PL}}(B)$. Similarly, the differential d_B of $A_{\mathrm{PL}}(B)$ extends uniquely to an antiderivation on C annihilating $\mathbb{Q} \otimes H^*(G)$, which we again call d_B. We consider C as a $\mathbb{Q}$-CDGA with respect to the unique differential $d_C := d_B + \chi^* i^* \tau$ extending both d_B and $\chi^* i^* \tau$. See Figure 8.1.4.

Figure 8.1.4: The differential of the algebra $C = A_{\mathrm{PL}}(B) \otimes H^*(G)$ as defined on generators



Let (z_ℓ) be a basis of $P(G)$ and set $\beta_\ell = (\chi^* i^* \tau) z_\ell$ for each ℓ, so that we have

$$d_C(\alpha \otimes 1) = d_B \alpha \otimes 1, \quad \alpha \in A_{\mathrm{PL}}(B);$$
$$d_C(1 \otimes z_\ell) = \beta_\ell \otimes 1.$$

The cochain maps $\big(A_{\mathrm{PL}}(B), d_B\big) \to (C, d_C) \to \big(H^*(G), 0\big)$ induce ring homomorphisms $H^*(B) \to H^*(C) \to H^*(G)$.

Theorem 8.1.5 (Chevalley [Car51][Kos51][Bor53, Thm 24.1, 25.1]). *Let $G \xrightarrow{j} E \xrightarrow{\pi} B$ be a principal G-bundle, and let (C, d_C) be as above. Then there exists an isomorphism $\lambda^*\colon H^*(C, d_C) \xrightarrow{\sim} H^*(E)$ making* *the following diagram commute:*

$$\begin{array}{ccccc} & & H^*(C) & & \\ & \nearrow & \Big\downarrow{\scriptstyle \lambda^*} & \searrow & \\ H^*(B) & & & & H^*(G). \\ & {\scriptstyle \pi^*}\searrow & & \nearrow{\scriptstyle j^*} & \\ & & H^*(E) & & \end{array}$$

Proof. We want to construct a cochain map $\lambda\colon C \longrightarrow A_{\mathrm{PL}}(E)$ into the algebra of polynomial differential forms on E (any CDGA calculating $H^*(E)$ would do), which we will then show to be a quasi-isomorphism by showing it induces an isomorphism between later pages of the associated filtration spectral sequences. The spectral sequence corresponding to $H^*\big(A_{\mathrm{PL}}(E)\big) \cong H^*(E)$ will be the Serre spectral sequence (E_r, d_r) of $G \xrightarrow{j} E \xrightarrow{\pi} B$ with respect to A_{PL} cochains.

Note that by construction and by Corollary 7.4.3 a primitive $z_\ell \in H^{r-1}(G)$ transgresses in E_r to $d_r[z_\ell] = [\beta_\ell]$. By the description in Proposition 2.2.21 of the transgression in the Serre spectral sequence, this means there exists a form $\gamma_\ell \in A_{\mathrm{PL}}(E)$ such that $[j^*\gamma_\ell] = z_\ell \in H^*(G)$ and $d_E\gamma_\ell = \pi^*\beta_\ell \in A_{\mathrm{PL}}(E)$. Define λ on algebra generators by

$$\begin{array}{rcl} \lambda\colon A_{\mathrm{PL}}(B) \otimes H^*(G) & \longrightarrow & A_{\mathrm{PL}}(E), \\ \alpha \otimes 1 & \longmapsto & \pi^*\alpha, \\ 1 \otimes z_\ell & \longmapsto & \gamma_\ell. \end{array} \tag{8.1.6}$$

Then λ is a cochain map by construction, for following through the formulas on generators,

$$d_E\lambda(\alpha \otimes 1) = d_E\pi^*\alpha = \pi^* d_B\alpha = \lambda d_C(\alpha \otimes 1);$$
$$d_E\lambda(1 \otimes z_\ell) = d_E\gamma_\ell = \pi^*\beta_\ell = \lambda d_C(1 \otimes z_\ell).$$

Filter $B = \bigcup B^p$ by its p-skeleta, E by the preimages $\pi^{-1}B^p$ of these, and C and $A_{\mathrm{PL}}(E)$ by

$$F_pC = \bigoplus_{i \geqslant p} A^i_{\mathrm{PL}}(B) \otimes H^*(G), \qquad F_pA_{\mathrm{PL}}(E) = \ker\big(A_{\mathrm{PL}}(E) \longrightarrow A_{\mathrm{PL}}(\pi^{-1}B^{p-1})\big).$$

Then λ preserves filtration degree for elements of $H^*(B)$, which is enough to see that it sends $F_pC \longrightarrow F_pA_{\mathrm{PL}}(E)$.

Write (E_r, d_r) still for the spectral sequence of the filtration on $A_{\mathrm{PL}}(E)$ and $('E_r, 'd_r)$ for that of the filtration on C. The former is just the SSS of $G \to E \to B$ using A_{PL} cochains (Theorem 2.2.2, Proposition 2.2.3),

$$E_2 = H^*(B) \otimes H^*(G).$$

On the other hand, following through the reasoning in Corollary 2.6.8 in this case, $'E_0$ is the associated graded algebra $\operatorname{gr} C \cong C$, and $'d_0$ is the differential induced by $d_C = d_B + \chi^*i^*\tau$. Since $\chi^*i^*\tau$ is induced by the transgression τ, it has filtration degree $\geqslant 2$ on all elements it fails to annihilate outright, and so vanishes under the associated graded algebra construction, and likewise d_B adds one to the filtration degree, so $'d_0 = 0$ and $'E_1 = 'E_0 \cong C$. Thus $'d_1 = d_B$ and

$$'E_2 \cong H^*(B) \otimes H^*(G) \cong E_2.$$

Now that we know these pages can both be identified with $H^*(B) \otimes H^*(G)$ in a natural way, it remains to show $\lambda_2\colon 'E_2 \longrightarrow E_2$ becomes the identity map under these identifications. But this is also the case by construction: the base elements $\alpha \in A_{\mathrm{PL}}(B) \otimes 1$ and $\lambda(\alpha \otimes 1) = \pi^*\alpha \in A_{\mathrm{PL}}(E)$ both become $[\alpha] \otimes 1$ in ${}'E_2 = E_2$ and the fiber elements $1 \otimes z_\ell \in 1 \otimes H^*(G)$ and $\lambda(1 \otimes z_\ell) = \gamma_\ell \in A_{\mathrm{PL}}(E)$ each become $1 \otimes [j^*\gamma_\ell] = 1 \otimes z_\ell$.

Since λ_2 is a cochain isomorphism, it follows from the general principle Proposition 2.7.2 that

$$\lambda^* = H^*(\lambda)\colon H^*\big(A_{\mathrm{PL}}(B) \otimes H^*(G), d_C\big) \longrightarrow H^*(E)$$

is an isomorphism. □

Remark 8.1.7. We are committed to a very classical viewpoint in this work, but those with some grounding in rational homotopy theory might note that $(SQ(BK) \otimes \Lambda PG, d)$ is a *pure Sullivan model*.

Remark 8.1.8. If we are willing to sacrifice multiplicative structure, we can take coefficients in a ring k of arbitrary characteristic, subject only to the condition $H^*(F;k)$ be a free k-module [Hir53].[1] Given a fibration $F \to E \xrightarrow{\pi} B$ with trivial $\pi_1(B)$-action on $H^*(F;k)$, assign to each element y of a basis of $H^*(F;k)$ a representing cocycle in $C^*(F;k)$ and extend this to a cochain $\gamma(y) \in C^*(E;k)$. There is a k-linear map λ', the analogue of λ from (8.1.6), taking $C' = C^*(B;k) \otimes_k H^*(F;k) \longrightarrow C^*(E;\lambda)$ via $b \otimes 1 \longmapsto \pi^*b$ for $b \in C^*(B;k)$ and $1 \otimes y \longmapsto \gamma(y)$. A differential can be defined on C' [Find the article to determine how] such that the obvious filtration induces an isomorphism of $H^*(B;k)$-modules on the E_2 page, so that $H^*(C') \cong H^*(E;k)$ on the level of graded $H^*(B;k)$-modules.

The algebra $C = A_{\mathrm{PL}}(B) \otimes H^*(G)$, although simpler than $A_{\mathrm{PL}}(E)$, still involves the algebra $A_{\mathrm{PL}}(B)$ of polynomial forms on the base B, which though graded-commutative and hence simpler than the algebra of singular cochains on B, is still typically a large ring (if B is a CW complex of positive dimension, then $\dim_{\mathbb{Q}} A_{\mathrm{PL}}(B) \geqslant 2^{\aleph_0}$), which we would rather replace with $H^*(B)$.

The E_2 page of the filtration spectral sequence associated to the filtration induced from the "horizontal" grading on $A_{\mathrm{PL}}(B)$ is the algebra we want, namely $H^*(B) \otimes H^*(G)$ equipped with the differential d_2 vanishing on $H^*(B)$ and sending $z \in PG$ to $(\chi^*\tau)z = \big[(\chi^* i^* \tau)z\big] \in H^{|z|+1}(B)$.

Definition 8.1.9. The algebra $C = H^*(B) \otimes H^*(G)$ equipped with the antiderivation d extending

$$P(G) \xrightarrow{\tau} Q(BG) \hookrightarrow H^*(BG) \xrightarrow{\chi^*} H^*(B)$$

is the ***Cartan algebra*** of the principal bundle $G \to E \to B$.

Remark 8.1.10. Observe that the Cartan algebra of a principal bundle $G \to E \to B$ is the Koszul complex (Definition 7.3.6) of a sequence $\vec{a}$ in $H^*(B)$ of images of generators of $H^*(BG)$ under the characteristic map $\chi^*\colon H^*(BG) \longrightarrow H^*(B)$. This follows because indeed $H^*(BG) = S\big[Q(BG)\big]$ by Borel's Theorem 7.4.1 and $\Lambda PG \otimes SQ(BG)$, equipped with $\tau\colon PG \xrightarrow{\sim} Q(BG)$, is the Koszul complex of PG. In particular, one has the following isomorphism.

Proposition 8.1.11. *Let $G \to E \to B$ be a principal bundle and C its Cartan algebra. Then there is an isomorphism*

$$H^*(C) \cong \operatorname{Tor}^{\bullet,\bullet}_{H^*_G}\big(\mathbb{Q}, H^*(B)\big).$$

Proof. By Remark 8.1.10, C is the Koszul complex of the map $\chi^*\colon H^*(BG) \longrightarrow H^*(B)$, and by Proposition 7.3.10, the cohomology of this complex is $\operatorname{Tor}^{\bullet,\bullet}_{H^*_G}\big(\mathbb{Q}, H^*(B)\big)$. □

[1] Hirsch actually wants k to be a field.

We would like to find a zig-zag of quasi-isomorphisms linking $\big(A_{\mathrm{PL}}(B) \otimes H^*(G), d_C\big)$ with $C = \big(H^*(B) \otimes H^*(G), d\big)$.[2] Recall from Definition 4.1.1 that in this case the space B and the complex $\big(A_{\mathrm{PL}}(B), d_{A_{\mathrm{PL}}(B)}\big)$ are both called *formal*.

Proposition 8.1.12. *If the base B of a principal bundle $G \to E \to B$ is formal, then the Cartan algebra of Definition 8.1.9 computes the cohomology H^*E of the total space.*

Proof. This is an application of the later lemma lemma 8.4.11 to the zig-zag of quasi-isomorphisms connecting $(A, d) = A_{\mathrm{PL}}(B)$ and $H^*(A) = \big(H^*(B), 0\big)$. In the lemma, let $V = P(G)$ and $\xi \colon P(G) \longrightarrow A_{\mathrm{PL}}(B)$ a lifting of

$$P(G) \xrightarrow{\tau} Q(BG) \hookrightarrow H^*(BG) \longrightarrow H^*(B). \qquad \square$$

The ring endomorphism ψ in the above proof can actually be seen to be an automorphism by a filtration argument; if one filters by B-degree, then ψ induces the identity map on the associated graded algebras.

We will be able to use this result later to discuss bundles over formal homogeneous spaces G/K, but the case of critical interest to us, of course, is the Borel fibration $G \longrightarrow G_K \longrightarrow BK$.

Definition 8.1.13. The Cartan algebra of the Borel fibration $G \longrightarrow G_K \longrightarrow BK$, given by $C = H^*(BK) \otimes H^*(G)$ equipped with antiderivation d extending $\rho^* \circ \tau \colon P(G) \longrightarrow Q(BG) \longrightarrow H^*(BK)$, is the ***Cartan algebra of the pair*** (G, K).

The key theorem, due to Cartan, is that the Cartan algebra of a compact pair (G, K) does indeed compute $H^*(G/K)$.

Theorem 8.1.14 (Cartan, [Car51, Thm. 5, p. 216][Bor53, Thm. 25.2])**.** *Given a compact pair (G, K), there is an isomorphism $H^*\big(H^*(BK) \otimes H^*(G)\big) \xrightarrow{\sim} H^*(G/K)$ making the following diagram commute:*

$$\begin{array}{ccccc} & & H^*\big(H^*(BK)\otimes H^*(G)\big) & & \\ & \nearrow & \downarrow & \searrow & \\ H^*(BK) & & \downarrow & & H^*(G) \\ & \searrow^{\chi^*} & \downarrow & \nearrow_{j^*} & \\ & & H^*(G/K) & & \end{array} \tag{8.1.15}$$

Proof. Because $H^*(BK) \cong S\big[Q(BG)\big]$ is a free CGA, it is formal and Proposition 8.1.12 applies.

To avoid use of Lemma 8.4.11 in full generality, note that picking generators for $H^*(BK)$ defines a CDGA quasi-isomorphism $H^*(BK) \longrightarrow A_{\mathrm{PL}}(BK)$ and apply the spectral sequence of the filtration with respect to the grading of $A_{\mathrm{PL}}(BK)$ to the induced CDGA map $H^*(BK) \otimes H^*(G) \longrightarrow A_{\mathrm{PL}}(BK) \otimes H^*(G)$ to get an isomorphism on E_2 pages. $\square$

Remark 8.1.16. If B is not formal, the Cartan algebra of a bundle can indeed fail to compute the cohomology of the total space. For an example of this phenomenon, see Section 3 of Baum and Smith [BS67, p. 178].

[2] There exists a single quasi-isomorphism $\big(H^*(BG), 0\big) \longrightarrow \big(A_{\mathrm{PL}}(BG), d_{A_{\mathrm{PL}}(BG)}\big)$, but for general B, a chain of quasi-isomorphisms is required.

Corollary 8.1.17. *There is an isomorphism*

$$H^*(G/K) \cong \operatorname{Tor}^{\bullet,\bullet}_{H^*(BG)}\big(\mathbb{Q}, H^*(BK)\big).$$

Proof. By Theorem 8.1.14 and Proposition 8.1.11, $H^*(G/K) \cong H^*(C) \cong \operatorname{Tor}^{\bullet,\bullet}_{H^*(BG)}\big(\mathbb{Q}, H^*(B)\big)$. □

Remark 8.1.18. If we set $K = G$, this statement makes precise our motivating claim in the introduction to Section 7.3 that the differentials in the SSS of the universal bundle $G \to EG \to BG$ filter an antiderivation τ extending the transgression which can be seen as the "one true differential." In the same way, the SSS of the Borel fibration $G \to G_K \to BK$ filters the differential on the Cartan algebra. This does not make this SSS, which we have already exploited to such effect, any less valuable: we will see examples in the next section where the Cartan algebra is unpleasantly complicated and it behooves us to look at the associated graded algebra $E_\infty = \operatorname{gr} H^*(G/K)$ instead. Moreover, in precisely the complement of this "bad" case, the associated graded construction is an isomorphism, so that the SSS of the Borel fibration calculates $H^*(G/K)$ on the algebra level. Rather than one description being more powerful, it is the *equivalence* of these two descriptions that turns out to be critical.

Remark 8.1.19. It is only fair to say at one point why we insist so fervently that K be connected. The main issue is that if K is not connected, then BK will not be simply-connected, and the Serre spectral sequence of the Borel fibration is calculated with local coefficients. One can still say some things, for if $K_0 < K$ is the identity component, then $BK_0 \longrightarrow BK$ and $G/K_0 \longrightarrow G/K$ are finite coverings, so if $|\pi_0 K|$ is invertible in k, one can embed $H^*(G/K)$ as the $\pi_0 K$-invariants of $H^*(G/K_0)$ by Proposition B.2.1 and likewise H^*_K as the $\pi_0 K$-invariants of $H^*_{K_0}$.

That G be connected, on the other hand, is not a real restriction if we insist K be connected, for then K will lie in the identity component G_0 of G and G/K will factor homeomorphically as $\pi_0 G \times G_0/K$, a finite disjoint union of copies of G_0/K.

Historical remarks 8.1.20. The original, unpublished statement of Chevalley's theorem [Kos51, p. 70][Bor53, p. 183][Car51, p. 61], as best the author can tell, applied to the de Rham cohomology of a smooth principal G-bundle with compact total space. This statement is cited by Cartan and Koszul both (without proof) in the *Colloque* proceedings. Borel's generalization of this result, as proved in his thesis, removes the smoothness hypotheses by relying, instead of on forms, on an object of Leray's creation known as a *couverture*, which was superseded so quickly and so thoroughly by the ring of global sections of a fine $\mathbb{R}$-CDGA resolution of the constant sheaf $\underline{\mathbb{R}}$ that it never acquired an English translation. Borel's statement of the result still requires compactness of the base because it relies on (what is essentially) sheaf cohomology with compact supports and a result of Cartan which in modern terms can be interpreted as saying a resolution of the constant sheaf $\underline{\mathbb{R}}$ on a paracompact Hausdorff space by a fine sheaf of $\mathbb{R}$-CDGAs always exists. Neither the principal bundle $G \to EG \to BG$ nor a $\mathbb{Q}$-CDGA model of cohomology was available to Borel at the time, so in his statement [Bor53, Thm 24.1] of Chevalley's theorem, our $H^*(B)$ is replaced with (essentially, again) a fine resolution $\mathscr{B}$ of the real constant sheaf on B.

As we have noted in Historical remarks 7.4.8, the unavailability of BK available, complicated Borel's proof, which hence needed to invoke n-universal K-bundles $E(n,K) \longrightarrow B(n,K)$ for n sufficiently large. Borel's proof also applied not the Serre spectral sequence as we do, but the *Leray spectral sequence*, applied simultaneously to an early formulation of a sheaf and a *couverture*. We will reproduce a less drastic modernization of Borel's original argument in Appendix C.3, and delve slightly further there into the meaning of the Leray spectral sequence, fine sheaves, and *couvertures*.

8.2. The structure of the Cartan algebra, I

The Cartan algebra makes a few results on $H^*(G/K)$ easy which would require more sophistication if attacked with the map of spectral sequences that was the subject of Section 8.1.1. We reproduce here the important bundle diagram (8.1.2) whose induced spectral sequence map the Cartan algebra encodes.

$$\begin{array}{ccc} G & = & G \\ {\scriptstyle j}\downarrow & & \downarrow \\ G_K & \longrightarrow & EG \\ {\scriptstyle \chi}\downarrow & & \downarrow \\ BK & \xrightarrow[\rho]{} & BG. \end{array}$$

One important subobject of the Cartan algebra is related to the image of the map $j^*\colon H^*(G/K) \longrightarrow H^*(G)$ induced by $j\colon G \longrightarrow\!\!\!\!\rightarrow G/K \simeq G_K$.

Definition 8.2.1. The image of $j^*\colon H^*(G/K) \longrightarrow H^*(G)$ is called the ***Samelson subring*** of $H^*(G)$. It meets the primitives $PH^*(G) \leqslant H^*(G)$ in the ***Samelson subspace*** $\widehat{P}$.

The importance of the Samelson subspace is that in fact it generates $\operatorname{im} j^*$.

Proposition 8.2.2. *The Samelson subring is the exterior algebra* $\Lambda\widehat{P}$.

Proof (Borel [Bor53, Prop. 21.1, p. 179]). By definition $\widehat{P} \leqslant \operatorname{im} j^*$, so that $\Lambda\widehat{P} \leqslant \operatorname{im} j^*$, and we want to see the reverse inclusion. Because primitives are involved, we will need the coproduct on $H^*(G)$. Recall that the left translation action of G on G/K descends from the multiplication of G, in the sense that the left diagram below commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle \mathrm{id}\times j}\downarrow & & \downarrow{\scriptstyle j} \\ G \times G/K & \xrightarrow{\psi} & G/K, \end{array} \qquad \begin{array}{ccc} H^*(G)\otimes H^*(G) & \xleftarrow{\mu^*} & H^*(G) \\ {\scriptstyle \mathrm{id}\otimes j^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ H^*(G)\otimes H^*(G/K) & \xleftarrow{\psi^*} & H^*(G/K). \end{array}$$

The right diagram is that induced in cohomology, applying the Künneth theorem and assuming the torsion of G is invertible in k. From commutativity of the diagram

Suppose that we have shown the inclusion in degrees less than that of $y \in H^*(G/K)$. Fix an ordered basis $(z_i)_{i=1}^{\operatorname{rk} G}$ of $PH^*(G)$, so that monomials $z^I = \prod_{i\in I} z_i$ for $I \subseteq \{1,\ldots,\operatorname{rk} G\}$ form a basis of $H^*(G)$. Then we can write $p^*(y) = ax + \sum b_K z^K$ with $x \in PH^*(G)$ and $a, b_K \in k$, and

$$(\mathrm{id}\otimes j^*)\psi^*(y) = \mu^* p^*(y) = a(1\otimes x + x\otimes 1) + \sum_K \sum_{I\amalg J=K} \pm b_K x^I \otimes x^J$$

in the resulting basis for $H^*(G)\otimes H^*(G)$. In particular, for each K with $b_K \neq 0$, and each $i \in K$ the sum contains the term $\pm b_K z^{K\setminus\{i\}} \otimes z_i \in H^*(G)\otimes H^*(G)$, which implies that $z_i \in \widehat{P} = PH^*(G) \cap \operatorname{im} j^*$. Thus $\sum b_K z^K \in \operatorname{im} j^*$, so $ax = p^*(y) - \sum b_K z^K \in \operatorname{im} j^*$. But x was assumed primitive, so $x \in \widehat{P}$ and $p^*(y) \in \Lambda\widehat{P}$. ☐

Proposition 8.2.3. *If $H_K^{\geqslant 1}$ denotes the augmentation ideal of H_K^*, then one has* $\widehat{P} = d^{-1}\big(H_K^{\geqslant 1} \cdot \operatorname{im} d\big)$.

Proof. By construction, the Serre spectral sequence of $G \to G_K \to BK$ is the filtration spectral sequence of the Cartan algebra $(H_K^* \otimes H^*G, d)$ with respect to the grading of H_K^*. Elements z of $\widehat{P}^{r-1} = P^{r-1}H^*G \cap \operatorname{im} j^*$ are represented by elements $1 \otimes z \in E_2^{0,\bullet}$ which survive to E_∞, meaning all differentials vanish on the class of $1 \otimes z$. This means that the image under the Cartan differential, $x \otimes 1 = d(1 \otimes z) \in H_K^* \otimes \mathbb{Q}$, represents zero in the quotient $E_r^{\bullet,0}$, or in other words lies in the kernel of the quotient map $H_K^* \otimes \mathbb{Q} = E_2^{\bullet,0} \twoheadrightarrow E_r^{\bullet,0}$. This kernel is, by induction, the ideal generated by the lifts to E_2 of the images of previous transgressions $d_i \colon E_i^{0,i-1} \longrightarrow E_i^{i,0}$. Since these transgressions lie in degree $< r$, it follows $dz \in H_K^{\geqslant 1} \cdot \operatorname{im} d$.

On the other hand, if $dz \in H_K^{\geqslant 1} \cdot \operatorname{im} d$, say $dz = \sum a_j \, dz_j$ with $a_j \in H_K^{\geqslant 1}$ and $z_j \in PH^*(G)$, then $|z_j| = |z| - |a_j| < |z|$, so $E_{|z|}^{\bullet,0}$ is a quotient of $H_K^*/(dz_j)$ and particularly $dz \otimes 1$ represents 0 in E_r, meaning $1 \otimes z$ survives to E_∞ in the filtration spectral sequence and $z \in \widehat{P}$. □

The Samelson subring is in fact a tensor factor of $H^*(G/K)$.

Definition 8.2.4. Let (G, K) be a compact pair. We write $\check{P} := PG/\widehat{P}$, and call this the ***Samelson complement***; the notation is supposed to indicate its complementarity to $\widehat{P}$.

Proposition 8.2.5. *The Cartan algebra admits a coproduct decomposition*

$$(H_K^* \otimes \Lambda PG, d) \cong \big(H_K^* \otimes \Lambda \check{P}, d\big) \otimes (\Lambda \widehat{P}, 0).$$

The proof is just what one would naively hope; we paraphrase from Greub *et al* [GHV76, 3.15 Thm. V, p. 116].

Proof. Choose some $\mathbb{Q}$-linear section

$$\widehat{P} \longrightarrow \ker d \leqslant H_K^* \otimes \Lambda PG$$

of the column projection $\ker d \twoheadrightarrow H^*(G/K) \xrightarrow{j^*} H^*(G)$. This section extends uniquely to a ring injection $f \colon \Lambda \widehat{P} \longrightarrow \ker d$ which we can extend further to a ring map

$$\begin{aligned} (H_K^* \otimes \Lambda \check{P}) \otimes \Lambda \widehat{P} &\longrightarrow H_K^* \otimes \Lambda PG \\ (a \otimes \check{z}) \otimes \hat{z} &\longmapsto (a \otimes \check{z}) \cdot f(\hat{z}). \end{aligned}$$

This ring map is also a cochain map, since it is the identity on the first tensor-factor of its domain and since for $\hat{z} \in \Lambda \widehat{P}$ we have $0 = d(f\hat{z}) = f\big(0(\hat{z})\big)$.

It remains to see f is bijective. Note that f is the identity on $HK \otimes \Lambda \check{P}$ and that given an element $z \in \widehat{P}$, since f is defined to be a section of the projection to the leftmost column, we have $f(z) \equiv 1 \otimes z \pmod{H_K^{\geqslant 1}}$. Thus f preserves the the horizontal filtration induced by the filtration $F_p H_K^* = \bigoplus_{i \geqslant p} H_K^p$ on the base H_K^* and induces an isomorphism $\operatorname{gr}_\bullet f$ on associated graded algebras. By Proposition 2.7.2, f is an isomorphism. □

Corollary 8.2.6. *Let (G, K) be a compact pair. Then there exists a tensor decomposition*

$$H^*(G/K) \cong H^*(H_K^* \otimes \Lambda \check{P}, d) \otimes \Lambda \widehat{P},$$

where the subring $\Lambda \widehat{P} = \operatorname{im} j^* \leqslant H^*(G)$ is induced from the projection $j \colon G \longrightarrow G/K$.

We write the first factor as J.

Corollary 8.2.7. *The factor J satisfies Poincaré duality.*

Proof. Since G/K is a compact manifold, $H^*(G/K)$ is a PDA by Theorem A.2.10, and the exterior algebra $\Lambda\widehat{P}$ is a PDA, so by Proposition A.2.12, so also must be the remaining factor J. □

The same way that $\operatorname{im} j^*$ admits a description as the leftmost column of E_∞ for the SSS of $G \to G_K \to BK$, so also the image of χ^* admits a description as the bottom row $E_\infty^{\bullet,0}$.

Definition 8.2.8. The map $\chi^*\colon H_K^* \longrightarrow H^*(G/K)$ is traditionally called the ***characteristic map*** and $\operatorname{im}\chi^* \cong H_K^* /\!\!/ H_G^*$ the ***characteristic subring*** of the pair (G, K). The factor $J = H^*\big(H_K^* \otimes \Lambda\check{P}, d\big)$ of $H^*(G/K)$ in the decomposition Corollary 8.2.6 is called the ***characteristic factor***.

The name *characteristic subring* arises because, up to homotopy, the classifiying map $G/K \longrightarrow BK$ of the principal K-bundle $K \to G \to G/K$ is the projection $\chi\colon G_K \longrightarrow BK$ of the Borel fibration (see (8.0.1)), and the characteristic classes of the former K-bundle bundle lie in $\operatorname{im}\chi^*$. The *characteristic factor* is so called because $H_K^* \longleftrightarrow H_K^* \otimes H^*(G)$ factors through $H_K^* \otimes \Lambda\check{P}$, making clear the following containment.

Proposition 8.2.9. *The characteristic ring* $\operatorname{im}\chi^*$ *is contained in the characteristic factor J.*

The cohomology sequence (8.0.2) is coexact at H_K^*, yielding the following pleasing description of the characteristic subring.

Proposition 8.2.10. *The characteristic subring is given by* $\operatorname{im}\chi^* \cong H_K^* /\!\!/ H_G^*$.

Proof. The bottom row H_K^* lies in the kernel of the Cartan differential d_C, and meets the image $\operatorname{im} d_C$ in the ideal $\mathfrak{j}$ generated by $\rho^*(\operatorname{im}\tau)$. Since $\tau\colon P(G) \xrightarrow{\sim} Q(BG)$ surjects onto generators of H_G^*, it follows that the ideal $\mathfrak{j}$ which is the kernel of $H_K^* \longrightarrow H^*\big(H_K^* \otimes H^*(G)\big)$ is generated by the image $\rho^* H_G^{\geqslant 1}$ of the augmentation ideal, so this image is $H_K^*/(\rho^* H_G^{\geqslant 1}) = H_K^* /\!\!/ H_G^*$, the ring-theoretic cokernel. By the commutativity of the diagram (8.1.15), this image subalgebra corresponds to $\operatorname{im}\chi^*$ in $H^*(G/K)$. □

This information is already enough to compute $H^*(G/K)$ in many cases of interest.

8.3. Cohomology computations, I

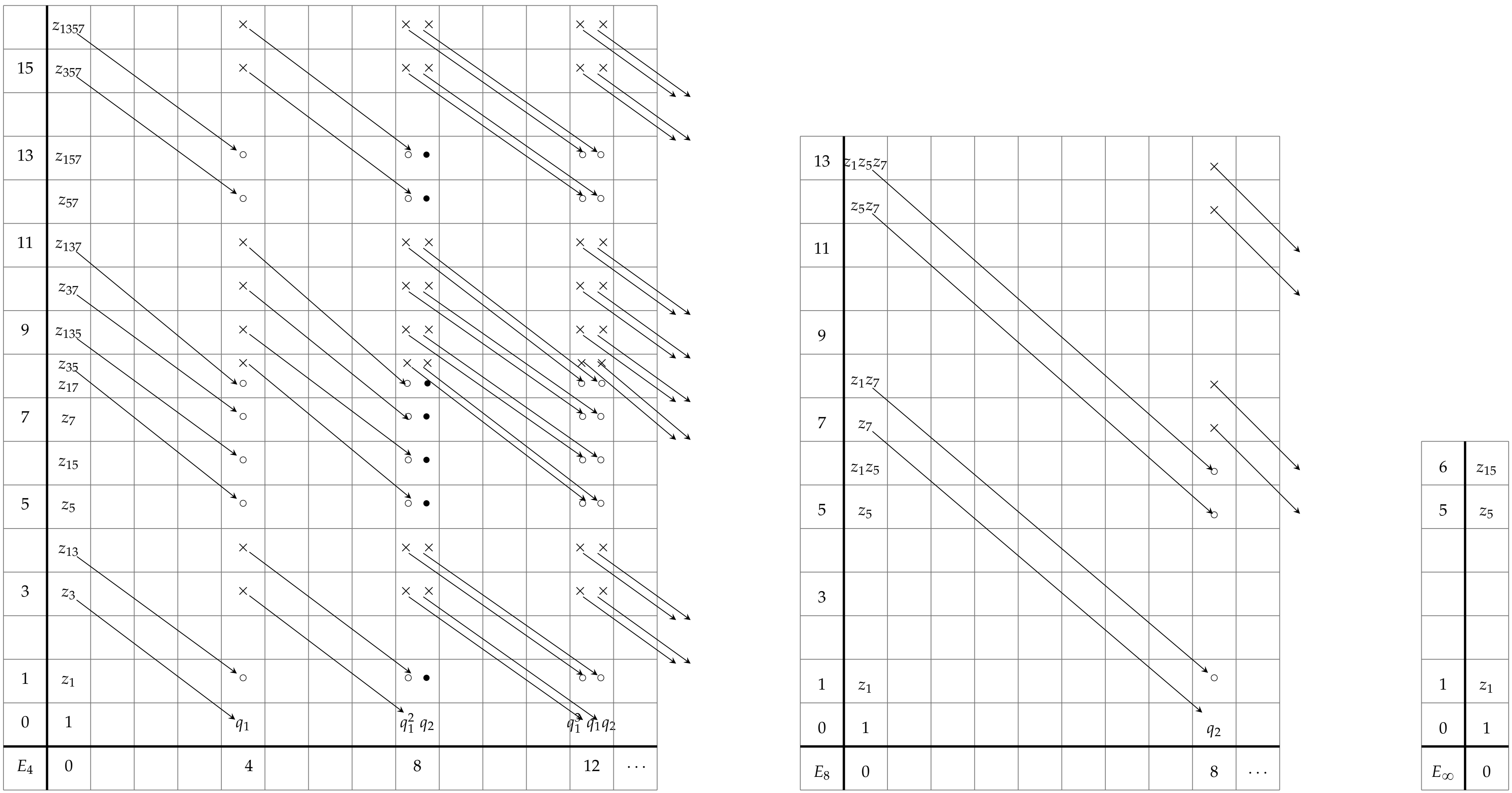
Lest we miss the trees for the forest in fleshing out our general description of the Cartan algebra, we take a detour to describe the cohomology of two popular classes of homogeneous spaces G/K, namely those for which $H^*(G) \to H^*(K)$ is surjective and those for which $\operatorname{rk} G = \operatorname{rk} K$.

8.3.1. Cohomology-surjective pairs

The map (8.1.2) of spectral sequences lets us easily reobtain Hans Samelson's classic theorem that $H^*(G) \cong H^*(K) \otimes H^*(G/K)$ whenever $H^*(G) \twoheadrightarrow H^*(K)$. Pictorially, this means the Serre spectral sequence of $G \to G_K \to BK$ looks like that of $\mathrm{U}(4) \to \mathrm{U}(4)_{\mathrm{Sp}(2)} \to B\mathrm{Sp}(2)$, as pictured in Figure 8.3.3; for now, just look at the E_∞ page, on the right.

Definition 8.3.1. If (G, K) is a compact pair such that $K \hookrightarrow G$ induces a surjection $H^*(G) \longrightarrow H^*(K)$ in cohomology, we call (G, K) a ***cohomology-surjective pair***.

Figure 8.3.3: The Serre spectral sequence of $U(4) \to U(4)_{Sp(2)} \to BSp(2)$; nonzero differentials (shown) send $\times \mapsto \circ$, whereas $\bullet$s survive to the next page



Theorem 8.3.2 (Samelson [Sam41, Satz VI(b), p. 1134]). *Suppose that (G, K) is a cohomology-surjective pair. Then*

1. $\rho^*\colon H_G^* \longrightarrow H_K^*$ *is surjective,*
2. $\chi^*\colon H_K^* \longrightarrow H^*(G/K)$ *is trivial,*
3. *the Samelson subspace $\widehat{P}$ is complementary to $P(K)$ in $P(G)$,*
4. *$H^*(G/K)$ is the exterior algebra $\Lambda\widehat{P} \cong \Lambda P(G) /\!/ \Lambda P(K)$, and*
5. $H^*(G) \cong H^*(K) \otimes H^*(G/K)$.
6. *If the Poincaré polynomials of PG and PK are respectively $p(PG) = \sum_{j=1}^n t^{d_j}$ and $p(PK) = \sum_{j=1}^{\ell} t^{d_j}$, then $p(G/K) = \prod_{j=\ell+1}^n (1 + t^{d_j})$.*

Proof. By Proposition 1.0.11 the fact $i\colon K \longleftrightarrow G$ is a group homomorphism implies $i^*\colon H^*(G) \twoheadrightarrow H^*(K)$ takes the primitives $P(G) \longrightarrow P(K)$. Because we have assumed i^* surjective, it follows $i^*P(G) = P(K)$ and because i^* is a ring homomorphism that $\ker i^* \cong \Lambda\big[P(G)/P(K)\big]$.

The outer columns of (8.1.1) are a bundle map between the universal principal K- and G-bundles, inducing a map of Serre sequences interleaving the transgressions. Restricting to primitives, one has the commutative diagram

$$\begin{array}{ccc} P(K) & \xleftarrow{\ i^*\ } & P(G) \\ \wr\downarrow \tau_K & & \tau_G \downarrow\wr \\ Q(BK) & \xleftarrow[\ Q\rho^*\ ]{} & Q(BG), \end{array} \qquad (8.3.4)$$

which implies that $Q(\rho^*)Q(BG) = Q(BK)$ and hence that $\rho^*\colon H^*(BG) \longrightarrow H^*(BK)$ is also surjective. It follows from the triviality of $\chi^* \circ \rho^*$ that the characteristic subring $\operatorname{im}\big(\chi*\colon H_K^* \longrightarrow H^*(G/K)\big)$ is $\mathbb{Q}$.

If we embed $P(K) \rightarrowtail P(G)$ by taking a section of i^*, we see from the transgression square (8.3.4) that the complement of $P(K)$ is annihilated by $\rho^* \circ \tau_G$, so that the Samelson subspace $\widehat{P} \leqslant P(G)$ is a complement to $P(K)$, or $\widehat{P} \cong P(G)/P(K)$.

Because $\rho^* \circ \tau$ ends $P(K)$ onto $Q(BK)$ and annihilates $\widehat{P}$, we have a ring factorization of $E_2 \cong H^*(BK) \otimes H^*(G)$ as

$$\big[H^*(BK) \otimes H^*(K)\big] \otimes \Lambda\widehat{P},$$

which respects the transgression in that all differentials are trivial on $\widehat{P}$, and the left tensor factor is the beginning of the filtration spectral sequence corresponding to the Koszul complex on $Q(BK)$ (cf. Proposition 7.4.2). It follows $E_\infty = E_\infty^{0,\bullet} \cong \Lambda\widehat{P}$. Thus we can identify the short coexact sequence $H^*(K) \underset{i^*}{\leftarrow} H^*(G) \underset{j^*}{\leftarrow} H^*(G/K)$ with

$$0 \leftarrow \Lambda P(K) \longleftarrow \Lambda\big[P(K) \oplus \widehat{P}\big] \longleftarrow \Lambda\widehat{P} \leftarrow 0;$$

the tensor factorization is valid simply because by Proposition A.4.3 the free CGA $\Lambda P(K)$ is projective.

The result on Poincaré polynomials follows from the statements in Appendix A.2.3, since $p(\Lambda PG) = \prod_{j=1}^n (1 + t^{d_i})$ and $p(\Lambda PK) = \prod_{j=1}^{\ell} (1 + t^{d_i})$. □

Remarks 8.3.5. (*a*) With the benefit of hindsight, our calculations of the cohomology rings of $\mathrm{SU}(n)$ in Proposition 3.1.6 and of $V_j(\mathbb{C}^n)$ and $V_j(\mathbb{H}^n)$ in Proposition 3.1.8 can all be seen to be of this form.

(*b*) The Samelson isomorphism $H^*(G) \cong H^*(G/K) \otimes H^*(K)$ also follows directly from Corollary 1.0.7 independent of any consideration of classifying spaces.

[Introduce minimal models here.]

Proposition 8.3.6 ([Car51, 1°, p. 69][Bor53, Corollaire, p. 179])**.** *Let $i\colon K \hookrightarrow G$ be an inclusion of compact, connected Lie groups. Then $\rho^*\colon H^*_G \longrightarrow H^*_K$ is surjective if and only if $i^*\colon H^*(G) \longrightarrow H^*(K)$ is.*

Proof. This follows immediately from the commutative square (8.3.4) in the proof of Theorem 8.3.2 since the vertical maps are isomorphisms. □

Most of these conditions are clearly equivalent. In fact, a weaker dimension condition on $H^*(G/K)$ is equivalent to cohomology-surjectivity.

Proposition 8.3.7 ([GHV76, Thm. 10.19.X(6) p. 466])**.** *Let (G,K) be a compact pair. One has*

$$h^\bullet(G) \leqslant h^\bullet(G/K) \cdot h^\bullet(K),$$

with equality if and only if (G,K) is cohomology-surjective.

Proof [GHV76, Cor. to Thm. 3.18.V, p. 125]. This follows from Corollary 2.3.5 as applied to the Serre spectral sequence of $K \to G \to G/K$, evaluating the Poincaré polynomials at $t = 1$. □

Example 8.3.8. Recall from Example 7.6.2 that $H^*\big(B\mathrm{U}(4)\big) \longrightarrow H^*\big(B\mathrm{Sp}(2)\big)$ is surjective. From Proposition 8.3.6, we see as well that $H^*\big(\mathrm{U}(4)\big) \longrightarrow H^*\big(\mathrm{Sp}(2)\big)$, as promised. We had

$$\begin{aligned} c_1 &\longmapsto 0, \\ c_2 &\longmapsto -q_1, \\ c_3 &\longmapsto 0, \\ c_4 &\longmapsto q_2, \end{aligned}$$

so in the primitive subspace $P\big(\mathrm{U}(4)\big) = \mathbb{Q}\{z_1, z_3, z_5, z_7\}$ we have $P\mathrm{Sp}(2) = \mathbb{Q}z_3 \oplus \mathbb{Q}z_7$ and $\widehat{P} = \mathbb{Q}z_1 \oplus \mathbb{Q}z_5$. It follows from Section 8.3.1 that

$$H^*\big(\mathrm{U}(4)/\mathrm{Sp}(2)\big) \cong \Lambda[z_1, z_5], \quad \deg z_j = j.$$

The resulting spectral sequence, Figure 8.3.3, appears complicated, but this complexity is only apparent. Staring closely at the picture, one sees that $\Lambda\widehat{P} = \Lambda[z_1, z_5]$ is a tensor-factor, to which nothing ever happens, and the massive simplifications after the 4^{th} and 8^{th} pages just witness that the Koszul complexes $K[z_3]$ and $K[z_7]$ are other tensor-factors.

Alternately, not bothering with the picture, the transgression in the universal principal $\mathrm{U}(4)$-bundle takes $z_1 \longmapsto c_1$ and $z_5 \longmapsto c_3$, this means that $\Lambda\widehat{P} = \Lambda[z_1, z_5]$ splits off in the Cartan algebra immediately, and $S[q_1, q_2] \otimes \Lambda[z_3, z_7]$ is a Koszul complex, so acyclic.

A little more work shows that $H^*_{\mathrm{U}(2n)} \longrightarrow H^*_{\mathrm{Sp}(n)}$ is surjective for all n with kernel the odd Chern classes, and it follows

$$H^*\big(\mathrm{U}(2n)/\mathrm{Sp}(n)\big) \cong \Lambda[z_1, \ldots, z_{4n-3}], \quad \deg z_j = j.$$

As an example application of Samelson's theorem, we prove a result which will be of use to us later in investigating equivariant formality of isotropy actions.

Lemma 8.3.9. *Let S be a torus in a compact, connected Lie group G and $Z = Z_G(S)$ its centralizer in Z. The cohomology of Z decomposes as*

$$H^*(Z) \cong H^*(S) \otimes H^*(Z/S).$$

Consequently, $H^*(Z/S)$ is an exterior algebra on $\operatorname{rk} G - \operatorname{rk} S$ generators and $h^\bullet(Z/S) = 2^{\operatorname{rk} G - \operatorname{rk} S}$.

Proof. By Theorem 8.3.2, it will be enough to show the inclusion $S \hookrightarrow Z$ surjects in cohomology. Since S is normal in Z, the quotient Z/S is another Lie group, so $\pi_2(Z/S) = 0$ by Corollary 1.0.12 and in the long exact homotopy sequence (Theorem B.1.4) of the bundle $S \to Z \to Z/S$ we find the fragment $0 = \pi_2(Z/S) \to \pi_1 S \to \pi_1 Z$. Since S and Z are topological groups, their fundamental groups are abelian by Proposition B.4.3 and hence isomorphic to their first homology groups by Proposition B.1.5, so $H_1(S;\mathbb{Z}) \longrightarrow H_1(Z;\mathbb{Z})$ is injective. It follows from Theorem B.1.1 that $H_1(S;\mathbb{Q}) \longrightarrow H_1(Z;\mathbb{Q})$ is injective, and, dualizing, that $H^1(Z;\mathbb{Q}) \longrightarrow H^1(S;\mathbb{Q})$ is surjective. Since $H^1(S)$ generates $H^*(S)$, it must be that $H^*(Z) \longrightarrow H^*(S)$ is surjective as well.

The statement on Betti number follows because Z must have the same rank as G, since S is contained in some maximal torus of G by Theorem B.4.11. □

Historical remarks 8.3.10. Proposition 8.3.6 was first proven by Cartan [Car51, 1°, p. 69][Bor53, Corollaire, p. 179].

A surjection $H^*(G) \longrightarrow H^*(K)$ in cohomology corresponds dually to an injection $H_*(K) \longrightarrow H_*(G)$ in homology, and it was this condition Hans Samelson researched in the work in which the tensor decomposition Theorem 8.3.2.5 above was first proven [Sam41]. It has since been said that K is *totally nonhomologous to zero* in G. Samelson said the *Isotropiegruppe U nicht homolog in der Gruppe G ist* or $U \not\sim 0$, the letter U for *Untergruppe* (our K), and showed if the fundamental class $[K] \in H_*(K)$ did not become zero in $H_*(G)$, then $H_*(K) \longrightarrow H_*(G)$: the fundamental class $[K] \in H_*(K;\mathbb{Q}) \cong \Lambda(PK)^*$ is the product of a set of algebra generators, so if $\rho_*[K] \neq 0$ in $H_*(G)$, then ρ_* is injective. The "totally" is redundant and sometimes dropped for this reason.

When the cohomology ring rather than the homology ring became the primary actor, later expositors named the condition, by analogy, *totally noncohomologous to zero*, though that name taken literally would imply the surjection $H^*(G) \longrightarrow H^*(K)$ should be injective. These conditions have been abbreviated variously *TNHZ*, *TNCZ*, and *n.c.z.* For safety's sake, in dealing with this situation we will always simply say a map surjects in cohomology.

8.3.2. Pairs of equal rank

We recast some of the results from Chapter 6 in this framework.

Definition 8.3.11. A compact, connected pair (G, K) is an ***equal-rank pair*** if $\operatorname{rk} G = \operatorname{rk} K$.

Theorem 8.3.12 (Leray)**.** *Let (G, K) be an equal-rank pair. Then*

1. *$\rho^* \colon H^*_G \longrightarrow H^*_K$ is injective,*
2. *$\chi^* \colon H^*_K \longrightarrow H^*(G/K)$ is surjective,*
3. *the Samelson subspace $\widehat{P}$ is trivial,*

4. *$H^*(G/K) \cong H_K \,/\!/\, H_G^* \cong H_T^{W_K} \,/\!/\, H_T^{W_G}$.*

5. *If the Poincaré polynomials of $PH^*(G)$ and $PH^*(K)$ are respectively given by $p(PG) = \sum_{j=1}^n t_j^{2g_j-1}$ and $p(PK) = \sum_{j=1}^n t_j^{2k_j-1}$, then the Poincaré polynomial of G/K is*

$$p(G/K) = \frac{p(BK)}{p(BG)} = \prod_{j=1}^{n} \frac{1-t^{2k_j}}{1-t^{2g_j}}. \tag{8.3.13}$$

Proof. The inclusion $K \hookrightarrow G$ induces an injection of Weyl groups $W_K \rightarrowtail W_G$ and in turn an inclusion $H_T^{W_G} \hookrightarrow H_T^{W_K} \hookrightarrow H_T^*$ of Weyl invariants. Recalling from Corollary 6.3.7 that $H_G^* \cong H_T^{W_G}$, this means $\rho^* \colon H_G^* \longrightarrow H_K^*$ are injective.[3] Since the transgression $\tau \colon PG \xrightarrow{\sim} Q(BG)$ is also injective, the composition $\rho^* \circ \tau \colon PG \longrightarrow H_K^*$ is as well, so its kernel $\widehat{P}$ is 0. The injectivity of ρ^* combined with the fact $\operatorname{im}\chi^* \cong H_K^* \,/\!/\, H_G^*$ means $H_K^* \cong H_G^* \otimes \operatorname{im}\chi^*$ as an H_G^*-module, so the Cartan algebra $H^*(BK) \otimes H^*(G)$ factors as

$$(\operatorname{im}\chi^*, 0) \otimes \big(H_G^* \otimes H^*(G), d\big).$$

Since the second term is a Koszul complex, which has trivial cohomology by Proposition 7.3.4, we have $H^*(G/K) \cong \operatorname{im}\chi^* = H_K^* \,/\!/\, H_G^*$ by the Künneth theorem.

As far as Poincaré polynomials are concerned, the statements assume the results of Chapter 1, that $H^*(G)$ and $H^*(K)$ are exterior algebras, and by Theorem 7.4.1 we know $Q(BG) \cong \Sigma PG$ is spanned by generators of degree $2g_j$ and $H^*(BG) = S\big[Q(BG)\big]$ is a polynomial ring on these generators. By the results of Appendix A.2.3, we have

$$p(BG) = \prod_j \frac{1}{1-t^{2g_j}} \qquad \text{and} \qquad p(BK) = \prod_j \frac{1}{1-t^{2k_j}}.$$

The H_G^*-module isomorphism $H_K^* \cong H_G^* \otimes H^*(G/K)$ reduces on the the level of graded vector spaces to

$$p(BK) = p(BG) \cdot p(G/K).$$

Multiplying through by $p(BG)^{-1} = \prod_j (1-t^{2g_j})$ yields the claimed formula. □

Corollary 8.3.14 (Leray [Bor53, Prop. 29.2, p. 201])**.** *Let G be a compact, connected Lie group and T a maximal torus. Then the characteristic map $\chi^* \colon H^*(BT) \longrightarrow H^*(G/T)$ is surjective, and if the Poincaré polynomial of $P(G)$ is $p(PG) = \sum_{j=1}^n t_j^{2g_j-1}$, then*

$$p(G/T) = \prod_{j=1}^{n} \frac{1-t^2}{1-t^{2g_j}}.$$

[3] We have proved this from abstract results about invariants, but these maps arise from the cohomology of the base spaces in the sequence

$$\begin{array}{ccccc} G_T & \longrightarrow & G_K & \longrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ BT & \longrightarrow & BK & \longrightarrow & BG, \end{array}$$

of principal G-bundles maps, where the maps of total spaces can be conceived as "further quotient" maps among quotients of $EG \times G$.

We have also a converse.

Proposition 8.3.15. *If $H^*(G/K)$ is concentrated in even degrees, then K and G are of equal rank.*

Proof. If $H^*(G/K)$ is concentrated in even degrees, then the Euler characteristic $\chi(G/K) > 0$. Thus the result follows from Corollary 6.2.5; if we had $\operatorname{rk} K < \operatorname{rk} G$, then we would also have $\chi(G/K) = 0$. $\square$

This result also admits a purely algebraic proof involving commutative algebra and the Samelson subspace.

Corollary 8.3.16 (Borel [Bor53, Corollaire, p. 168])**.** *Suppose (G, K) is a pair of compact, connected Lie groups such that the characteristic homomorphism $\chi^*\colon H^*_K \longrightarrow H^*(G/K)$ is surjective. Then for every principal G-bundle $G \to E \to B$, the fiber inclusion of the quotient bundle $G/K \longrightarrow E/K \to B$ is surjective in cohomology.*

Proof. The principal bundle $G \to E \to B$ is classified by a map $B \longrightarrow BG$, inducing a bundle map to the universal bundle $G \to EG \to BG$. Taking the right quotient of the total spaces of both bundles by K yields a bundle map

$$\begin{array}{ccc} G/K & = & G/K \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \chi} \\ E/K & \xrightarrow{h} & BK \\ \downarrow & & \downarrow \\ B & \longrightarrow & BG. \end{array}$$

But the existence of this map puts us in the situation of Theorem 2.4.1, so one has $H^*(E/K) \longrightarrow H^*(G/K)$ surjective, and moreover

$$H^*(E/K) \cong H^*(B) \underset{H^*_G}{\otimes} H^*_K. \qquad \square$$

Example 8.3.17. Consider the pair $\big(\mathrm{U}(n), T^n\big)$. The Weyl group $W_{\mathrm{U}(n)}$ is the symmetric group S_n acting on $H^*_T = \mathbb{Q}[t_1, \ldots, t_n]$ by permuting the generators $t_j \in H^2(BT)$, so $H^*_{\mathrm{U}(n)} = \mathbb{Q}[c_1, \ldots, c_n]$ is generated by the elementary symmetric polynomials $c_j = \sigma_j(\vec{t})$. It follows that the cohomology of the complex flag manifold $\mathrm{U}(n)/T^n$ is

$$H^*\big(\mathrm{U}(n)/T^n\big) \cong \mathbb{Q}[t_1, \ldots, t_n]/(c_1, \ldots, c_n),$$

with Poincaré polynomial

$$p\big(\mathrm{U}(n)/T^n\big) = {(1-t)^n}\Big/{\prod_{j=1}^n (1-t)^j} = 1(1+t)(1+t+t^2)\cdots(1+t+t^2+\cdots+t^{n-1}),$$

which, evaluated at $t = 1$, yields rational dimension $n! = |S_n| = |W_{\mathrm{U}(n)}|$. We will see this is no coincidence.

If we take $n = 2$, then

$$\mathrm{U}(2)/T^2 = {\mathrm{U}(2)}\Big/{\mathrm{U}(1) \times \mathrm{U}(1)} \approx G(1, \mathbb{C}^2) = \mathbb{C}\mathrm{P}^1 \approx S^2,$$

so we know what to expect. Indeed, $c_1 = t_1 + t_2$ and $c_2 = t_1t_2$ in $H^*_T = \mathbb{Q}[t_1, t_2]$, so

$$H^*\big(\mathrm{U}(2)/T^2\big) \;\cong\; H^*_{T^2} /\!\!/ H^*_{\mathrm{U}(2)} \;=\; \mathbb{Q}[t_1, t_2]\Big/(t_1 + t_2, t_1t_2) \;\cong\; \mathbb{Q}[t_1]/(t_1^2)$$

as predicted.

For a less trivial example, take $n = 3$, so that $c_1 = t_1 + t_2 + t_3$ and $c_2 = t_1t_2 + (t_1 + t_2)t_3$ and $c_3 = t_1t_2t_3$. Since we are setting each $c_j \equiv 0$, we can eliminate out the generator $t_3 \equiv -(t_1 + t_2)$ and know $0 \equiv c_2 \equiv t_1t_2 - (t_1 + t_2)^2 = -(t_1^2 + t_2^2 + t_1t_2)$. Simplifying $c_3 \equiv 0$ yields $t_1t_2^2 + t_1^2t_2 \equiv 0$, so

$$H^*\big(\mathrm{U}(3)/T^3\big) \;\cong\; \mathbb{Q}[t_1, t_2]\Big/(t_1^2 + t_2^2 + t_1t_2,\; t_1^2t_2 + t_1t_2^2).$$

See Figure 8.3.18.

Figure 8.3.18: The E_∞ page for $\mathrm{U}(3)/T^3$

0	1		t_1, t_2		t_1^2, t_2^2		$t_1^3 + t_2^3$
	0		2		4		6

Example 8.3.19. Consider the pair $\big(\mathrm{Sp}(n), \mathrm{Sp}(k) \times \mathrm{Sp}(n-k)\big)$, yielding as quotient the quaternionic Grassmannian $G(k, \mathbb{H}^n)$. The Weyl group $W_{\mathrm{Sp}(n)}$ is the signed perrmution group $\{\pm 1\}^n \rtimes S_n$: in the semidirect product, S_n acts by permuting the entries of $\{\pm 1\}^n$, and $W_{\mathrm{Sp}(n)}$ acts on $H^*_T = \mathbb{Q}[t_1, \ldots, t_n]$ by permuting and negating the generators $t_j \in H^2(BT)$, so $H^*_{\mathrm{Sp}(n)} = \mathbb{Q}[q_1, \ldots, q_n]$ is generated by the elementary symmetric polynomials $q_j = \sigma_j(t_1^2, \ldots, t_n^2)$ in the squares $t_j^2 \in H^4(BT)$. The factors of the Weyl group $W_{\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)} = W_{\mathrm{Sp}(k)} \times W_{\mathrm{Sp}(n-k)}$ separately permute the tensor factors $\mathbb{Q}[t_1, \ldots, t_k]$ and $\mathbb{Q}[t_{k+1}, \ldots, t_n]$, so

$$H^*\big(G(k, \mathbb{H}^n)\big) \;\cong\; \mathbb{Q}[t_1, \ldots, t_k]^{W_{\mathrm{Sp}(k)}} \otimes \mathbb{Q}[t_{k+1}, \ldots, t_n]^{W_{\mathrm{Sp}(n-k)}}\Big/(q_1, \ldots, q_n)\,.$$

We will calculate explicitly what happens if $n = 5$ and $k = 3$. For convenience, set $u_j = t_j^2$. The numerator ring $H^*_{\mathrm{Sp}(3)} \otimes H^*_{\mathrm{Sp}(2)}$ is the polynomial subring $\mathbb{Q}[r_1, r_2, r_3, s_1, s_2]$ of $\mathbb{Q}[u_1, u_2, u_3, u_4, u_5]$ generated by the five generators on the left, and the denominator ideal is generated by the elements on the right:

$$\begin{aligned}
r_1 &= u_1 + u_2 + u_3, & q_1 &= r_1 + s_1,\\
r_2 &= u_2(u_1, u_2, u_3), & q_2 &= r_1s_1 + r_2 + s_2,\\
r_3 &= u_1u_2u_3, & q_3 &= r_3 + r_2s_1 + r_1s_2,\\
s_1 &= u_4 + u_5, & q_3 &= r_3s_1 + r_2s_2,\\
s_2 &= u_4u_5; & q_5 &= r_3s_2.
\end{aligned}$$

Imposing the congruences generated by setting each $q_j \equiv 0$ and crunching relations a few times yields

$$H^*\big(G(3, \mathbb{H}^5)\big) \;\cong\; \mathbb{Q}[r_1, r_2]\Big/(r_1^4 - r_1^2r_2 - r_2^2, 2r_1^3r_2 + 3r_1r_2^2), \qquad |r_1| = 4,\; |r_2| = 8.$$

Historical remarks 8.3.20. Leray's determination of $H^*(G/T)$ dates back to 1946 in the event G is a compact, connected, classical simple group [Ler46b]. By 1949, he only requires that the universal compact cover (see Theorem B.4.5) $\widetilde{G}$ of G contain no exceptional factors [Ler49a]. His original statement of Theorem 8.3.12 requires no exceptional group to occur as factors of the universal compact cover $\widetilde{G}$ of G, but allows K to be any closed subgroup, not necessarily connected, of equal rank. His additional requirement on G is removed by the time of his contribution [Ler51] to the 1950 Brussels *Colloque de Topologie*. The formula (8.3.13) was first conjectured by Guy Hirsch and is hence traditionally called the *Hirsch formula*. According to Dieudonné [Die09, p. 448], Cartan and Koszul obtained this result independently around the same time. The initial proof that $H^*(G/T)$ is the regular representation of W_G also dates to Leray in the Bruxelles conference; he had earlier [Ler49a] shown the same result holds if G is finitely covered by a product of classical groups.

8.4. The structure of the Cartan algebra, II: formal pairs

Returning to our discussion of homogeneous spaces, let (G, K) be a compact pair and consider the Cartan algebra $H_K^* \otimes H^*(G)$ with differential d induced by $\rho^* \circ \tau$.

Recall that if the Samelson subspace $\widehat{P} \leqslant H^*(G)$ is the subspace of the primitives of G where d vanishes and $\check{P} = PG/\widehat{P}$ is the Samelson complement, we defined the *characteristic factor* to be $J := H^*\big(H_K^* \otimes \Lambda\check{P}, d\big)$ and found a tensor decomposition (Corollary 8.2.6)

$$H^*(G/K) \cong J \otimes \Lambda\widehat{P}.$$

One would like in a similar way to be able to tensor-factor out the characteristic subring $\operatorname{im}\chi^*$ from J, but *this is not generally possible*. The best we are able to do in this regard is the following.

Proposition 8.4.1. *The characteristic ring* $\operatorname{im}\chi^*$ *is simultaneously a subring and quotient ring of the characteristic factor* $J = H^*(H_K^* \otimes \Lambda\check{P})$.

Proof. Since the image of d meets H_K^* in $\rho^* H_G^*$, the composite projection

$$H_K^* \otimes H^*(G) \longrightarrow\!\!\!\!\!\rightarrow H_K^* \longrightarrow\!\!\!\!\!\rightarrow H_K^* /\!\!/ H_G^* = \operatorname{im}\chi^*$$

descends in cohomology to a homomorphism $H^*(G/K) \longrightarrow\!\!\!\!\!\rightarrow \operatorname{im}\chi^*$ split by the defining inclusion $\operatorname{im}\chi^* \hookrightarrow H^*(G/K)$. □

In this section, we explore the propitious case in which the characteristic subring $\operatorname{im}\chi^*$ *is* the characteristic factor J.

Definition 8.4.2. If $H^*(G/K) \cong \operatorname{im}\chi^* \otimes \Lambda\widehat{P}$, we call (G, K) a ***formal pair*** (traditionally, such a pair is called a *Cartan pair*).

Example 8.4.3. Suppose (G, K) is a cohomology-surjective pair. Then, by Theorem 8.3.2, the characteristic factor J is trivial.

Example 8.4.4. Suppose (G, K) is an equal-rank pair. Then, by Theorem 8.3.12, the Samelson subring $\Lambda\widehat{P}$ is trivial and the characteristic factor J is the characteristic ring $\operatorname{im}\chi^*$.

One can see formal pairs as the smallest class of cases that contains both these extreme examples. Another way of seeing it is this: the first interesting page of the Serre spectral sequence of the Borel fibration $G \to G_K \to BK$ is $E_2 = E_2^{\bullet,0} \otimes E_2^{0,\bullet} \cong H_K^* \otimes H^*(G)$, a coproduct of CGAs, with one tensor-factor each arising from the base and the fiber of the fibration. In our examples in Section 8.3.2 and Section 8.3.1, this tensor-product structure persisted throughout the entire sequence, in that the decomposition $E_r = E_r^{\bullet,0} \otimes E_r^{0,\bullet}$ continued to hold, and

$$E_\infty = E_\infty^{\bullet,0} \otimes E_\infty^{0,\bullet} = \big(H_K^* /\!\!/ H_G^*\big) \otimes \Lambda \widehat{P}$$

was the tensor product of the characteristic subring $\operatorname{im} \chi^*$ and the Samelson subring $\Lambda\widehat{P}$.[4] For a representative example, see Figure 8.7.4. This is also the optimal situation from a purely numerical perspective, because, in particular, the tensor decomposition yields a factorization

$$p(G/K) = p(E_\infty^{\bullet,0}) \cdot p(E_\infty^{0,\bullet}), \tag{8.4.5}$$

of Poincaré polynomials and in particular, setting the formal variable t to 1, a factorization

$$h^\bullet(G/K) = \dim_{\mathbb{Q}} E_\infty^{\bullet,0} \cdot \dim_{\mathbb{Q}} E_\infty^{0,\bullet}.$$

We will expound a number of properties of and equivalent characterizations of the formal pair condition, in the process justifying the nomenclature. The very fact that there are so many ways of stating this property should be a further argument, were one needed, for the naturality of the concept.

But first we introduce an important bound on the dimension of the Samelson subspace.

Definition 8.4.6 (Paul Baum)**.** The ***deficiency*** of a compact pair (G, K) is the integer

$$\mathrm{df}(G,K) := \operatorname{rk} G - \operatorname{rk} K - \dim \widehat{P}.$$

Proposition 8.4.7. *The deficiency is a natural number. That is, for any compact pair (G, K), one has*

$$\dim PG - \dim PK \geqslant \dim \widehat{P}.$$

Proof (Baum [Bau68, Lem. 3.7, p. 26]). Since $\check{P} \oplus \widehat{P} = PG$ by definition, it is enough to show $\dim \check{P} \geqslant \dim PK$. This can be shown through Poincaré polynomials. We may view H_K^* as an algebra over the polynomial ring $A = S\big[\tau(\check{P})\big]$ by restricting $\rho^* \colon H_G^* \longrightarrow H_K^*$. If we lift a basis of $H_K^* /\!\!/ H_G^* = H_K^* /\!\!/ A$ back to H_K^*, this basis spans H_K^* as an A-module (typically with some redundancy; we do not expect H_K^* to be a free A-module). Thus $p(H_K^* /\!\!/ H_G^*) \cdot p(A) \geqslant p(H_K^*)$ (in that each coefficient of t^n on the left is at least its counterpart on the right), or dividing through,

$$p(H_K^* /\!\!/ H_G^*) \geqslant \frac{p(H_K^*)}{p(A)}.$$

Both the numerator and denominator on the right-hand side are products of factors $1 - t^n$, by (A.2.13). There are $\dim PK$ such factors in the numerator and $\dim \check{P}$ in the denominator, so if we had $\dim PK > \dim \check{P}$, the rational function $p(H_K^*)/p(A)$ would have a pole at $t = 1$, but this is impossible because it is majorized by the polynomial $p(H_K^* /\!\!/ H_G^*)$. □

[4] We concede that in those examples, it was the tensor product of precisely one of those factors—there are historical reasons why those cases were studied first.

Theorem 8.4.8 ([Oni94, Thm. 12.2, p. 211]). *Let (G, K) be a compact pair. The following conditions are equivalent:*

1. *(G, K) is a formal pair.*
2. *The kernel $(\operatorname{im}\widetilde{\rho}^*)$ of the characteristic map $H_K^* \xrightarrow{\chi^*} H^*(G/K)$ is a regular ideal in the sense of Definition 7.3.6.*
3. *The sequence $H_K^* \xrightarrow{\chi^*} H^*(G/K) \xrightarrow{j^*} H^*G$ is coexact.*
4. *The characteristic factor J in the decomposition $H^*(G/K) \cong J \otimes \Lambda\widehat{P}$ is evenly-graded.*
5. *The deficiency $\operatorname{df}(G, K) = \dim PG - \dim PK - \dim \widehat{P}$ is zero.*

Proof. We always have $H^*(G/K) \cong J \otimes \Lambda\widehat{P}$, so the task is to prove the remaining conditions are equivalent to the statement $J = \operatorname{im}\chi^*$.

$1 \iff 2$. If we singly grade the CDGA $C = H_K^* \otimes \Lambda\check{P}$, by

$$\cdots \longrightarrow H_K^* \otimes \Lambda^2\check{P} \longrightarrow H_K^* \otimes \Lambda^1\check{P} \longrightarrow H_K^* \to 0, \tag{8.4.9}$$

where the differential d vanishes on H_K^* and is induced by

$$\check{P} \hookrightarrow PG \xrightarrow[\tau]{\sim} Q(BG) \xrightarrow{\rho^*} H_K^*$$

then $J = \operatorname{im}\chi^* = H_K^* /\!/ H_G^*$ if and only if $H^*(C) = H^0(C)$. But if we write $\vec{x}$ for a basis of $\tau(\check{P}) \leqslant H_G^*$, then C is the Koszul complex $K_{H_G^*}(\vec{x}, H_K^*)$ of Definition 7.3.6. Then Proposition 7.3.9 states this Koszul complex is acyclic if and only if the sequence is regular.

$1 \implies 3$. By the definition Definition 8.2.1 of the Samelson subring, j^* factors as $H^*(G/K) \twoheadrightarrow \Lambda\widehat{P} \hookrightarrow H^*G$, so one can replace H^*G by $\Lambda\widehat{P}$ in the coexact sequence above. Once we factor out $\Lambda\widehat{P}$, the new claim is that the sequence

$$H_K^* \xrightarrow{\chi^*} J \to \mathbb{Q}$$

is coexact, or that every class of positive degree in J has a representative in $H_K^* \otimes \Lambda\check{P}$ lying in the ideal $\chi(H_K^{\geqslant 1})$. But this is clearly the case if χ surjects onto J.

$3 \implies 1$. Assume every class of positive degree in J admits a representative in the ideal $(H_K^{\geqslant 1})$ of $H_K^* \otimes \Lambda\check{P}$. Then the quadruple

$$\mathfrak{a} := H_K^{\geqslant 1} \lhd A := H_K^*, \qquad V := \mathbb{Q} = J_0 < M := J$$

satisfies $M = \mathfrak{a}M + V$, so the corollary A.1.3 of Nakayama's lemma yields $J = M = AV = A \cdot 1$, meaning χ^* is surjective.

$1 \implies 4$. This is clear since $\operatorname{im}\chi^* = H_K^* /\!/ H_G^*$ inherits an even grading from H_K^*.

$4 \implies 2$. If J is evenly graded, then H^1 of the Koszul complex C of (8.4.9) above must be zero because $\check{P} \leqslant PG$ is oddly-graded. But by Proposition 7.3.9, this also means $J = H^*(C) = H^0(C) = H_K^* /\!/ H_G^* = \operatorname{im}\chi^*$.

2 $\iff$ 5. ([Oni94, p. 144]) Write $y_1, \ldots, y_n$ for a basis of $Q(BK)$ and $b_1, \ldots, b_\ell$ for a basis of $\tau(\check{P}) \leqslant S[y_i]$. Note that we know that $\mathrm{df}(G,K) \geqslant 0$ in any event by Proposition 8.4.7, and if $\mathrm{df}(G,K) = 0$, then $\dim \check{P} = \dim PK$.

Working over $k = \overline{\mathbb{Q}}$ or $\mathbb{C}$, the ring $k[y_i]/(b_j)$ is finite-dimensional as a k-module, so the variety $V = V(b_1, \ldots, b_\ell) \subseteq k^n$ is zero-dimensional. By a result of algebraic geometry [VA67, Ch. 16], the sequence (b_j) is regular if and only each component of V is $(n-\ell)$-dimensional. Thus (b_j) is regular if and only if $\operatorname{rk} K = n = \ell = \dim \check{P}$. □

To justification our choice of terminology, we need to bring in a concept from rational homotopy theory.

Theorem 8.4.10 ([Oni94, p. 145][GHV76, Thm. 10.17.VIII])**.** *A compact pair (G,K) is formal if and only if its Cartan algebra is formal in the sense of Definition 4.1.1.*

The proof needs a level of sophistication with models we have not needed elsewhere. The crux is the following lemma, distilled from the material in Section 3.7 of Greub *et al.* [GHV76, pp. 147–152].

Lemma 8.4.11. *Suppose $(B_i, d_i)_{i=0}^n$ is a zig-zag of quasi-isomorphic k-CDGAs as depicted in Definition 4.1.1, that $F = SQ \otimes \Lambda P$ is a free k-CGA on a strictly-positive graded subspace $V = Q \oplus P$ and that we are given a k-linear map $\xi_0 \colon V \longrightarrow Z(B_0) = \ker d_0$ increasing degree by one. Extend ξ_0 uniquely to a derivation on $B_0 \otimes F$ vanishing on B_0 and define a new derivation on $B_0 \otimes F$ by $\delta_0 = \xi + d_0$. Then there exist k-CDGA structures $(B_i \otimes F, \delta_i)$ extending the δ_i such that the rings $H^*(B_i \otimes F)$ are isomorphic through isomorphisms which preserve the images $H^*(B_0) \xrightarrow{\sim} H^*(B_i) \longrightarrow H^*(B_i \otimes F)$*

In particular, if (A,d) is formal, then there exists a k-CDGA structure on $H^*(A) \otimes F$ with isomorphic cohomology to that of $(A \otimes F, \xi)$ and such that the triangle

$$\begin{array}{ccc} & H^*(A) & \\ \swarrow & & \searrow \\ H^*\big(H^*(A) \otimes F\big) & \xrightarrow{\quad\sim\quad} & H^*(A \otimes F) \end{array}$$

commutes.

Proof. To guarantee the second condition, that the quasi-isomorphisms to be defined among the $(B_i \otimes F, \delta_i)$ preserve the image of $H^*(B_0)$, we stipulate at the beginning all the quasi-isomorphisms we construct must restrict on the bases B_i to the original quasi-isomorphisms. Now inductively suppose the construction has been established up to $B = B_i$ and that the differential $\delta = \delta_i$ on $B \otimes F$ is a derivation of degree 1. Write $C = B_{i+1}$. There are two cases for the induction step, quasi-isomorphisms $\varkappa \colon (B,d) \longrightarrow (C, d_C)$ or $\lambda \colon (B,d) \longleftarrow (C, d_C)$.

In the former case, using the assumed differential δ on $B \otimes F$ and the fact that $\varkappa$ is a cochain map, we extend

$$V \xrightarrow{\delta_B} Z(B) \xrightarrow{\varkappa} Z(C),$$

uniquely to a derivation ξ_C on $C \otimes F$. Then $\delta_C := d_C + \xi_C$ is again a derivation of degree one because δ is. The map $\varkappa \otimes \mathrm{id} \colon B \otimes F \longrightarrow C \otimes F$ is a ring map because $\varkappa$ was, and a cochain map because it is so on generators.

In the latter case, pick a homogeneous basis (v) of V. Since $\lambda \colon (B,d) \longleftarrow (C, d_C)$ is a quasi-isomorphism, for each v there is a unique class in $H^*(C)$ mapping onto $[\delta v] \in H^*(B)$ under

$H^*(\lambda)$, and we may choose an element $\xi_C v \in Z^*(C)$ representing this class. Since $\lambda\xi_C v$ and δv are cohomologous, we can then write

$$\lambda\xi_C v = \delta v + d\alpha(v) = \delta\big(v + \alpha(v)\big)$$

for some elements $\alpha(v) \in B$. These maps of the basis extend k-linearly to $\alpha\colon V \longrightarrow B$ and $\xi_C\colon V \longrightarrow Z^*(C)$. Uniquely extend ξ_C to a derivation on $C \otimes F$ and define $\delta_C = d_C + \xi_C$. An extension $\psi\colon C \otimes F \longrightarrow B \otimes F$ of λ to a ring map is determined uniquely by its restriction to V, and for this extension to also be a cochain map $(C \otimes F, \delta_C) \longrightarrow (B \otimes F, \delta)$, it is necessary and sufficient to demand that for $v \in V$ one have

$$\delta\psi(v) = \psi\delta_C(v) = \psi\underbrace{(\xi_C v)}_{\in C} = \lambda\xi_C v = \delta\big(v + \alpha(v)\big).$$

But we can achieve this by just letting $\psi(v) = v + \alpha(v)$ on V.

It remains to see $\varkappa \otimes \mathrm{id}_F$ and ψ are quasi-isomorphisms; we do this for ψ, the other case being slightly simpler without the complication of α. Filter $B \otimes F$ and $C \otimes F$ "horizontally" with respect to the degree of B- and C-tensor components respectively. Then it is clear that both δ and δ_C increase filtration degree and that ψ preserves filtration degree since the filtration degrees of v and $v + \alpha(v)$ are equal for $v \in V$, so ψ induces a map of filtration spectral sequences. Since $\deg\alpha(v) = 1$, the element $v + \alpha(v)$ becomes just v in the associated graded algebra, so the map of E_0 pages is just $\lambda \otimes \mathrm{id}\colon C \otimes F \longrightarrow B \otimes F$. Since elements of the generating space $V \leqslant F$ are sent forward at least two degrees in the filtration by δ_C, we find $E_1 = E_0$ in both sequences and the map of E_2 pages is $H^*(\lambda) \otimes \mathrm{id}\colon H^*(B) \otimes F \longrightarrow H^*(C) \otimes F$, which by assumption is an isomorphism. By Proposition 2.7.2, then, ψ is a quasi-isomorphism. □

Proof of Theorem 8.4.10 ([GHV76, Thm. 2.19.VIII, Thm. 3.30.XI, Thm. 10.17.VIII]). For the forward direction, one always has an algebra map

$$\begin{aligned}\lambda\colon (H_K^* \otimes \Lambda PG, d) &\longrightarrow \big((H_K^* /\!/ H_G^*) \otimes \Lambda\widehat{P}, 0\big),\\ a \otimes 1 &\longmapsto \big(a + (\widetilde{\operatorname{im}\rho^*})\big) \otimes 1,\\ 1 \otimes z &\longmapsto 1 \otimes \big(z + (\check{P})\big),\end{aligned}$$

which is in fact a DGA homomorphism since $d(1 \otimes \check{P})$ is contained in $\operatorname{im}\rho^*$. If (G,K) is a formal pair, so that $H^*(G/K) \cong (H_K^* /\!/ H_G^*) \otimes \Lambda\widehat{P}$, then λ is a quasi-isomorphism, so the Cartan algebra $(H_K^* \otimes \Lambda PG, d)$ is formal.

For the other direction, the strategy is to show the sequence

$$H_K^* \xrightarrow{\chi} H^*(G/K) \xrightarrow{j^*} H^*G$$

is coexact, this being one of the equivalent formulations in Theorem 8.4.8. Start by noting the Cartan CDGA $C = (H_K^* \otimes H^*G, d)$ is quasi-isomorphic to $A_{\mathrm{PL}}(G/K)$ by Theorem 8.1.14, and that by the assumption that G/K is formal there also exists a zigzag of quasi-isomorphisms connecting C with $H^*(C) \cong H^*(G/K)$ as equipped with the zero differential. Proposition 8.1.12 then allows us to connect a CDGA structure on $H^*(G/K) \otimes H^*K$ via a zig-zag of quasi-isomorphisms to the Chevalley algebra $\big(A_{\mathrm{PL}}(G/K) \otimes H^*K, d\big)$ of the bundle $K \to G \to G/K$, which calculates H^*G by Theorem 8.1.5.

Since this zigzag connects the subalgebra $A_{\mathrm{PL}}(G/K)$ of $A_{\mathrm{PL}}(G/K) \otimes H^*K$ with the factor $H^*(C) \cong H^*(G/K)$ of $H^*(C) \otimes H^*K$, when we take cohomology, we obtain an isomorphism $H^*\big(H^*(G/K) \otimes H^*K\big) \xrightarrow{\sim} H^*G$ such that the following triangle commutes:

$$\begin{array}{ccc} & H^*(G/K) & \\ \swarrow & & \searrow{\scriptstyle j^*} \\ H^*\big(H^*(G/K) \otimes H^*K\big) & \xrightarrow{\quad\sim\quad} & H^*G. \end{array}$$

Thus we can identify these two maps, the left being induced by the obvious inclusion $H^*(G/K) \otimes \mathbb{Q} \hookrightarrow H^*(G/K) \otimes H^*K$ and the right by the quotient map $j\colon G \to G/K$.

To show the sequence is coexact, it remains to show the common kernel of these maps is the ideal generated by $\chi(H_K^{\geqslant 1})$ in $H^*(G/K)$. But the differential in the algebra on the bottom left of the triangle is induced by the composition

$$PK \xrightarrow{\sim} Q(BK) \hookrightarrow H_K^* \xrightarrow{\chi^*} H^*(G/K).$$

It follows that the image of $H^*(G/K)$ in $H^*\big(H^*(G/K) \otimes H^*K\big)$ is the quotient of $H^*(G/K) \otimes \mathbb{Q}$ by the image of the generators of H_K^*, so the kernel is the ideal in $H^*(G/K)$ generated by $\chi^*(H_K^{\geqslant 1})$ as claimed. □

Proposition 8.4.12. *Let (G, K) be a formal pair of Lie groups. If the Poincaré polynomials of the Samelson subspace $\widehat{P}$, the Samelson complement $\check{P}$, and the primitive space PK are given respectively by*

$$p(\widehat{P}) = \sum_{j=1}^{\operatorname{rk} G - \operatorname{rk} K} t^{d_j}, \qquad p(\check{P}) = \sum_{\ell=1}^{\operatorname{rk} K} t^{c_j}, \qquad p(PK) = \sum_{\ell=1}^{\operatorname{rk} K} t^{k_j},$$

then the Poincaré polynomial of G/K is

$$p(G/K) = p(\Lambda \widehat{P}) \cdot \frac{p(BK)}{p\big(S[\Sigma \check{P}]\big)} = \prod_{j=1}^{\operatorname{rk} G - \operatorname{rk} K} (1 + t^{d_j}) \cdot \prod_{\ell=1}^{\operatorname{rk} K} \frac{1 - t^{c_j + 1}}{1 - t^{k_j + 1}}$$

and its total Betti number is

$$h^\bullet(G/K) = \frac{2^{\operatorname{rk} G}}{2^{\operatorname{rk} K}} \cdot \prod_{\ell=1}^{\operatorname{rk} K} \frac{c_\ell + 1}{k_\ell + 1} = \frac{\prod_{\ell=1}^{\operatorname{rk} K} (c_\ell + 1)}{|W_K|} 2^{\operatorname{rk} G - \operatorname{rk} K}.$$

Proof. Given the equations (8.4.5) and (A.2.13), all that remains to be shown is that $p(H_K^* /\!\!/ H_G^*) = p(BK)/p\big(S[\Sigma \check{P}]\big)$ as claimed. But Theorem 8.4.8, the generators of $\operatorname{im} \rho^*$ form a regular sequence of $\operatorname{rk} K$ elements of H_K^* of degrees $c_j + 1$. These generators are thus algebraically independent and form a polynomial subalgebra $S \cong S[\Sigma \check{P}]$ of H_K^* such that H_K^* is a free S-module. The result then follows from Proposition A.2.14. □

Proposition 8.4.13 ([Oni94, Rmk., p. 212])**.** *Suppose (G, K) is a compact pair and S a maximal torus of K. Then (G, K) is a formal pair if and only if (G, S) is.*

Proof. This follows from Corollary 6.3.6, with $X = G$. Write W for the Weyl group of K. If (G, S) is formal, then $H^*_S(G) = H^*(G/S) \cong (H^*_S /\!/ H^*_G) \otimes \Lambda\widehat{P}$. Since the W-action on $H^*(G)$ descends from the K-action, which is trivial since K is path connected, the action of W on $H^*_S(G)$ affects only the bottom row $H^*_S /\!/ H^*_G$, and we have

$$H^*(G/K) = H^*_K(G) \cong H^*_S(G)^W \cong \Big(H^*_S /\!/ H^*_G\Big)^W \otimes \Lambda\widehat{P} \cong \Big((H^*_S)^W /\!/ H^*_G\Big) \otimes \Lambda\widehat{P} \cong \big(H^*_K /\!/ H^*_G\big) \otimes \Lambda\widehat{P}.$$

On the other hand, if (G, K) is formal, so that $H^*_K(G) \cong (H^*_K /\!/ H^*_G) \otimes \Lambda\widehat{P}$, then

$$H^*(G/S) \cong H^*_S \underset{H^*_K}{\otimes} H^*(G/K) \cong H^*_S \underset{H^*_K}{\otimes} H^*_K /\!/ H^*_G \otimes \Lambda\widehat{P} \cong H^*_S /\!/ H^*_G \otimes \Lambda\widehat{P}. \qquad \square$$

Remarks 8.4.14. Though the formality condition on pairs (G, K) is convenient, is natural, has many equivalent formulations, is guaranteed by several commonly studied sufficient conditions, and is invariant under the act of replacing the isotropy group K with its maximal torus S, there still seems to be no simpler way of determining formality of a randomly given pair (G, K) than carefully examining the image of the map $\rho^*\colon H^*_G \longrightarrow H^*_S$, and our knowledge has arguably not improved in any major way since regular sequences were introduced into commutative algebra in the mid-1950s. Indeed, it seems computing the map ρ^* is an NP-hard problem [Ama13, Sec. 1].

Historical remarks 8.4.15. The deficiency first appears in Paul Baum's 1962 doctoral dissertation [Bau62], where it is shown *inter alia* that if $k = \mathbb{Z}$ or k is any field and $H^*(G;k)$ and $H^*(K;k)$ are exterior algebras and the analogue of the deficiency with k coefficients satisfies $\operatorname{df}(G, K) \leqslant 2$, then the Eilenberg–Moore spectral sequence of $G/K \to BK \to BG$ collapses at $E_2 = \operatorname{Tor}_{H^*_K}(k, H^*_G)$. The deficiency thus links our account with the Eilenberg–Moore spectral sequence analysis of the cohomology of homogeneous spaces discussed in Section 8.8.2. This deficiency is actually an invariant of the homogeneous space G/K and not just of the compact pair (G, K), according to a theorem of Arkadi Onishchik; see Onishchik [Oni72].

What we call a *formal pair* is traditionally called a *Cartan pair* (as seen, e.g., in the standard reference by Greub *et al.* [GHV76, p. 431]). The condition already arises in Cartan's classic transgression paper in the *Colloque* [Car51, (3) on p. 70], so the attribution is just, but the name is made inconvenient by the vast prolificacy of the Cartans: pursuant to the work of Cartan *père* on symmetric spaces, the pair $(\mathfrak{k}, \mathfrak{p})$ of ± 1-eigenspaces of the Lie algebra $\mathfrak{g}$ induced by an involutive Lie group automorphism $\theta\colon G \to G$ is also called a *Cartan decomposition* or a *Cartan pair*. (The author spent an embarrassingly long time in grad school finally convincing himself these two concepts of "Cartan pair" are entirely unrelated.)

The formal pair condition also appears in the (Russian-language) writings of Doan Kuin', where—at least as the translator would have it—K is said to be *in the normal condition* in G. This locution did not catch on. We hope that despite the existence of standard terminology, this section has made the case that ours is preferable.

The proof of Theorem 8.4.10 is due to Steve Halperin, and in fact (personal communication) is the first result he proved as a graduate student. The first published proof was in Greub *et al.*

8.5. Cohomology computations, II: symmetric spaces

Now we are able to discuss the cohomology of a famous class of homogeneous spaces which has been intensively studied since the early 1900s, the so-called *symmetric spaces*. The irreducible examples have been completely classified and we will be able to study them thoroughly. It is possible to discuss *generalized homogeneous spaces* in the same breath, so we do.

Definition 8.5.1. Let G be a connected Lie group and $\theta \in \operatorname{Aut} G$ a smooth automorphism of finite order. Then the fixed point set $G^{\langle\theta\rangle}$ is a closed subgroup of G. Let K be a subgroup of $G^{\langle\theta\rangle}$ containing the identity component $(G^{\langle\theta\rangle})_0$. Then G/K is called a ***generalized symmetric space***. In the event θ is an involution, G/K is a ***symmetric space***. If in addition G and K are compact and connected, we call (G, K) a ***generalized symmetric pair***.

It turns out all symmetric pairs are formal. The argument, already due in its essence to Élie Cartan [FIND CITATION], turns into a proof G/K is *geometrically* formal if one verifies that the representing forms we find are in fact harmonic.

Proposition 8.5.2. *Suppose (G, K) is a compact pair such that G/K is a symmetric space. Then (G, K) is a formal pair.*

Proof. Recall from Proposition 6.1.1 that elements of $H^*(G/K; \mathbb{R})$ can be represented by G-invariant differential forms on G/K, which are determined by their values at the identity coset, elements of the exterior algebra $\Lambda(\mathfrak{g}/\mathfrak{k})^\vee$. Recall further, from Proposition 6.1.2, that G-invariance on $\Omega^\bullet(G/K)$ translates to $\operatorname{Ad}^*(K)$-invariance in $\Lambda(\mathfrak{g}/\mathfrak{k})^\vee$. Thus elements of $H^*(G/K)$ are represented by elements of $\big(\Lambda(\mathfrak{g}/\mathfrak{k})^\vee\big)^K$. Let $\theta \in \operatorname{Aut} G$ be the involution fixing K, so that $\mathfrak{g}$, viewed as an $\langle\theta\rangle$-representation decomposes as the direct sum of the with 1-eigenspace $\mathfrak{k}$, the Lie algebra of K, a (-1)-eigenspace $\mathfrak{p}$. This $\mathfrak{p}$ is orthogonal to $\mathfrak{k}$ under the Killing form B, for θ_* is an isometry, and if $u \in \mathfrak{k}$ and $v \in \mathfrak{p}$, then $B(u, v) = B(\theta_* u, \theta_* v) = B(u, -v)$. Since $\mathfrak{k}$ is $\operatorname{Ad}^*(K)$-invariant so also is $\mathfrak{p}$, so that $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ as an $\operatorname{Ad}(K)$-representation, and hence $\big(\Lambda(\mathfrak{g}/\mathfrak{k})^\vee\big)^K \cong \Lambda[\mathfrak{p}^\vee]^K$.

We claim every one of these elements corresponds to a closed differential form. Indeed, because θ is a Lie group automorphism, the induced map θ^* on $\Omega^\bullet(G/K)$ commutes with the exterior derivative d, and hence with the induced differential on $\Lambda[\mathfrak{p}^\vee]$. Now, since θ_* acts as $-\mathrm{id}$ on $\mathfrak{p}$, its dual θ^* acts as $-\mathrm{id}$ on $\mathfrak{p}^\vee$ and so acts as $(-1)^\ell \cdot \mathrm{id}$ on $\Lambda^\ell[\mathfrak{p}^\vee]$, which is spanned by wedge products of ℓ elements of $\mathfrak{p}^\vee$. Let ω be one such element. Then, since $d \circ \theta^* = \theta^* \circ d$, we have

$$(-1)^{\ell+1} d\omega = \theta^* d\omega = d\theta^*\omega = (-1)^\ell d\omega,$$

so $d\omega = 0$. Thus all elements of $\big(\Lambda[\mathfrak{p}^\vee]^K, d\big)$ are closed. Translating back, every element of $\Omega^\bullet(G/K)^G < \Omega^\bullet(G/K)$ is closed, so $\big(H^*(G/K; \mathbb{R}), 0\big) \cong \big(\Omega^\bullet(G/K)^G, d\big)$ and G/K is formal over $\mathbb{R}$. □

Corollary 8.5.3. *Let B be a generalized symmetric space in the sense of Definition 8.5.1 and $G \to E \to B$ a principal G-bundle over B. Then the Cartan algebra calculates $H^*(E)$.*

Proof. By Remark 8.5.5, a generalized symmetric space is formal, so Proposition 8.1.12 applies. □

Corollary 8.5.4 (Koszul, [Kos51]). *Let (G, K) be a pair such that G/K is a symmetric space. Then the Cartan algebra of $K \to G \to G/K$ calculates $H^*(G)$.*

[THE REST OF THIS SECTION WILL INCLUDE ALL IRREDUCIBLE SYMMETRIC SPACES AS EXAMPLES, WITH SOME OF THE CALCULATIONS LEFT AS EXERCISES.]

Remark 8.5.5. Svjetlana Terzić [Ter01] and independently Zofia Stępień [Stęb] have also shown that compact generalized symmetric spaces G/K with isotropy group K connected are formal. It is *not*, however, the case that wedge products of harmonic forms on such spaces are again harmonic (that such should happen is called *geometric formality*); see Terzić's later joint article with Dieter Kotschick [KT03].

8.6. Cohomology computations, III: informal spaces

This section comprises a pair of computations demonstrating the case the pair is informal, in this case of deficiency 1. The first example, $\mathrm{Sp}(5) > \mathrm{SU}(5)$, is also done in Paul Baum's thesis. Both also appear in the book of Greub *et al.* [GHV76, pp. 488–9].

8.6.1. Sp(5)/SU(5)

This is an example Paul Baum says Armand Borel showed him in the '60s. We understand $H^*_{\mathrm{Sp}(5)} \longrightarrow H^*_{\mathrm{SU}(5)}$ in terms of invariants of $H^*_{T^5} = \mathbb{Q}[t_1, t_2, t_3, t_4, t_5]$ under the actions of the Weyl groups of $\mathrm{Sp}(5)$ and $\mathrm{U}(5)$, which are respectively $\{\pm 1\}^5 \rtimes S_5$ and S_5 acting on the t_j in the expected way. We find that generators of $H^*_{\mathrm{Sp}(5)}$ are given by elementary symmetric polynomials p_n of degree $4n$ in the variables $-t_j^2$ and those of $H^*_{\mathrm{U}(5)}$ by elementary symmetric polynomials c_n of degree $2n$ in the t_j. These are of course the symplectic Pontrjagin classes and Chern classes. The restriction maps between them are a matter of combinatorics: Write $\bar{c}_n$ for the elementary symmetric polynomials in the $-t_j$, so that $\bar{c}_n = (-1)^n c_n$, and set $p_0 = c_0 = \bar{c}_0 = 1$. Then the total Pontrjagin and Chern classes satisfy

$$\begin{aligned} c &= \sum c_n = \prod (1 + t_j), \\ p &= \sum p_n = \prod (1 - t_j^2) = \prod (1 + t_j)(1 - t_j) = c\bar{c}, \end{aligned}$$

from which, collecting terms of like degree, we read off $p_n = \sum_{j=0}^{2n} c_j \bar{c}_{2n-j}$. Recalling the map $H^*_{\mathrm{U}(\ell)} \longrightarrow H^*_{\mathrm{SU}(\ell)} = \mathbb{Q}[c_2, \ldots, c_\ell]$ induced by the inclusion is given by $c_1 \mapsto 0$ and $c_n \mapsto c_n$ for $n > 1$, we can strip out all the c_1 from the expressions for the p_n and finally compute $H^*_{\mathrm{Sp}(5)} \longrightarrow H^*_{\mathrm{SU}(5)}$ as

$$\begin{aligned} p_1 &\longmapsto c_2 + \bar{c}_2 = 2c_2, \\ p_2 &\longmapsto c_4 + \bar{c}_4 = 2c_4, \\ p_3 &\longmapsto c_2\bar{c}_4 + c_3\bar{c}_3 + c_4\bar{c}_2 = 2c_2c_4 - c_3^2, \\ p_4 &\longmapsto c_3\bar{c}_5 + c_4\bar{c}_4 + c_5\bar{c}_3 = c_4^2 - 2c_3c_5, \\ p_5 &\longmapsto c_5\bar{c}_5 = -c_5^2. \end{aligned} \tag{8.6.1}$$

One observes the image ring is

$$\mathbb{Q}[c_2, c_4, c_3^2, c_3c_5, c_5^2].$$

Now to compute the cohomology of $\mathrm{Sp}(5)/\mathrm{SU}(5)$ is to determine the cohomology of the resulting Cartan algebra

$$C := \mathbb{Q}[c_2, c_3, c_4, c_5] \otimes \Lambda[\sigma p_1, \sigma p_2, \sigma p_3, \sigma p_4, \sigma p_5],$$

where the σp_n are suspensions of the Pontrjagin classes, living in $H^{4n-1}\mathrm{Sp}(5)$, and the differential is the unique one taking σp_n to the image of p_n in $H^*_{\mathrm{SU}(5)}$. A clever choice of generators helps compute the cohomology of C, but we will find it easier to filter C by the base degree in $H^*_{\mathrm{SU}(5)}$ and run the filtration spectral sequence. This is stable until $E_4 = C$, and then the first nonzero differential cancels σp_1 against c_2 and we get

$$E_5 = \mathbb{Q}[c_3, c_4, c_5] \otimes \Lambda[\sigma p_2, \sigma p_3, \sigma p_4, \sigma p_5]$$

with differentials

$$\begin{aligned} \sigma p_2 &\longmapsto 2c_4, \\ \sigma p_3 &\longmapsto -c_3^2, \\ \sigma p_4 &\longmapsto c_4^2 - 2c_3c_5, \\ \sigma p_5 &\longmapsto -c_5^2. \end{aligned}$$

The next differential is on E_8, and after we get

$$E_9 = \mathbb{Q}[c_3, c_5] \otimes \Lambda[\sigma p_3, \sigma p_4, \sigma p_5]$$

with

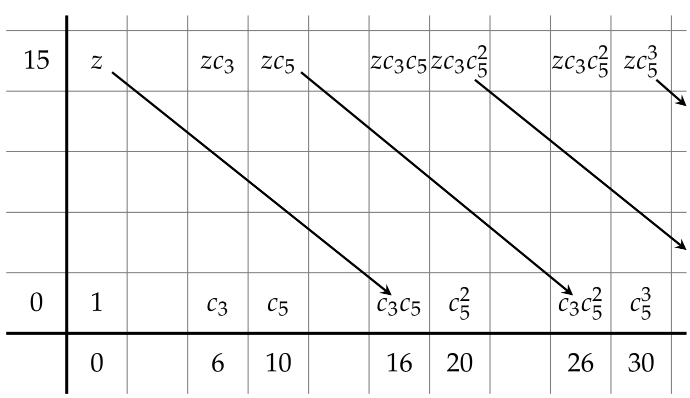
$$\begin{aligned} \sigma p_3 &\longmapsto -c_3^2, \\ \sigma p_4 &\longmapsto -2c_3c_5, \\ \sigma p_5 &\longmapsto -c_5^2. \end{aligned} \tag{8.6.2}$$

Up to an irrelevant rescaling of the generators σp_n, this is Baum's presentation.[5]

Everything we have done so far could have been done on the algebra level. To see what happens next, we prefer to proceed via the spectral sequence. Although this should destroy multiplication, in fact we will be able to reconstruct it through degree considerations. The next page of the spectral sequence is the last at which we can afford not to draw a picture. The differential d_{12} cancels σp_3 and c_3^2, so the next page is

$$E_{13} = \mathbb{Q}[c_3, c_5] \big/ (c_3^2) \otimes \Lambda[\sigma p_4, \sigma p_5]$$

Figure 8.6.3: The E_{16} page for Sp(5)/SU(5)



.

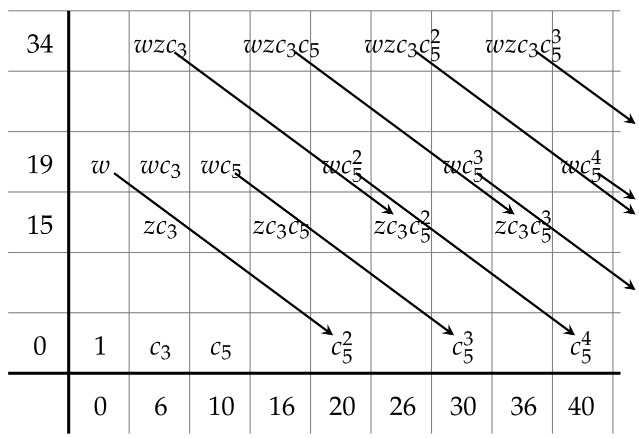
The next nontrivial differential, d_{16}, annihilates σp_5, and so leaves the tensor-factor $\Lambda[\sigma p_5]$ inert, so we will just look at the other factor $\mathbb{Q}[c_3, c_5] \big/ (c_3^2) \otimes \Lambda[\sigma p_4]$. Since by this page we have

5 Morally, this process has factored a Koszul algebra $\mathbb{Q}[c_2, c_4] \otimes \Lambda[\sigma p_1, \sigma p_2]$ out of C.

$c_3^2 = 0$, any differential of a term divisible by c_3 vanishes, so the nontrivial differentials originate from terms divisible by z and end at terms divisible by c_3c_5. Here we have made the abbreviation $z = -\sigma p_4/2$. The parallel copy, σp_5 times the displayed part, is omitted.

The last differential is on the page E_{20}. The nonzero differentials on this page come from generators divisible by $w = -\sigma p_5$ and land in squares divisible by c_5^2, as follows:

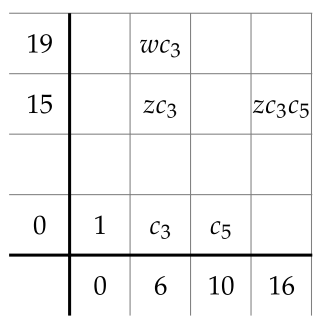
Figure 8.6.4: The E_{20} page for Sp(5)/SU(5)



.

What remains on $E_{21} = E_\infty$ is the following:

Figure 8.6.5: The E_∞ page for Sp(5)/SU(5)



.

The degrees of the surviving vector space generators are

$$0, \qquad 6, \qquad 10, \qquad 21, \qquad 25, \qquad 31$$

and the only nonzero products are those determined by Poincaré duality. The bottom row of the E_∞ page represents the image

$$\mathbb{Q}[c_3, c_5] \big/ (c_3, c_5)^2$$

of $H^*_{\mathrm{SU}(5)} \longrightarrow H^*\big(\mathrm{Sp}(5)/\mathrm{SU}(5)\big)$. We can see from the picture that the familiar (base)$\otimes$(fiber) structure that obtains in the formal examples has been destroyed by the decomposable differentials.

8.6.2. SU(6)/SU(3)^2

In this example we consider the inclusion of the block-diagonal subgroup $\mathrm{SU}(3) \times \mathrm{SU}(3)$ of SU(6).

We understand $H^*_{\mathrm{SU}(n)}$ in terms of the classifying space of its maximal torus T^{n-1} as the subring of invariants of $H^*_{T^{n-1}}$ under the action of the Weyl group S_{n-1}. It will be easier to think about this in terms of $H^*_{\mathrm{U}(n)}$ and $H^*_{T^n}$ first, and then restrict. So before considering $H^*_{\mathrm{SU}(6)} \longrightarrow H^*_{\mathrm{SU}(3)\times\mathrm{SU}(3)}$ we will assess $H^*_{\mathrm{U}(6)} \longrightarrow H^*_{\mathrm{U}(3)\times\mathrm{U}(3)}$. Since U(6) and U(3) × U(3) share the diagonal unitary matrix subgroup T^6 as maximal torus, we can think about this map as

$$(H^*_{T^6})^{S_6} \hookrightarrow (H^*_{T^6})^{S_3\times S_3}.$$

Writing $H^*_{T^6} = \mathbb{Q}[t_1, t_2, t_3, t'_1, t'_2, t'_3]$, the total Chern class whose components are the symmetric polynomials on all six variables is

$$\widetilde{c} = \sum \widetilde{c}_n := \prod(1+t_j)\prod(1+t'_j) =: \sum c_n \sum c'_n = c \cdot c'$$

Gathering terms one finds

$$\widetilde{c}_n = \sum_{j=0}^{n} c_j c'_{n-j}.$$

Recalling that $H^*_{\mathrm{SU}(n)} \cong H^*_{\mathrm{U}(n)}/(c_1)$, we find the map we want is given from the preceding by setting all of $\widetilde{c}_1, c_1, c'_1$ to 0 and $c_n, c'_n = 0$ for $n > 3$. Explicitly, $H^*_{\mathrm{SU}(6)} \longrightarrow H^*_{\mathrm{SU}(3)\times\mathrm{SU}(3)}$ can be identified with

$$\begin{aligned} \mathbb{Q}[\widetilde{c}_2, \widetilde{c}_3, \widetilde{c}_4, \widetilde{c}_5, \widetilde{c}_6] &\longrightarrow \mathbb{Q}[c_2, c_3, c'_2, c'_3]: \\ \widetilde{c}_2 &\longmapsto c_2 + c'_2, \\ \widetilde{c}_3 &\longmapsto c_3 + c'_3, \\ \widetilde{c}_4 &\longmapsto c_2 c'_2, \\ \widetilde{c}_5 &\longmapsto c_3 c'_2 + c_2 c'_3, \\ \widetilde{c}_6 &\longmapsto c_3 c'_3. \end{aligned}$$

It can be observed that the image is precisely the subring invariant under the involution given by $c_j \longleftrightarrow c'_j$. The resemblance to (8.6.1) will not escape the watchful reader.

To compute the cohomology we just need to find the cohomology of the Cartan algebra

$$C := \mathbb{Q}[c_2, c_3, c'_2, c'_3] \otimes \Lambda[\sigma\widetilde{c}_1, \sigma\widetilde{c}_2, \sigma\widetilde{c}_3, \sigma\widetilde{c}_4, \sigma\widetilde{c}_5],$$

where the σc_n are suspensions of the Chern classes $\widetilde{c}_n$ living in $H^{2n-1}\mathrm{SU}(5)$, and the differential is the unique one taking $\sigma\widetilde{c}_n$ to the image in $H^*_{\mathrm{SU}(3)\times\mathrm{SU}(3}$ just determined. We filter C by the base degree in $H^*_{\mathrm{SU}(3)\times\mathrm{SU}(3)}$ and run the filtration spectral sequence. This is stable until $E_4 = C$, and then the first nonzero differential cancels $\sigma\widetilde{c}_2$ against $c_2 + c'_2$. The result is that $c'_2 \equiv -c_2$ in E_5. Writing $\bar{c}_2$ for the class $c_2 \mod c_2 + c'_2$, one has

$$E_5 = \mathbb{Q}[\bar{c}_2, c_3, c'_3] \otimes \Lambda[\sigma\widetilde{c}_3, \sigma\widetilde{c}_4, \sigma\widetilde{c}_5, \sigma\widetilde{c}_6]$$

with differentials

$$\begin{aligned} \widetilde{c}_3 &\longmapsto c_3 + c'_3, \\ \widetilde{c}_4 &\longmapsto -\bar{c}_2^2 \\ \widetilde{c}_5 &\longmapsto \bar{c}_2(c'_3 - c_3), \\ \widetilde{c}_6 &\longmapsto c_3 c'_3. \end{aligned}$$

The next differential is on E_6, and cancels $\sigma\widetilde{c}_3$ against $c_3 + c_3'$. Writing $\bar{c}_3$ for the class c_3 mod $c_3 + c_3'$, we get

$$E_7 = \mathbb{Q}[\bar{c}_2, \bar{c}_3] \otimes \Lambda[\sigma\widetilde{c}_4, \sigma\widetilde{c}_5, \sigma\widetilde{c}_6]$$

with

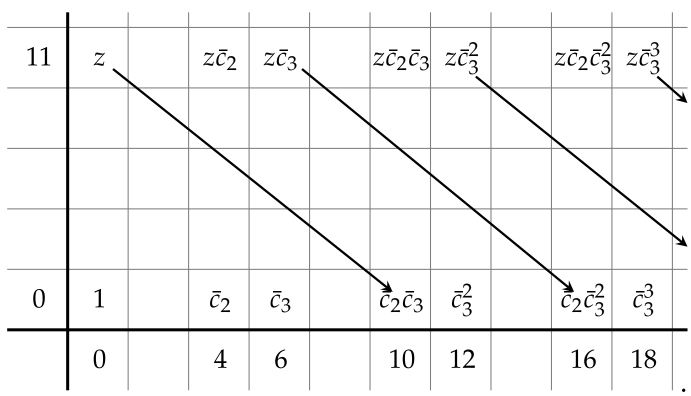
$$\begin{aligned} \sigma\widetilde{c}_4 &\longmapsto -\bar{c}_2^2, \\ \sigma\widetilde{c}_5 &\longmapsto -2\bar{c}_2\bar{c}_3, \\ \sigma\widetilde{c}_6 &\longmapsto -\bar{c}_3^2. \end{aligned}$$

This, of course, looks exactly like (8.6.2), and what happens in the spectral sequence from this point on will be the same up to grading. For thoroughness, we include the entire calculation. The differential d_8 cancels $\sigma\widetilde{c}_4$, and $\bar{c}_2^2$, so

$$E_9 \cong \mathbb{Q}[\bar{c}_2, \bar{c}_3] \big/ (\bar{c}_2^2) \otimes \Lambda[\sigma\widetilde{c}_5, \sigma\widetilde{c}_6].$$

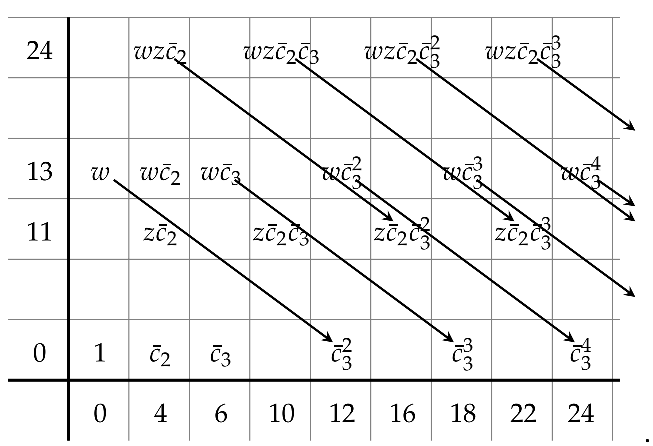
The next nontrivial differential, d_{10}, annihilates $\sigma\widetilde{c}_6$ and takes $z = -\sigma\widetilde{c}_5/2 \longmapsto \bar{c}_2\bar{c}_3$. We show this in Figure 8.6.6, omitting the parallel copy, which is $\sigma\widetilde{c}_6$ times the displayed part.

Figure 8.6.6: The E_{10} page for $\mathrm{SU}(6)/\mathrm{SU}(3)^2$



Now take $w = -\sigma\widetilde{c}_6$.

Figure 8.6.7: The E_{12} page for $\mathrm{SU}(6)/\mathrm{SU}(3)^2$



Finally, E_∞ is as follows:

Figure 8.6.8: The E_∞ page for $\mathrm{SU}(6)/\mathrm{SU}(3)^2$.

13		$w\bar{c}_2$		
11		$z\bar{c}_2$		$z\bar{c}_2\bar{c}_3$
0	1	$\bar{c}_2$	$\bar{c}_3$	
	0	4	6	10

.

The degrees of the surviving vector space generators are

$$0, \quad 4, \quad 6, \quad 15, \quad 17, \quad 21$$

and the products are determined by Poincaré duality. The bottom row of the E_∞ page represents

$$\mathbb{Q}[\bar{c}_2, \bar{c}_3] \big/ (\bar{c}_2, \bar{c}_3)^2,$$

the image of $H^*_{\mathrm{SU}(3)\times\mathrm{SU}(3)}$ in $H^*\big(\mathrm{SU}(6)/\mathrm{SU}(3)^2\big)$.

Remark 8.6.9. Aleksei Tralle [Tra93] commented that one similarly has informality for the same $K = \mathrm{SU}(3) \times \mathrm{SU}(3)$ embedded in the top-left 6×6 entries of $G = \mathrm{SU}(n)$ for $n \geqslant 6$.[6] The point is that the differentials of the first five generators of $PH^*\mathrm{SU}(n)$ are always the same, so formality is always destroyed, and one cannot partition 7 into two integers $\leqslant 3$, so the differentials of $\sigma\widetilde{c}_n$ for $n \geqslant 7$ are zero. Thus

$$H^*\big(\mathrm{SU}(n)/\mathrm{SU}(3)^2\big) \cong H^*\big(\mathrm{SU}(6)/\mathrm{SU}(3)^2\big) \otimes \overbrace{\Lambda[\sigma\widetilde{c}_7, \ldots \sigma\widetilde{c}_n]}^{H^*(\mathrm{SU}(n)/\mathrm{SU}(6))}.$$

Remark 8.6.10. It is possible to show that $\mathrm{SU}(3n)/\mathrm{SU}(3)^n$ is always of deficiency $n-1$.

Remark 8.6.11. Manuel Amann has a general theorem constructing many informal pairs, all of deficiency 1 [Ama13, Thm. E, Table 2]. In particular, he has an example in every dimension $\geqslant 72$.

8.7. Cohomology computations, IV: G/S^1

In order to obtain what was arguably the main result of the thesis this monograph evolved from, we needed a grasp on the cohomology rings $H^*(G/S;\mathbb{Q})$ of homogeneous spaces G/S for G compact connected and S a circle. It is not hard with the tools we have developed to describe these completely. In 2014, the author found the following dichotomy; note these are the only two options because $\dim_{\mathbb{Q}} H^1(S) = 1$.

Proposition 8.7.1. *Let G be a compact, connected Lie group and S a circle subgroup. Then the rational cohomology ring $H^*(G/S)$ has one of the following forms.*

[6] His point is actually to exhibit a nontrivial Massey product: the generator of order thirteen above represented by $z\bar{c}_2 + F_5$ lies in the product $\big\langle[\bar{c}_2], [\bar{c}_2], d[\bar{c}_3]\big\rangle$. In terms of the generators on the E_7 page, which is a DGA factor of the Cartan algebra, we find $d(-\sigma\widetilde{c}_4) = \bar{c}_2^2$ and $d(\sigma\widetilde{c}_5) = -2\bar{c}_2\bar{c}_3$, so $d(c_2\sigma\widetilde{c}_5 - 2c_3\sigma\widetilde{c}_4) = 0$.

1. *If $H^1(G) \longrightarrow H^1(S)$ is surjective, then there is $z_1 \in H^1(G)$ such that*

$$H^*(G/S) \cong H^*(G)/(z_1).$$

In terms of total Betti number, $h^\bullet(G) = \frac{1}{2}h^\bullet(G/S)$.

2. *If $H^1(G) \longrightarrow H^1(S)$ is zero, there are $z_3 \in H^3(G)$ and $s \in H^2(G/S)$ such that*

$$H^*(G/S) \cong \frac{H^*(G)}{(z_3)} \otimes \frac{\mathbb{Q}[s]}{(s^2)}.$$

In terms of total Betti number, $h^\bullet(G) = h^\bullet(G/S)$.

As it happens, we were not here first. General statements on the cohomology of a homogeneous space were already available to Jean Leray in 1946, the year after his release from prison [Mil00, sec. 3, item (4)]. In the second of his four *Comptes Rendus* announcements from that year [Ler46a, bottom of p. 1421], he states the following result.[7]

Theorem 8.7.2 (Leray, 1946)**.** *Let G be a compact, simply-connected, Lie group and S a closed, one-parameter subgroup [viz. a circle]. Then there exist an $n \in \mathbb{N}$, a primitive element $z_{2n+1} \in H^{2n+1}(G)$, and a nonzero $s \in H^2(G/S)$ such that*

$$H^*(G/S) \cong \frac{H^*(G)}{(z_{2n+1})} \otimes \frac{\mathbb{Q}[s]}{(s^{n+1})}$$

The following year, Jean-Louis Koszul published a note [Kos47b, p. 478, display] in the *Comptes Rendus* regarding Poincaré polynomials for these spaces.

Theorem 8.7.3 (Koszul, 1947)**.** *Let G be a semisimple Lie group and S a circular subgroup. Then the Poincaré polynomials (in the indeterminate t) of G/S and G are related by*

$$p(G/S) = p(G)\frac{1+t^2}{1+t^3}.$$

This result implies that in fact $n = 1$ in Leray's theorem. This enhanced version of Leray's result follows from Proposition 8.7.1 simply because $H^1(G) \cong H^2_G = 0$ for semisimple groups. The author is unaware of any published proof of the Leray and Koszul results, which is part of the motivation for including a proof of Proposition 8.7.1 here.

Before doing so, we illustrate the result with a representative example. Let S be a circle contained in the first factor $\mathrm{Sp}(1)$ of the product group $G = \mathrm{Sp}(1) \times \mathrm{U}(2)$. The cohomology of G is the exterior algebra

$$H^*(G) = \Lambda[q_3, z_1, z_3], \quad \deg z_1 = 1,\ \deg z_3 = \deg q_3 = 3,$$

and that of BS is

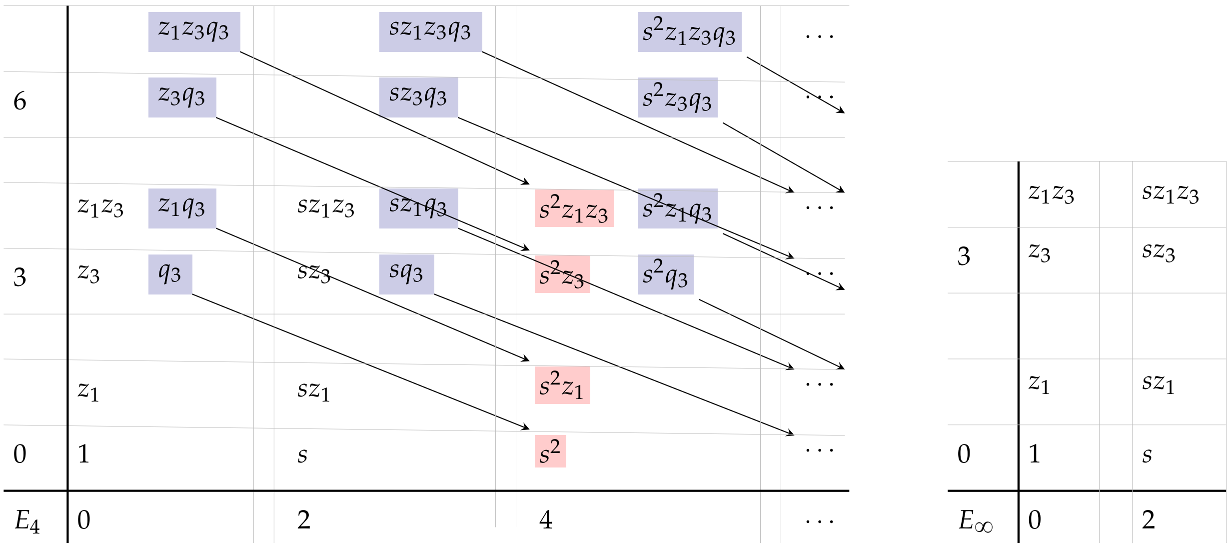
$$H^*_S = \mathbb{Q}[s], \qquad \deg s = 2.$$

The spectral sequence (E_r, d_r) associated to $G \to G_S \to BS$ is as follows. Its E_2 page is the tensor product $H^*_S \otimes H^*(G)$. Because the map $H^1(G) \longrightarrow H^1(S)$ is zero, the differential d_2 is zero,

[7] See also Borel [Bor98, par. 12]; only owing to Borel's summary are we confident "compact Lie group" is the contextually-correct interpretation of Leray's *groupe bicompact*, which translated literally would mean only that the group be compact Hausdorff.

and d_3 is zero for lacunary reasons, so $E_4 = E_2$. The differential d_4 annihilates each of s, z_1, z_3 and takes $q_3 \longmapsto s^2$.

Figure 8.7.4: The Serre spectral sequence of $\mathrm{Sp}(1) \times \mathrm{U}(2) \to \big(\mathrm{Sp}(1) \times \mathrm{U}(2)\big)_S \to BS$



Because d_4 is an antiderivation, its kernel is the subalgebra $\mathbb{Q}[s] \otimes \Lambda[z_1, z_3]$ and its image the ideal (s^2) in that subalgebra. Elements mapped to a nonzero element by d_4 are marked as blue in the diagram and elements in the image in red; the vector space spanned by these elements vanishes in E_5. Thus $E_5 = \Delta[s] \otimes \Lambda[z_1, z_3]$, where $\Delta[s] = \mathbb{Q}[s]/(s^2) \cong H^*S^2$. For lacunary reasons, $E_5 = E_\infty$. In fact,

$$G/S \;=\; {}^{\mathrm{Sp}(1)}\!/_{S} \times \mathrm{U}(2) \;\approx\; S^2 \times \mathrm{U}(2), \tag{8.7.5}$$

so this tensor decomposition was not unexpected.

This example has all the features of the general case; the pair is always formal, and either it is cohomology-surjective or else d_4 is a nontrivial differential taking some $z_3 \longmapsto s^2 \in H^4_S$, which then collapses the sequence at E_5. If $H^1(G) \neq 0$, then the exterior subalgebra of $H^*(G)$ generated by $H^1(G)$, an isomorphic $H^*(A)$, is in the Samelson subring, and can be split off before running the spectral sequence; the factoring out of this subalgebra is the algebraic analogue of the product decomposition (8.7.5) of G/S.

Lemma 8.7.6. *A compact pair (G, S^1) is formal.*

Proof. Consider the map $\rho^* \colon H^*_G \longrightarrow H^*_S$ in the sequence

$$H^*_G \xrightarrow{\rho^*} H^*_S \xrightarrow{\chi^*} H^*(G/S).$$

Because ρ^* is a homomorphism of graded rings and $H^*_S \cong \mathbb{Q}[s]$ is a polynomial ring in one variable, the cokernel $(\rho^*\widetilde{H}_S)$ of χ^* is generated by a single homogeneous element and hence is a regular ideal (s^n) for some n. By Theorem 8.4.8, it follows (G, S) is a formal pair. ☐

Proof of Proposition 8.7.1. If $H^1(G) \longrightarrow\!\!\!\!\rightarrow H^1(S)$, then Samelson's Corollary 1.0.7 applies and yields the result, so assume instead this map is zero. By Lemma 8.7.6, (G, S) is a formal pair, so

$$H^*(G/S) \cong {H^*_S}/\!\!/{H^*_G} \otimes \Lambda\widehat{P}$$

with $\dim \widehat{P} = \operatorname{rk} G - \operatorname{rk} S = \operatorname{rk} G - 1$ and $\dim \check{P} = 1$. It follows that $\rho^* \circ \tau$ takes $\check{P} \xrightarrow{\sim} \mathbb{Q}s^\ell$ for some ℓ, yielding Leray's theorem. To obtain Koszul's, it remains to show $\ell = 2$.

By Proposition B.2.4, we may replace G with its universal compact cover $A \times K$, where A is a torus and K simply-connected, and S with the identity component of its lift in this cover. If $H^1(G) \longrightarrow H^1(S)$ is trivial, then because $H^*(A)$ is generated by $H^1(A)$, it follows $H^*(A) \leqslant \Lambda\widehat{P}$ splits out of the Cartan algebra, so we may as well assume $G = K$ is semisimple.

[Update from published eqf_torus]

We now return to the map of spectral sequences described in Section 8.1.1. Recall the differentials in the spectral sequence (E_r, d_r) of the Borel fibration $K \to K_S \to BS$ vanish on H_S^* and are otherwise completely determined by by the composition

$$\rho^* \circ \tau \colon PK \longrightarrow H_K^* \longrightarrow H_S^*.$$

Because K is semisimple, $H^1(K) = 0$, so it follows $H_K^2 = 0$ as well by Borel's calculation from Section 7.4 of the spectral sequence of $K \to EK \to BK$. The edge homomorphisms d_2 and d_3 then must be zero, so

$$E_4 = E_2 = H_S^* \otimes H^*(K)$$

and the first potentially nontrivial differential is

$$d_4 \colon H^3(K) \xrightarrow{\sim} H_K^4 \longrightarrow H_S^4.$$

By Lemma 7.6.5, this is surjective, so $dz = \rho^*\tau z = s^2$ for some $z \in P^3(K)$. Thus $\widetilde{(\operatorname{im} \rho^*)}$ is generated by s^2 as claimed, concluding the proof. □

8.8. Valediction

At this point we have completed the exposition the author wished was available when he started work on his dissertation problem. We hope we have been able to do justice to the material so that the reader may find some measure of the beauty in it that the author does. This is of course neither the end nor the beginning of this story. We round out our account with some historical remarks and connections.

8.8.1. Cartan's approach to the Cartan algebra

Our presentation of the Cartan algebra computation of the cohomology ring $H^*(G/K; \mathbb{Q})$ of a homogeneous space G/K in this work introduced what we believe to be the least possible algebraic overhead, but is not the original version.

Cartan's account [Car51] was cast in Lie-algebraic terms, with the "choice of transgression" we have been somewhat casual about explicitly determined by a connection and induced from an $\mathbb{R}$-cdga called the ***Weil algebra***, $W\mathfrak{k} = S\Sigma\mathfrak{k}^* \otimes \Lambda\mathfrak{k}^*$, where $\mathfrak{k}^*$ is the dual to the Lie algebra of K, equipped with natural actions of $\mathfrak{k}$ by inner multiplications ι_ξ and the Lie derivative $\mathscr{L}_\xi$. The Weil algebra, as an algebra, is the Koszul algebra of Definition 7.3.3 but outfitted with a different differential which incorporates the adjoint action of the Lie algebra of G. It does this to emulate the behavior of connection and curvature forms determined by a connection on a principal bundle, and these in turn arise due to a desire to understand the cohomology of the total space of a principal bundle in terms of forms arising from pullback in its base. Thus it is an algebraic model of the cohomology of $EG \to BG$ and the homotopy quotient predating the general (1956) discovery of these objects. In particular, $H^*(BG)$ had not been calculated before this note.[8]

Given a principal K-bundle $K \to E \xrightarrow{\pi} B$, Cartan views a connection, as a linear map $\mathfrak{k}^* \longrightarrow \Omega^1(E)$ respecting both actions of $\mathfrak{k}$. Using the fact (Proposition 6.1.1) that there exist K-invariant representative forms for the classes on $H^*(E;\mathbb{R})$, Cartan constructs the ***Weil model*** $\big(S\Sigma\mathfrak{k}^* \otimes \Lambda\mathfrak{k}^* \otimes \Omega^\bullet(E)\big)_{\mathrm{bas}}$ of $H_K^*(E;\mathbb{R}) \cong H^*(B;\mathbb{R})$; here the subscript denotes the *basic subalgebra* annihilated by all ι_ξ and $\mathscr{L}_\xi$. The idea is that this should serve as a model for the base B, and indeed Cartan shows the natural inclusion of $\pi^*\Omega^\bullet(B) \cong \Omega^\bullet(B)$ in the Weil model is a quasi-isomorphism. He then shows the Weil model is quasi-isomorphic to the ***Cartan model*** $\big(S\Sigma\mathfrak{k}^* \otimes \Omega^\bullet(E)\big)^K$.[9] This in turn, when our principal bundle is $K \to G \to G/K$ for G another compact, connected Lie group, is quasi-isomorphic to a DGA with underlying algebra $(S\Sigma\mathfrak{k}^*)^K \otimes H^*(G)$.[10] This is the original version of the Cartan algebra.

8.8.2. The Eilenberg–Moore approach

There is a later chapter in the story of the cohomology of a homogeneous space, due to authors including Paul Baum, Peter May, Victor Gugenheim, Hans Munkholm, and Joel Wolf, using the Eilenberg–Moore spectral sequence.

The issue is that we only have a Cartan algebra over a field of characteristic zero. Without strictly commuting cochain models, we are not able to pick representatives for $H^*(G)$ in $C^*(G_K)$ in such a way as to get a ring structure, and in general torsion makes commutativity impossible.

Proposition 8.8.1 ([Bor51, Thm. 7.1]). *Let p be a positive prime. Then there is no functorial $\mathbb{F}_p$-CDGA model (A,d) for $H^*(-;\mathbb{F}_p)$ such that a closed inclusion $i\colon F \hookrightarrow X$ induces a surjection $A(X) \longrightarrow A(F)$.*

Proof. Suppose there were such a model. Let $F = \mathbb{C}\mathrm{P}^n$ for $n \geqslant p$ and $X = CF \simeq *$ be the cone over it. Let $a \in A^2(\mathbb{C}\mathrm{P}^n)$ represent a generator α in cohomology, so that $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{F}_p) \cong \mathbb{F}_p[\alpha]/(\alpha^{n+1})$, and let $\widetilde{a} \in A^2(X)$ be some extension of a to X. Then $d(\widetilde{a}^p) = p\widetilde{a}^{p-1} = 0$, so $\widetilde{a}^p$ represents a class in $H^{2p}(X;\mathbb{F}_p) = 0$ and hence $\widetilde{a}^p = d\widetilde{b}$ for some $\widetilde{b} \in A^{2p-1}(X)$. But then we would have $di^*\widetilde{b} = i^*d\widetilde{b} = i^*(\widetilde{a}^p) = (i^*\widetilde{a})^p = a^p$, so that $\alpha^p = 0$ in $H^{2p}(\mathbb{C}\mathrm{P}^n;\mathbb{F}_p)$, a contradiction. □

The last step in our journey to the Cartan algebra that worked with arbitrary coefficients was the map Section 8.1.1 of spectral sequences. If k is chosen such that $H^*(G;k)$ is an exterior algebra, then Theorem 7.4.1 does go through in characteristic $\neq 2$, so one still have $H^*(BG;k)$ a polynomial algebra on the transgressions and the map does still control many of the differentials in the Serre spectral sequence of $G \to G_K \to BK$. Because the Serre spectral sequence with $\mathbb{Q}$ coefficients is the filtration spectral sequence of the Cartan algebra by construction, we are able to recover what happens to elements that come from the free part of $H^*(G/K;\mathbb{Z})$ but rather little about the torsion.

[8] There also seems to have been a desire to stay in the realm of manifolds, so that finite-dimensional truncations of BK are mentioned instead. In Chevalley's review of this work, he states that *BG does not exist*, a statement that only makes sense if one demands finite-dimensionality.

[9] Cartan credits this reduction to Hirsch, as clarified by Koszul, but this point of view is not evident in Hirsch's *Comptes Rendus* announcement [Hir48] and Koszul's reworking is unpublished.

[10] For generic E, one can find a differential on the graded vector space $S\Sigma\mathfrak{k}^* \otimes H^*(B;\mathbb{R})$ whose cohomology is $H_K^*(E;\mathbb{R}) \cong H^*(B;\mathbb{R})$, but this isomorphism does not generally respect multiplication.

The cohomological Eilenberg–Moore spectral sequence starts from a pullback square and its resulting square of cochain algebras and cohomology rings

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} C^*(X \times_B Y) & \longleftarrow & C^*(Y) \\ \uparrow & & \uparrow \\ C^*(X) & \longleftarrow & C^*(B) \end{array} \qquad \begin{array}{ccc} H^*(X \times_B Y) & \longleftarrow & H^*(Y) \\ \uparrow & & \uparrow \\ H^*(X) & \longleftarrow & H^*(B). \end{array} \tag{8.8.2}$$

The commutativity of the last square makes $H^*(X \times_B Y)$ a module over $H^*(X) \otimes_{H^*(B)} H^*(Y)$; if B is a point and k a field the Künneth theorem says this is an isomorphism. The bundle eq. (2.4.1) says this map is an isomorphism if $F \to Y \to B$ is a bundle and $H^*(Y) \longrightarrow H^*(F)$ is surjective. To generalize this, consider the middle square, which allows us to make the observation that $C^*(X \times_B Y)$ a module over $C^*(X) \otimes_{C^*(B)} C^*(Y)$ in a differential-preserving manner.

This means the following. In general, a differential graded k-module $(M^\bullet, d_M)$ can be said to be a differential module *over a k-DGA* $(A^\bullet, d)$ if $d_M(ax) = da \cdot x + (-1)^{|a|} a \cdot d_M(x)$ for $a \in A^\bullet$ and $x \in M^\bullet$. One can construct a so-called *proper projective* $(A^\bullet, d)$-module resolution $(P_p^\bullet, d_p)$ of such a (M, d_M) conducive to the differential homological algebra setting. This carries both internal differentials d_p and resolution maps $P_p \longrightarrow P_{p+1}$, and filtering the total complex by the internal degree, one has $E_0 = P_\bullet^\bullet$ and $E_1 \cong M$, so $E_\infty = E_2 = H^*(M)$ and $P_\bullet^\bullet$ is a projective replacement for M. One uses this to define a *differential Tor*, written $\mathrm{Tor}^{p,n}_{(A,d)}(M^\bullet, N^\bullet)$, as the cohomology of the total complex of $P^\bullet \otimes_A N^\bullet$, analogously to the conventional Tor.

Filtering the algebra by filtration degree p yields a filtration spectral sequence with $E_1 \cong P_\bullet^\bullet \otimes_{H^*(A)} H^*(N)$ and $E_2 \cong \mathrm{Tor}^{\bullet,\bullet}_{H^*(A)}\big(H^*(M), H^*(N)\big)$ the traditional non-differential Tor. Because we resolve projectively, p is nonpositive, so this is a *left-half plane* spectral sequence and any square can receive arbitrarily many differentials, so convergence to the intended target, the differential $\mathrm{Tor}^{\bullet,\bullet}_{(A^\bullet,d)}\big((M^\bullet, d_M), (N^\bullet, d_N)\big)$, is not ensured.

Back in the motivating case, assume $F \to Y \longrightarrow B$ is a Serre fibration, so that $F \to X \times_B Y \longrightarrow X$ is as well. Pick a proper projective resolution $P_\bullet^\bullet$ of $C^*(X)$; then there is an induced DGA map $\phi\colon P_\bullet^\bullet \otimes_{C^*(B)} C^*(Y) \longrightarrow C^*(X \times_B Y)$ factoring through $P_0^\bullet \otimes_{C^*(B)} C^*(Y)$. If we filter $C^*(X \times_B Y)$ by the Serre filtration over X, $C^*(Y)$ by the Serre filtration over B, $P_\bullet^\bullet$ by total degree, and $P_\bullet^\bullet \otimes_{C^*(B)} C^*(Y)$ by the sum of degrees, then ϕ is filtration-preserving and so induces a map of spectral sequences. It is not hard to check that if $\pi_1 B$ acts trivially on $H^*(F)$, then $E_2(\phi)$ is the identity on $H^*\big(X; H^*(F)\big)$, so that ϕ is a quasi-isomorphism and $\mathrm{Tor}_{C^*(B)}\big(C^*(X), C^*(Y)\big) \cong H^*(X \times_B Y)$. The filtration spectral sequence of the previous paragraph in this case has $E_2 = \mathrm{Tor}_{H^*(B)}\big(H^*(X), H^*(Y)\big)$, and, if $\pi_1 B = 0$, the sequence converges. This is the ***Eilenberg–Moore spectral sequence***.

Our case of interest is given by $(Y \to B) = (BK \to BG)$ and $X = *$, so that $X \times_Y B \simeq G/K$. In this case the E_2 page is $\mathrm{Tor}^{\bullet,\bullet}_{H^*(BK;k)}\big(k, H^*(BG)\big)$, which in case $\mathbb{Q} \leqslant k$ is exactly the cohomology of the Cartan algebra, so the spectral sequence collapses and even gives the correct result at the algebra level. The desired generalization is that if $H^*(BK; k)$ and $H^*(BG; k)$ are polynomial rings, then the sequence should collapse at E_2. This is not at all obvious. The main line of approach runs through the following result.

Proposition 8.8.3. *If the vertical maps in a commutative diagram of differential graded k-modules*

$$\begin{array}{ccccc} A & \longleftarrow & \Gamma & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ M & \longleftarrow & \Lambda & \longrightarrow & N, \end{array} \tag{8.8.4}$$

are additive quasi-isomorphisms, then they induce an isomorphism $\mathrm{Tor}_\Gamma(A,B) \xrightarrow{\sim} \mathrm{Tor}_\Lambda(M,N)$*. If the vertical maps are multiplicative, this is an algebra isomorphism.*

Remark 8.8.5. We do not in fact need A, B, M, and N to be algebras for the algebra automorphism, just differential modules equivariant with respect to the map $\Gamma \to \Lambda$.

Proof. The map of algebraic Eilenberg–Moore spectral sequences is an isomorphism on E_2. □

Since $\mathrm{Tor}_{C^*(BG;k)}\big(k, C^*(BK;k)\big) \cong H^*(G/K;k)$, if we had quasi-isomorphisms between $C^*(BG;k)$ and $H^*(BG;k)$ making (8.8.4) commute, we would have a collapse result. It was only known how to construct such quasi-isomorphisms for K a torus, although it is now known they exist generally [Fra06, Prop. 1.3], and when they could be constructed, (8.8.4) did not usually commute. The proofs that emerged relied on extending the category k-DGA to a "homotopy version" requiring less than a DGA map but still inducing quasi-isomorphisms and showing (8.8.4) could be taken to commute up to homotopy. The strongest of these results is the following.

Theorem 8.8.6 (Munkholm [Mun74])**.** *Let k be a principal ideal domain such that $H^*(X;k)$, $H^*(Y;k)$, and $H^*(B;k)$ in* (8.8.2) *are polynomial rings in at most countably many variables. If* $\mathrm{char}\, k = 2$*, assume further that the Steenrod square* Sq^1 *vanishes on $H^*(X;k)$ and $H^*(Y;k)$. Then the Eilenberg–Moore spectral sequence of the square collapses at E_2, and $H^*(X \times_B Y;k) \cong E_2$ as a graded k-module.*

Thus the graded additive structure and bigraded multiplicative structure of the associated graded of $H^*(G/K;k)$ agree with $\mathrm{Tor}^{\bullet,\bullet}_{H^*_G}(k, H^*_K)$.

8.8.3. Biquotients and Sullivan models

Our expression for the cohomology of a homogeneous space generalizes to the quotient of G by the two-sided action $(u,v)\cdot g := ugv^{-1}$ of a subgroup U of $G \times G$, and one can consider the Borel fibration $G \to G_U \to BU$. If U acts freely on G, then $G_U \sim G/U$ is a ***biquotient***, a sort of space intensely studied in positive-curvature geometry, but if not, the algebra still makes sense, and if $U = K \times H$, then $H^*(G_U) = H^*_K(G/H)$ is the Borel K-equivariant cohomology of G/H, as discussed in Remark 5.5.5, whose study was the purpose of the dissertation this book emerged from.

The new Borel fibration looks like the bundle leading to the Cartan algebra but is no longer a principal G-bundle because G is not free on either side. Particularly, there is not a classifying map to BG-bundle. On the other hand, Eschenburg [Esc92] noticed that since $U \leqslant G \times G$, there is still a map $BU \longrightarrow BG \times BG$. Moreover, let us write $E(G \times G) = EG \times EG$, with the action $(g,h)\cdot(e,e') := (eg^{-1}, he')$. Then there is a natural map

$$\begin{array}{rcl} G_U = G \underset{U}{\otimes} (EG \times EG) & \longrightarrow & EG \underset{G}{\otimes} EG = BG, \\ g \otimes (e,e') & \longmapsto & e \otimes e', \end{array}$$

where the object on the right is BG because it is the quotient of the contractible total space of a principal G-bundle by G. The map

$$\begin{aligned} \Delta\colon EG \underset{G}{\otimes} EG &\longrightarrow BG \times BG, \\ e \otimes e' &\longmapsto (eG, Ge') \end{aligned}$$

then makes the following diagram commute:

$$\begin{array}{ccc} G & = & G \\ \downarrow & & \downarrow \\ G_U & \longrightarrow & BG \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ BU & \xrightarrow[Bi]{} & BG \times BG. \end{array} \tag{8.8.7}$$

One can actually check that G_U is isomorphic to the pullback. We would like to use this map the same way we used the Borel map before.

Exercise 8.8.8. Convince yourself that the map we called Δ can be identified up to homotopy with the diagonal map $BG \longrightarrow BG \times BG$.

The map Δ^* induced in cohomology is exactly the cup product, which, when k is taken such that $H^*(BG;k) \cong k[\vec{x}]$, has kernel the ideal generated by $x_j \otimes 1 - 1 \otimes x_j$, so one expects $\tau z_j = x_j \otimes 1 - 1 \otimes x_j$ in the Serre spectral sequence of the bundle Δ. One can check this guess by including the universal bundle in Δ two ways, via $EG \xrightarrow{\sim} EG \otimes_G Ge_0 \hookrightarrow EG \otimes_G EG$, which induces $BG \xrightarrow{\sim} BG \times \{Ge_0\} \hookrightarrow BG \times BG$ on the base, and via $EG \xrightarrow{\sim} \{e_0\} \otimes_G EG$. One of the projection picks up a sign due to the fact that one of the maps takes a right G-action to a left.

So the Serre spectral sequence of Δ is the filtration sequence of the CDGA $(H_G^* \otimes H_G^* \otimes H^*G, d)$ with $dz = 1 \otimes \tau z - \tau z \otimes 1$ on generators. Borel, in deriving the Chevalley algebra of Theorem 8.1.5, makes a generalization [Bor53, Thm. 24.1′] extracting a submodel $\Omega^\bullet(B) \otimes H^*F$ of $\Omega^\bullet(E)$, for a fiber bundle $F \to E \to B$, so long as H^*F is an exterior algebra on generators that transgress in the Serre spectral sequence, as this part of the argument no longer needs that the bundle is principal. Thus, using the same argument we used to obtain the Cartan model, then, we can use the map (8.8.7) to construct a model

$$(H_U^* \otimes H_G^*, d)$$

of G/U where d vanishes on H_U^* and takes a primitive $z \in PH^*G$ to

$$(Bi)^*(1 \otimes \tau z - \tau z \otimes 1).$$

It turns out Vitali Kapovitch discovered this model ten years before the author by more general considerations [Kap04, Prop. 1][FOT08, Thm. 3.50], which we will now elaborate.

Definition 8.8.9. We adopt the new convention that $\Lambda Q := SQ$ as well if Q is an evenly-graded rational vector space, so that any $\mathbb{Q}$-CDGA can be written as ΛV for V a graded rational vector space. A ***Sullivan algebra*** is a CDGA $(\Lambda V, d)$ such that V is an increasing union of graded subspaces $V(\ell)$ such that $V(-1) = 0$ and $dV(\ell) \leqslant \Lambda V(\ell - 1)$. (The effect is that any finitely generated subalgebra is annihilated by some power of d.) A ***Sullivan model*** of a space X is a quasi-isomorphism $(\Lambda V, d) \longrightarrow A_{\mathrm{PL}}(X)$ from a Sullivan algebra.

A ***pure Sullivan algebra*** is a Sullivan algebra $(\Lambda V = \Lambda Q \otimes \Lambda P, d)$ with Q evenly graded such that $V(0) = Q$ and P oddly-graded such that $V(1) = Q \oplus P$. That is, $dQ = 0$ and $dP \leqslant \Lambda Q$. All the finitely generated models we have discussed in this book have been pure Sullivan models.

Sullivan models behave well with respect to fibrations and pullbacks.

Theorem 8.8.10 ([FHT01, Prop. 15.5,8]). *Given a map of Serre fibrations*

$$\begin{array}{ccc} F & \longrightarrow & F' \\ \downarrow & & \downarrow \\ E & \longrightarrow & E' \\ \downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[f]{} & B' \end{array}$$

and Sullivan models $(\Lambda V_{B'}, d) \longrightarrow (\Lambda V_B, d)$ *for* f *and* $(\Lambda V_{B'}, d) \longrightarrow (\Lambda V_{B'} \otimes \Lambda V_{F'}, d)$ *for* q*, if* $H^*F' \longrightarrow H^*F$ *is an isomorphism,* $\pi_1 B$*,* $\pi_1 B'$*,* $\pi_0 E$*, and* $\pi_0 E' = 0$ *are zero and either* H^*F *or both of* H^*B *and* H^*B' *are of finite type, then* E *admits a Sullivan model*

$$(\Lambda V_E, d) = (\Lambda V_B, d) \underset{(\Lambda V_{B'}, d)}{\otimes} (\Lambda V_{B'} \otimes \Lambda V_{F'}, d) \cong (\Lambda V_B \otimes \Lambda V_{F'}, d).$$

The Cartan algebra is probably the first instance of this theorem, and Kapovitch derives his model as a consequence. It is clear this amalgamation of models has great flexibility. Here is another classical example.

Theorem 8.8.11 (Baum–Smith [BS67]). *Given a bundle* $G/H \to E \to B$ *induced from a principal* G*-bundle, with* G *and* H *connected Lie groups and* B *a formal space, one has a rational isomorphism*

$$H^*(E) \cong \mathrm{Tor}_{H^*_G}\big(H^*(B), H^*_K\big)$$

of graded algebras.

Baum and Smith actually additionally assume B is a Riemannian symmetric space, because they know these are formal Proposition 8.5.2, and that G is compact.

Proof. The assumption of the theorem is that there is some principal G-bundle $G \to \widetilde{E} \to B$ such that $E = \widetilde{E}/H$. Let $(\mathrm{X}, \chi)\colon (E \to B) \longrightarrow (EG \to BG)$ be both components of the classifying map, so that $\chi \circ p = \rho \circ \mathrm{X}$. Then reducing X modulo H induces the map $\widetilde{E}/H \longrightarrow EG/H$ in the diagram below.

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \longrightarrow & BH \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ B & \xrightarrow[\chi]{} & BG \end{array}$$

A model for G/H is given by the Cartan algebra $(H^*_K \otimes H^*G, d_{G/H})$. To extend this to a model of BH inducing the right map to G, take $A = (H^*_G \otimes H^*_K \otimes H^*G, d_{BH})$, with $d_{BH}z = \tau z \otimes 1 + 1 \otimes \rho^*\tau z \in H^*_G \otimes H^*_K$ for $z \in PH^*G$, where τ is a choice of transgression in the Serre spectral sequence of $G \to EG \to BG$. Filtering by H^*_K degree and running the filtration spectral sequence, one sees $H^*(A) = H^*_K$.

To get a model for χ, start with $A_{\mathrm{PL}}(\chi)\colon A_{\mathrm{PL}}(BG) \longrightarrow A_{\mathrm{PL}}(B)$ and precompose with $H^*_G \longrightarrow A_{\mathrm{PL}}(BG)$. Each generator of $H^*_G \cong \Lambda Q H^*_G$ goes to some cocycle in $A_{\mathrm{PL}}(B)$; lifting these to any Sullivan model $(\Lambda V_B, d_b)$ of B gives a map $\chi^\#\colon H^*_G \longrightarrow \Lambda V_B$ inducing χ^*. Applying Theorem 8.8.10 yields a model

$$\Lambda V_B \underset{H^*_G}{\otimes} (H^*_G \otimes H^*_K \otimes H^*G) = \Lambda V_B \otimes H^*_K \otimes H^*G.$$[11]

The factor $H^*_G \otimes H^*_K \otimes H^*G \cong (H^*_G \otimes H^*G \otimes H^*_G) \otimes_{H^*_G} H^*_K$ can be seen as a free H^*_G-module resolution of H^*_K, so the cohomology E, which is the cohomology of our model, can be identified with (differential) Tor:

$$H^*(E) \cong \mathrm{Tor}_{(H^*_G,0)}\big((\Lambda V_B, d_B), (H^*_K, 0)\big).$$

Now, since we assume B is formal, we can take $(\Lambda V_B, d_B) = \big(H^*(B), 0\big)$, so this collapses to the regular Tor of the claim. ☐

Remark 8.8.12. Baum and Smith of course did not use this language, but recalled the Eilenberg–Moore theorem that $H^*(E;\mathbb{R}) \cong \mathrm{Tor}_{\Omega^\bullet(BG)}\big(\Omega^\bullet(B), \Omega^\bullet(BK)\big)$. Here they have taken real coefficients to be able to use harmonic forms as representatives of $H^*(B;\mathbb{R})$ and used finite approximations of BG and BK to be able to describe their cohomology via forms. They take our model $H^*_G \otimes H^*_K \otimes HG$ as an H^*_G-module resolution of H^*_K and then use the three DGA quasi-isomorphisms $\big(H^*(B;\mathbb{R}), 0\big) \longrightarrow \Omega^\bullet(B)$, etc.

[EMAIL JOEL WOLF ABOUT THAT BIZARRE PAPER]

8.8.4. Further reading

The story of understanding the cohomology of the base of a bundle through invariant forms starts with the work of Élie Cartan in the early 1900s and continues through the work of Henri Cartan and his school (Koszul, Borel, and for a time Leray, with major unpublished contributions by Chevalley and Weil) in the late 1940s and early 1950s. The main and classical source for these developments is the conference proceedings [Cen51] to the 1950 *Colloque de Topologie (espaces fibrés)*, held in Bruxelles, with contributions by Beno Eckmann, Heinz Hopf, Guy Hirsch, Koszul, Leray, and Cartan. The second of the two papers by Cartan in this volume, “La transgression dans un groupe de Lie et dans un espace fibré principal” [Car51], promulgates in Lie-algebraic terms what we have called the Cartan algebra, as summarized in Section 8.8.1. This was later responsible for the institution of the *Cartan model* of equivariant cohomology, a full ten years before the Borel model gained currency. The classic sketched proof of the *equivariant de Rham theorem* showing the equivalence between these two models of equivariant cohomology is also contained in this terse paper.

[11] We do not need this level of detail, but the differential d restricts to d_B on V_B and to 0 on H^*_K, and sends $z \in PH^*G$ to $\chi^\#\tau z \otimes 1 + 1 \otimes \rho^*\tau z \in \Lambda V_B \otimes H^*_K$.

There is also no shortage of secondary sources for the work of this school [And62, Ras69, GHV76, Oni94], especially as it applies to the Cartan model of equivariant cohomology [GS99, GLS96, GGK02].

Appendix A

Algebraic background

In this appendix we gather a ragtag assortment of algebraic preliminaries. Notationally, in all that follows we denote containment of an algebraic substructure by "$\leqslant$," containment of an ideal by "$\trianglelefteq$," isomorphism by "$\cong$," and bijection by "$\leftrightarrow$." The restriction of a map $f\colon A \longrightarrow B$ to a subset $U \subseteq A$ is written $f|_U$.

A.1. Commutative algebra

We will take tensor products, direct products, and modules as given. Beyond this, we only need a very little pure commutative algebra, a corollary of Nakayama's lemma and a version of the Krull intersection theorem.

Lemma A.1.1 (Nakayama's lemma; [AM69, Cor. 2.7, p. 22])**.** *Let A be a commutative ring, M a finitely generated A-module, N a submodule of M, and $\mathfrak{a} \trianglelefteq A$ an ideal contained in the Jacobson radical. If $M = \mathfrak{a}M + N$, then $M = N$.*

Proposition A.1.2 ([AM69, Cor. 10.19, p. 110])**.** *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal contained in its Jacobson radical, and M a finitely-generated A-module. Then $\bigcap_{n=0}^{\infty} \mathfrak{a}^n M = 0$.*

Corollary A.1.3 ([GHV76, Lemma 2.8.I, p. 62])**.** *Let k be a commutative ring and $A = k[x_1, \ldots, x_n]$ a polynomial ring in finitely many indeterminates, and write $\mathfrak{a} = (x_1, \ldots, x_n) \triangleleft A$ for the ideal of positive-degree polynomials. Let M be a finitely-generated A-module and V a k-submodule of M, and suppose $M = \mathfrak{a}M + V$. Then $M = AV$.*

This is just an application of Nakayama's lemma A.1.1 to the case $N = AV$.

Alternate proof. Iteratively substituting the entire left-hand side of $M = \mathfrak{a}M + V$ in for the occurrence of M on the same-side, one inductively finds

$$\begin{aligned} M &= \mathfrak{a}M + V \\ &= \mathfrak{a}^2 M + \mathfrak{a}V + V \\ &\cdots \\ &= \mathfrak{a}^{n+1} M + \sum_{j=0}^{n} \mathfrak{a}^j V. \end{aligned}$$

Intersecting all right-hand sides yields $M = \bigcap_{n=0}^{\infty} \mathfrak{a}^n M + \sum_{n=0}^{\infty} \mathfrak{a}^n V$, but by Proposition A.1.2, $\bigcap_n \mathfrak{m}^n M$ is zero. □

A.2. Commutative graded algebra

A $\mathbb{Z}$-***graded*** k***-module*** is an $A \in k$-Mod admitting a direct sum decomposition $A = \bigoplus_{n\in\mathbb{Z}} A_n$. An element $a \in A$ is ***homogeneous*** if there exists some integer $|a| = \deg a$, the ***degree*** of A, such that $a \in A_{\deg a}$. We blur the distinction between $0 \in A_n$ and $0 \in A$, and leave the degree of the latter indeterminate. A k-module homomorphism $f\colon A \longrightarrow B$ between *graded* k-modules is said to be a ***graded*** k***-module homomorphism*** of ***degree*** $n = \deg f$ if

$$\deg f(a) = n + \deg a = \deg f + \deg a$$

for all homogeneous $a \in A$. We let gr-k-Mod be the category of graded k-modules and graded k-module homomorphisms.

A cohomology ring A will be a ***graded commutative*** k***-algebra***. This means A is a graded k-module, and additionally the product is such that

$$A_m \cdot A_n \leqslant A_{m+n};$$

and for all homogeneous elements $a, b \in A$, one has

$$ba = (-1)^{|a||b|}ab.$$

For us, these rings will actually be $\mathbb{N}$-***graded***, so that $A_n = 0$ for $n < 0$, and the absolute cohomology rings $H^*(X)$ (as opposed to relative cohomology rings $H^*(X, Y)$) will be unital, so that the map $x \mapsto x \cdot 1$ embeds $k \rightarrowtail A_0 \hookrightarrow A$ and the k-algebra structure can be seen as the restriction of the ring multiplication $A \times A \longrightarrow A$. We will call these k-CGAs for short, and the category of graded commutative k-algebras and degree-preserving k-algebra homomorphisms will be written k-CGA. The product in k-CGA is the ring product $A \times B$, graded by $(A \times B)_n = A_n \times B_n$.

Some k-algebras A we will encounter will have a ***bigrading***:

$$A = A^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}$$

in such a way that the ***bidegrees*** (p, q) add under multiplication:

$$A^{i,j} \cdot A^{p,q} \leqslant A^{i+p,j+q}.$$

We conventionally visualize such a ring as a grid in the xy-plane, with the p^{th} ***column***

$$A^{p,\bullet} = \bigoplus_{q} A^{p,q}$$

residing in the strip $p \leqslant x \leqslant p+1$ and the q^{th} ***row***

$$A^{\bullet,q} = \bigoplus_{p} A^{p,q}$$

residing in the strip $q \leqslant y \leqslant q+1$. For us, such gradings will always reside in the first quadrant: we demand $(p, q) \in \mathbb{N} \times \mathbb{N}$. A linear map $f\colon A \to B$ of bigraded algebras is said to have bidegree $\operatorname{bideg}(f) = (p, q)$ if $f(A^{i,j}) \leqslant B^{i+p,j+q}$. The associated singly-graded k-algebra of a bigraded algebra is $A^\bullet = \bigoplus_n A^n$, graded by $A^n := \bigoplus_{p+q=n} A^{p,q}$, and a bigraded algebra will be said to be ***commutative*** if this associated singly-graded algebra is a CGA.

As a particular example, given two graded k-algebras A, B, we can form the ***graded tensor product***: this is $A \otimes_k B$ as a group, equipped with the bigrading $(A \otimes_k B)^{p,q} = A^p \otimes_k B^q$. The associated singly graded algebra is also written $A \otimes_k B$ and is the coproduct in k-CGA. The resulting commutation rule is $(1 \otimes b)(a \otimes 1) = (-1)^{|a||b|} a \otimes b$ for $a \in A_{|a|}$ and $b \in B_{|b|}$. As often as feasible, we suppress ring subscripts on tensor signs, and in elements, we omit the tensor signs themselves, letting $a \otimes b =: ab$, so that for example we recover the reassuring expression $ba = (-1)^{|a||b|} ab$.

Given a graded unital k-algebra A with a preferred basis (a_j) of $A_0 \neq 0$, the map

$$\begin{aligned} A_0 \xrightarrow{\sim} k\{a_j\} &\longrightarrow k, \\ \sum \gamma_j a_j &\longmapsto \sum \gamma_j \end{aligned}$$

induces a natural ring homomorphism $A \twoheadrightarrow A_0 \longrightarrow k$ called the ***augmentation***. Its kernel $\widetilde{A}$ is called the ***augmentation ideal***; the notation is in analogy with reduced cohomology.[1] If $A_0 \cong k$, we say A is ***connected***; the terminology is because the singular cohomology of a connected space satisfies this condition. In this case, the augmentation ideal is $\bigoplus_{n \geqslant 1} A_n$.

Given a degree-zero homomorphism $f \colon A \longrightarrow B$ of connected augmented k-algebras, write

$$B /\!\!/ A := B/\big(f(\widetilde{A})\big).$$

This is the right conception of cokernel for maps between cohomology rings: one wants the 0-graded component to stay the same and the rest of the image of f to vanish. This sort of quotient will become relevant to us in Section 8.4, where it will be found that an important subring of the cohomology ring $H^*(G/K;\mathbb{Q})$, of a compact homogeneous space, namely the image of the characteristic map $\chi^* \colon H^*(BK;\mathbb{Q}) \longrightarrow H^*(G/K;\mathbb{Q})$, is of this form.

If A is a graded subalgebra of B, then one wants to think of

$$0 \to A \to B \to A /\!\!/ B \to 0$$

as a "short exact sequence" of rings, but of course this doesn't make sense: the sequence $A \to B \to C$ of k-modules is exact at B if $\operatorname{im}(A \to B) = \ker(B \to C)$, but the image of a ring map is a ring, while the kernel is an ideal, a different type of algebraic object. The appropriate modification is the following.

Definition A.2.1. A sequence $A \to B \to C$ of homomorphisms of unital k-algebras is said to be ***exact*** at B if

$$\ker(B \to C) = \big(\operatorname{im}(\widetilde{A} \to \widetilde{B})\big).$$

One should think of this as the ring-theoretic substitute for exactness in sequences of groups.

Example A.2.2. Let A be a graded k-subalgebra of a graded k-algebra B. Then $0 \to A \to B \to A /\!\!/ B \to 0$ is a short exact sequence, by design. If A and C are k-algebras, free as k-modules (in the applications we care most about, $k = \mathbb{Q}$), then taking $B = A \otimes C$, we see the sequence

$$0 \to A \longrightarrow A \otimes C \longrightarrow C \to 0$$

is short exact.

[1] Industry standard seems to be $\overline{A}$, but I have resisted this because $\widetilde{H}^*$ is the kernel of the augmentation in cohomology and I am used to overbar notation referring to quotients.

Remark A.2.3. This condition is usually called *coexactness* [MS68, p. 762]. The idea is that in any category C equipped with a zero object 0, there is a unique zero map $0_{A\to B}$ between two any objects, and one can define the (co)kernel of any map $A \longrightarrow B$ to be the (co)equalizer of it and $0_{A\to B}$. Suppose a composition $A \xrightarrow{f} B \xrightarrow{g} C$ is zero. Then f factors as $(\ker g)\circ \bar{f}$ for some morphism $\bar{f}$ and dually g factors as $\bar{g}\circ(\operatorname{coker} f)$. One says the sequence is ***exact*** at B if $\bar{f}$ is an epimorphism and ***coexact*** at B if $\bar{g}$ is an monomorphism. However [Car15], these notions are equivalent in the category of k-cdgas equipped with zero object the field k.

A.2.1. Free graded algebras

Suppose that $\operatorname{char} k \neq 2$. As with modules, there are free objects in the category of k-cgas, which have the following description. Given a free graded k-module V if we separate it into even- and odd-degree factors V_{even} and V_{odd}, then the ***free graded commutative k-algebra*** on V is the graded tensor product

$$SV_{\text{even}} \underset{k}{\otimes} \Lambda V_{\text{odd}}$$

of the symmetric algebra SV_{even} on the even-degree generators and the exterior algebra ΛV_{odd} on the odd-degree generators. Given k-bases $\vec{t} = (t_1, \ldots, t_m)$ of V_{even} and $\vec{z} = (z_1, \ldots, z_n)$ of V_{odd}, we also write these as

$$S[\vec{t}\,] := SV_{\text{even}};$$
$$\Lambda[\vec{z}\,] := \Lambda V_{\text{odd}}.$$

Write

$$\Delta[z_m] := k\{1, z_m\},$$

for the unique rank–two unital k-algebra with elements of degrees zero and m, which is the cohomology of an m-sphere. This is $\Lambda[z_m]$ for m odd and $S[z_m]/(z_m^2)$ for m even.

In the event $\operatorname{char} k = 2$, the graded commutativity relation $xy = (-1)^{|x|\,|y|}yx$, or equivalently $xy \pm yx = 0$, forces genuine commutativity $xy = yx$ for all elements since $1 = -1$ in k. Thus a free k-cga is a symmetric algebra SV in characteristic 2, independent of the grading on V. Algebras which merely *resemble* ΛV still play an important role in characteristic two.

Definition A.2.4. Let k be a commutative ring. A k-algebra A (*not* assumed graded commutative), free as a k-module, is said to have a ***simple system of generators*** $V = (v_1, \ldots, v_n, \ldots)$ if a k-basis for A is given by the monomials

$$v_{j_1} \cdots v_{j_\ell}, \qquad j_1 < \cdots < j_\ell,$$

where each generator occurs at most once. If A has a simple system of generators, we write

$$A =: \Delta V =: \Delta[v_1, \ldots, v_n, \ldots]$$

despite the fact that this description does not specify A up to algebra isomorphism.

Example A.2.5. The exterior algebra $\Lambda[z_1, \ldots, z_n]$ admits $z_1, \ldots, z_n$ as a simple system of generators.

This is of course the motivating example. Polynomial rings also afford examples.

Example A.2.6. The polynomial ring $k[x]$ admits $x, x^2, x^4, x^8, \ldots$ as a simple system of generators, as consequence of the binary representability of natural numbers.

Example A.2.7. The property of admitting a simple system of generators is preserved under tensor product (e.g., $k[x, y]$ admits $x^{2^i}y^{2^j}$ for $i + j > 0$ and $k[x] \otimes \Lambda[z]$ admits $x^{2^j} \otimes 1$ and $x^{2^j} \otimes z$) so in fact all free CGAs are examples.

The multiplication in a ΔV need not be anticommutative, as one can see from the following example.

Example A.2.8 ([Bor54, Théorème 16.4]). Borel found that the mod 2 *homology* ring of Spin(10) is given by

$$H_*(\mathrm{Spin}(10); \mathbb{F}_2) = \Delta[v_3, v_5, v_6, v_7, v_9, v_{15}],$$

where all $v_j^2 = 0$ and all pairs of v_j commute except for (v_6, v_9), which instead satisfies

$$v_6 v_9 = v_9 v_6 + v_{15}.$$

A.2.2. Poincaré duality algebras

The real cohomology ring of a compact manifold exhibits an important phenomenon which we generalize to an arbitary CGA.

Definition A.2.9. Let A be a k-CGA, free as a k-module. Suppose there exists a maximum $n \in \mathbb{N}$ such that $A_n \neq 0$, that $A_n \cong k$, and that for all $j \in [0, n]$ the natural pairing

$$A_j \times A_{n-j} \longrightarrow A_n$$

obtained by restricting the multiplication of A is nondegenerate. Then we call A a ***Poincaré duality algebra*** (or PDA) and a nonzero element of A_n a ***fundamental class*** for A, which we write as $[A]$. If we fix a homogeneous basis (v_j) of A, we can define a linear map $a \longmapsto a^*$ on A by setting $v_j^* := v_{n-j}$ whenever $v_j v_{n-j} = [A]$ and extending linearly. Such a linear map is called a ***duality map***.

Theorem A.2.10 (Poincaré; [BT82, I.(5.4), p. 44])**.** *If M is a compact manifold, the real singular cohomology ring $H^*(M; \mathbb{R})$ is a* PDA.

Example A.2.11. Let V be a finitely generated, oddly-graded free k-module. Then the exterior algebra ΛV is a Poincaré duality algebra with fundamental class given by the product of a basis of V.

Poincaré duality is a severe restriction on the structure of a ring, with powerful consequences, and it is inherited by tensor-factors.

Proposition A.2.12. *Let A and B be k-CGAs, free as k-modules, and suppose B is a* PDA. *Then $A \otimes B$ exhibits Poincaré duality just if A does.*

Sketch of proof. If A and B are PDAs with duals given by $a \longmapsto a^*$ and $b \longmapsto b^*$, then $a \otimes b \longmapsto a^* \otimes b^*$ is easily seen to be a duality on $A \otimes B$ up to sign. If, on the other hand, $b \longmapsto b^*$ is a duality on A and $a \otimes b \longmapsto \overline{a \otimes b}$ is a duality on $A \otimes B$, then for any homogeneous $a \in A$ one has $\overline{a \otimes 1} = a^* \otimes [B]$ for some $a^* \in A$, and $a \longmapsto a^*$ is a duality on A. □

A.2.3. Polynomials and numbers associated to a graded module

A graded k-module A is said to be of ***finite type*** if each graded component A_n has finite k-rank. Given a graded k-module A of finite type, we define the ***Poincaré polynomial*** of A to be the formal power series

$$p(A) := \sum_{n\in\mathbb{Z}} (\operatorname{rk}_k A_n)t^n.$$

The sum $p(X)|_{t=1} = \sum \operatorname{rk}_k A_n$ is the ***total rank*** or ***total Betti number*** of A. If the total Betti number of A is finite, then when we evaluate at $t=-1$ instead, we get the ***Euler characteristic*** $\chi(A) := p(X)|_{t=-1} = \sum(-1)^n \operatorname{rk}_k A_n$; otherwise the Euler characteristic is undefined.

In most cases we care about, the Poincaré polynomial will applied to a nonnegatively-graded k-CGA of finite type. The Poincaré polynomial is a homomorphism $\mathsf{gr}\text{-}k\text{-}\mathsf{Mod} \longrightarrow k[t]$ in the sense that

$$p(A\times B) = p(A) + p(B), \qquad p(A\otimes B) = p(A)\cdot p(B).$$

Usually the CGA in question will be the cohomology ring $H^*(X;k)$ of a space, and we will write

$$p(X) := p\big(H^*(X;k)\big) = \sum_{n\in\mathbb{N}} \operatorname{rk}_k H^n(X;k)t^n.$$

The individual ranks $h^k(X) := \dim_{\mathbb{Q}} H^k(X;\mathbb{Q})$ are called the ***Betti numbers*** of X; the associated total rank $p(X)|_{t=1} = \sum h^n(X)$ is called the ***total Betti number*** of the space and denoted $h^\bullet(X)$. The Euler characteristic $p(X)|_{t=-1} = \sum(-1)^n h^n(X)$ of $H^*(X;k)$ is called the Euler characteristic *of the space*, and written $\chi(X)$; it does not depend on k. If we write $h^{\text{even}}(X) = \sum h^{2n}(X)$ and $h^{\text{odd}}(X) = \sum h^{2n+1}(X)$, then

$$\begin{aligned} h^\bullet(X) + \chi(X) &= 2\cdot h^{\text{even}}(X); \\ h^\bullet(X) - \chi(X) &= 2\cdot h^{\text{odd}}(X) \end{aligned}$$

Free CGAs behave pleasantly under the Poincaré polynomial because $p(-)$ is multiplicative. If $\deg x = n$ is odd, then $p\big(\Lambda[x]\big) = 1+t^n$. Thus given an exterior algebra ΛV on an oddly-graded free k-module V of finite type, with Poincaré polynomial $p(V) = \sum t^{n_j}$ (where it is fine if some n_j occur more than once), the tensor rule yields

$$p(\Lambda V) = \prod(1+t^{n_j}).$$

Likewise, if $\deg x = n$ is even, then $S[x] = k[x]$ is spanned by $1, x, x^2, \ldots$, so

$$p\big(S[x]\big) = \sum_{j\in\mathbb{N}} t^{jn} = \frac{1}{1-t^n}.$$

Given a symmetric algebra SV on an evenly-graded free k-module V of finite type with $p(V) = \sum t^{n_j}$, thenthe tensor rule yields

$$p(SV) = \prod \frac{1}{1-t^{n_j}}. \tag{A.2.13}$$

Proposition A.2.14. *Let k be a field, V be a positively-graded k–vector space, SV the symmetric algebra, and W a graded vector subspace of SV such that the subalgebra it generates is a symmetric algebra SW and SV is a free SW-module. Then*

$$p(SV /\!/ SW) = \frac{p(SV)}{p(SW)}.$$

Proof. Let (q_α) be a homogeneous A-basis for SV. Then $(q_\alpha \otimes 1)$ forms a graded basis for $SV \mathbin{/\!/} SW = SV \otimes_{\widetilde{SW}} k$, so on the level of graded k-modules, one has $SV \cong SW \otimes_k k\{q_\alpha \otimes 1\} \cong SW \otimes (SV \mathbin{/\!/} SW)$. Taking Poincaré polynomials and dividing through by $p(SV \mathbin{/\!/} SW)$ gives the result. □

A.3. Differential algebra

Our cohomology theories will always take coefficients in an ungraded, commutative ring k with unity; usually, k will be $\mathbb{Q}$ or $\mathbb{R}$. The category of k-modules and k-module homomorphisms is denoted k-Mod. A ***differential k-module*** is a pair (A, d), where $A \in k\text{-Mod}$ is a k-module and $d \in \operatorname{End}_k A$, the ***differential***, is a nilsquare endomorphism, so that the composition $d^2 := d \circ d = 0$ is the constant map to the zero element. A homomorphism $f\colon (A, d) \to (B, \delta)$ in the category of differential k-modules, a group homomorphism $f\colon A \to B$ such that $fd = \delta f$.

A ***cochain complex*** (A, d) is a differential k-module such that $A \in \text{gr-}k\text{-Mod}$ and additionally d is of *degree* 1. A homomorphism of cochain complexes, as described in the first paragraph of the subsection, is then called a ***cochain map***.[2] We write $d \restriction A_n =: d_n$. A map $f\colon (A, d) \to (B, \delta)$ of cochain complexes is a cochain map of differential k-modules that is additionally a graded map of *degree* 0, so that $fA_n \leqslant B_n$. We let k-Ch denote the category of cochain complexes and cochain maps of k-modules,

The ***cohomology*** $H(A, d)$ of a differential k-module (A, d) is the quotient $(\ker d)/(\operatorname{im} d)$, which makes sense because $d^2 = 0$. We also write this as $H_d(A)$. The differential k-module is ***exact*** if $H_d(A) = 0$. A cochain map $f\colon (A, d) \to (B, \delta)$ induces a homomorphism $f^*\colon H(A, d) \to H(B, \delta)$ of k-modules. If this map is an isomorphism, then one says f is a ***quasi-isomorphism***.

If A is a chain complex, then $H(A, d)$ is graded by

$$H^n(A, d) := H^*(A, d)_n := \ker d_n / \operatorname{im} d_{n-1}.$$

Then a (graded) cochain map induces a map of graded modules, so cohomology is a functor $k\text{-Ch} \longrightarrow \text{gr-}k\text{-Mod}$. A cochain complex (A, d) is said to be ***acyclic*** if $H^*(A, d) = H^0(A, d) = k$, meaning $H^n(A, d) = 0$ for $n \neq 0$.

We will say a map $A \longrightarrow B$ of differential k-modules ***surjects in cohomology*** or is ***H^*-surjective*** if it induces a surjection $H^*(A) \twoheadrightarrow H^*(B)$. In the opposite extreme case, that the map $H^*(A) \longrightarrow H^*(B)$ is zero in dimensions $\geqslant 1$ and is the isomorphism $H^0(A) \longrightarrow H^0(B)$ in dimension 0, we call this map ***trivial***, and say the map $X \longrightarrow Y$ is ***trivial in cohomology***. If $A \longrightarrow B$ is the map $f^*\colon H^*(Y) \longrightarrow H^*(X)$ in cohomology induced by a continuous map $f\colon X \longrightarrow Y$, then we likewise say f is surjective in cohomology or trivial in cohomology if f^* is.

Given a chain complex, Euler characteristic is preserved under cohomology: one has the following corollary of the fundamental rank–nullity theorem of linear algebra, as applied to the differential d.

Proposition A.3.1. *Let (A, d) be a chain complex over k of finite total Betti number. Then*

$$\chi(A) = \chi\big(H^*(A, d)\big).$$

[2] A ***chain complex*** is a graded differential group $(A^\bullet, d)$ with $\deg d = -1$; a homomorphism of chain complexes is a ***chain map***. Chain and cochain complexes are mirror images of each other under the reindexing $A^n = A_{-n}$, and we will focus our attention on cochain complexes.

A.3.1. Differential graded algebras

A cohomology ring is a commutative graded algebra, and it is defined as the cohomology of a chain complex which is itself a graded algebra. We set out some commonplaces of these objects.

A chain complex $(A^\bullet, d)$ concentrated in nonnegative degree such that $A^\bullet$ is also a graded ring satisfying the product rule

$$d(ab) = da \cdot b + (-1)^{|a|} a \cdot db$$

for homogeneous elements a, b is a ***differential graded algebra*** (or k-DGA). A differential d on a graded ring satisfying this condition is called an ***derivation***.[3] An derivation on a unital k-algebra satisfies $d1 = 0$ and hence $d(k \cdot 1) = 0$. A morphism of DGAs is a k-algebra map that is simultaneously a cochain map. If A was a k-CGA, then we say (A, d) is a ***commutative differential graded algebra*** (henceforth k-CDGA).

The kernel of an derivation d is a subalgebra, because d is additive and because if $da = db = 0$, then $d(ab) = (da)b \pm a(db) = 0$. The image of d is an ideal of $\ker d$, because if $b = da \in B$ and $c \in \ker d$, then $b \in \ker d$ and $d(ac) = (da)c + a(dc) = bc$. It follows that $H^*(A^\bullet, d)$ is again a graded k-algebra.

The product in the category of DGAs is the graded ring direct product $A \times B$, equipped with the differential $d(a, b) := (da, db)$. The coproduct is the same tensor product $A \underset{k}{\otimes} B$ as for CGAs, equipped with the unique derivation given by

$$d(a \otimes b) = d_A a \otimes b + (-1)^{|a|} a \otimes d_B b$$

on pure tensors. If we omit the tensor signs, this gives back, formally, the same product rule.

A ***differential bigraded algebra*** (A, d) is a bigraded algebra such that d is an antideriviation on the associated singly-graded algebra $A^\bullet$ of degree 1. We make no additional demands as to how d interacts with the bigrading, but note that since $dA^n \leqslant A^{n+1}$, one has for each bidegree (i, j) that $dA^{i,j} \leqslant \bigoplus_\ell A^{i+\ell, j+1-\ell}$, and composing with projections to $A^{i+\ell, j+1-\ell}$, one obtains ***component maps*** $d^\ell \colon A^{i,j} \longrightarrow A^{i+\ell, j+1-\ell}$ of bidegree $(\ell, 1-\ell)$ such that

$$d = \sum_{\ell \in \mathbb{Z}} d^\ell.$$

A.3.2. The algebraic Künneth theorem

It is trivial that a product of DGAs induces a product decomposition in taking cohomology. In an ideal world, the same would remain true of coproducts, and this ideal world is achieved in the event one of the DGAs lacks torsion.

Theorem A.3.2. *Let k be a principal ideal domain and suppose A and C are free graded differential k-modules. Then*

$$H^n(A \underset{k}{\otimes} C) \cong \bigoplus_{0 \leqslant j \leqslant n} \big(H^j(A) \otimes H^{n-j}(C)\big) \oplus \bigoplus_{0 \leqslant j \leqslant n} \operatorname{Tor}_1^k\big(H^{j+1}(A), H^{n-j}(C)\big).$$

[3] Classically, this was an *antiderivation* and a *derivation* was required to satisfy $d(ab) = da \cdot b + a \cdot db$ independent of degree, but this is never the right notion in the graded context.

Proof. Write $Z^n = \ker(d^n \colon A^n \longrightarrow A^{n+1})$ and $B^n = \operatorname{im}(d^{n-1} \colon A^{n-1} \longrightarrow A^n)$. Then one has a short exact sequence

$$0 \to Z \longrightarrow A \longrightarrow B^{\bullet+1} \to 0$$

of complexes where the differentials on Z and $B^{\bullet+1}$ are 0. Since we have assumed C is flat, on tensoring these complexes with C, we obtain a short exact sequence

$$0 \to Z \otimes C \longrightarrow A \otimes C \longrightarrow B^{\bullet+1} \otimes C \to 0$$

of complexes, where the differentials on $Z^\bullet \otimes C$ and $B^{\bullet+1} \otimes C$ are both $\mathrm{id}_A \otimes d_C$ and the differential on $A^\bullet \otimes C$ is the expected $d^A \otimes \mathrm{id}_C \pm \mathrm{id}_A \otimes d_C$. Write $i^\bullet \colon B^\bullet \longrightarrow Z^\bullet$ for the inclusion; then it is not hard to see the the connecting map in the long exact sequence in cohomology is the map $(i \otimes \mathrm{id}_C)^* \colon B^\bullet \otimes H^*(C) \longrightarrow Z^\bullet \otimes H^*(C)$ induced by $i \otimes \mathrm{id}_C$. Thus we get a short exact sequence

$$0 \to \operatorname{coker}(i \otimes \mathrm{id}_C)^* \longrightarrow H^*(A \otimes C) \longrightarrow \ker(i^{\bullet+1} \otimes \mathrm{id}_C)^* \to 0.$$

Because $0 \to B^{\bullet+1} \longrightarrow Z^{\bullet+1} \longrightarrow H^{\bullet+1}(A) \to 0$ is exact, the first term is $H^*(A) \otimes_k H^*(C)$ and the last is $\operatorname{Tor}_1^k\big(H^{*+1}(A), H^*(C)\big)$. Re-sorting summands to gather equal total degrees yields the statement of the theorem. □

In particular, one has the following.

Corollary A.3.3. *Let A and C be k-DGAs free as k-modules and such that $H^*(C)$ is flat over k. Then*

$$H^*(A \underset{k}{\otimes} C) \cong H^*(A) \underset{k}{\otimes} H^*(C)$$

as k-algebras.

Proof. The hypotheses precisely ensure the Tor_1^k term vanishes. □

Note that it more than suffices k be a field.

A.4. Splittings

An epimorphism $A \twoheadrightarrow B$ is said to ***split*** if there exists a monomorphism $B \rightarrowtail A$, called a ***section***, such that the composition $B \to A \to B$ is the identity on B. This section is virtually never canonical, but it is frequently still useful to be able to lift the structure of B back into A in however haphazard a manner.

Surjective homomorphisms onto free objects always split in categories whose objects carry a group structure (we always assume the axiom of choice), and we use this simple fact repeatedly.

Proposition A.4.1. *Let $\pi \colon A \twoheadrightarrow F$ be a surjection in* gr-k-Mod *and suppose F is free. Then π splits.*

Proof. Let S be a k-basis for F and for each $s \in S$ pick a preimage $a_s \in \pi^{-1}\{s\}$. This assignment extends to the needed section. □

Restricting to the case everything lies in one graded component, one obtains the result in k-Mod. Specializing to the category S^1-Mod of modules over $S^1 \cong \mathbb{R}/\mathbb{Z}$ one obtains the following useful statement.

Proposition A.4.2. *Any exact sequence $0 \to A \longrightarrow B \longrightarrow C \to 0$ of tori splits: we can write $B \cong A \oplus \sigma(C)$ as an internal direct sum of topological groups for some suitable section $\sigma\colon C \rightarrowtail B$ of the projection to C.*

Alternate proof. Any short exact sequence of free abelian groups splits, and the functors

$$A \longmapsto A \underset{\mathbb{Z}}{\otimes} \mathbb{R}/\mathbb{Z},$$
$$\pi_1(T, 1) \longleftarrow T$$

furnish an equivalence of categories between finitely generated free abelian groups and tori. □

We will also need to apply this principle to CGAs.

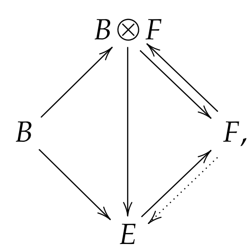
Proposition A.4.3. *Let F be a free k-CGA and $\pi\colon A \twoheadrightarrow F$ a surjective k-CGA homomorphism. Then there exists a section $i\colon F \rightarrowtail A$ of π.*

Proof. Suppose F is free on the graded k-module V. Since V is free as a graded module, there exists a section $i\colon V \rightarrowtail A$ of π over V by Proposition A.4.1. As π is a ring homomorphism, the subalgebra A' generated in A by iV projects back onto F under π. Were A' not itself a free k-CGA, there would be some relation between homogeneous elements of A' other than those ensured by the CGA axioms, and π would transfer that relation to F, so there is no such relation. Thus $\pi|_{A'}$ is a CGA isomorphism; now extend i to be its inverse. □

When we deal with principal bundles, the following simple proposition will be useful.

Proposition A.4.4. *Let $0 \to B \longrightarrow E \longrightarrow F \to 0$ be a exact sequence of k-CGA maps with F free and E of finite type. Suppose further that for each degree n we have $\mathrm{rk}_k\, E_n = \mathrm{rk}_k(B \underset{k}{\otimes} F)_n$. Then $E \cong B \otimes F$.*

Proof. The projection $E \longrightarrow F$ splits by Proposition A.4.3, and together with $\widetilde{B}$, the lift of $\widetilde{F}$ generates E as an algebra, so there is a commutative diagram



of ring maps with the vertical map surjective. If this vertical map failed to also be injective, the rank assumption would fail, so it is an isomorphism. □

Appendix B

Topological background

In this appendix we state some well-known results in algebraic topology and Lie theory. We will take homotopy groups and singular homology and cohomology groups as known concepts, and cite basic results in algebraic topology without proof, but will restate that the 0^{th} *homotopy set* $\pi_0 X$ of a space X is its set of path-components, which inherits a group structure if X is a group. We denote homotopy equivalences by "$\simeq$," homeomorphisms by "$\approx$," and Lie group isomorphisms by "$\cong$." If a group G acts on a space X via $\phi\colon G \times X \longrightarrow X$, we write $\phi\colon G \curvearrowright X$. The interior of a manifold M with boundary ∂M is $\mathring{M}$. The complement of a set $A \subseteq B$ is $B \backslash A$.

B.1. Algebraic topology grab bag

This section is just a collection of useful algebro-topological results we will need later, presented without much in the way of motivation, which one might have encountered in a first topology or Lie theory course.

Let Top be the category whose objects are pairs (X, A) of topological spaces, A closed in X, with morphisms $(X, A) \longrightarrow (Y, B)$ those continuous maps $f\colon X \longrightarrow Y$ such that $f(A) \subseteq B$. The category whose objects are individual topological spaces and morphisms continuous maps is included as a full category through the inclusion $X \longmapsto (X, \varnothing)$, where $\varnothing$ is the empty space.

B.1.1. Cell complexes

A ***CW complex*** is a topological space X equipped with a decomposition into a union of disks of increasing dimension. Less elliptically, such an X must admit a filtration (X^n) into ***n-skeleta*** meeting the following conditions:

- The 0-skeleton X^0 is a discrete space.

- Given the n-skeleton X^n, index a collection of distinct $(n+1)$-disks as $(D^{n+1}_\alpha)_{\alpha \in \mathrm{A}}$. From each boundary S^n_α, let a continuous map $\varphi_\alpha\colon S^n_\alpha \longrightarrow X^n$, the ***attaching map***, be given. These maps assemble into a map $\varphi\colon \coprod_{\alpha \in \mathrm{A}} S^n_\alpha \longrightarrow X^n$, and X^{n+1} is defined to be the quotient space

$$X^n \amalg \coprod_{\alpha \in \mathrm{A}} D^{n+1}_\alpha \Big/ s \sim \varphi(s)$$

  of the disjoint union: we've identified the boundaries of the D^{n+1}_α with their images in X^n.

- The entire space X is $X = \bigcup_{n \in \mathbb{N}} X^n$, the colimit, with the direct limit topology. This amounts to saying $U \subseteq X$ is open just if each $U \cap X^n$ is open in X^n.

A map $f\colon X \longrightarrow Y$ between two CW complexes is said to be ***cellular*** if it respects the skeleta: $f\colon X^n \longrightarrow Y^n$ for all n.

Write CW for the subcategory of Top whose objects are ***CW pairs*** consisting of a CW complex X and closed subcomplex A, and whose maps $(X, A) \longrightarrow (Y, B)$ are required to be cellular, meaning both $X \longrightarrow Y$ and the restriction $A \longrightarrow B$ are cellular. The category CW is a homotopy-theoretic skeleton of Top in the sense that given any $(X, A) \in$ Top there exists $(\widetilde{X}, \widetilde{A}) \in$ CW and a weak homotopy equivalence $(\widetilde{X}, \widetilde{A}) \longrightarrow (X, A)$ in Top. This map (or (X, A) itself) is called a *CW approximation* [Hat02, Example 4.15, p. 353]. Moreover, any map of pairs is the same up to homotopy as a map between CW complexes: given a map $(X, A) \longrightarrow (Y, B)$ of pairs there exists a map between CW approximations making the following square commute up to homotopy:

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{A}) & \to & (X, A) \\ \downarrow & & \downarrow \\ (\widetilde{Y}, \widetilde{B}) & \to & (Y, B). \end{array}$$

Although CW is unstable under the formation of mapping spaces, with judicious use of CW approximations, we may basically assume every space that follows is a CW complex.

The algebraic Künneth Theorem A.3.2 has at least two major topological repercussions.

Theorem B.1.1 (Universal coefficients [Hat02, Thms. 3.2, 3.A.3, pp. 195, 264])**.** *Let X be a topological space and k a principal ideal domain. For each $n \in \mathbb{N}$ one has the following short exact sequences of abelian groups:*

$$0 \to H_n(X; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} k \longrightarrow H_n(X; k) \longrightarrow \operatorname{Tor}_1^k\big(H_{n-1}(X; \mathbb{Z}), k\big) \to 0,$$
$$0 \to \operatorname{Ext}_k^1\big(H_{n-1}(X; \mathbb{Z}), k\big) \longrightarrow H^n(X; k) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}\big(H_n(X; \mathbb{Z}), k\big) \to 0.$$

Proof. The homology sequence follows from Theorem A.3.2 by taking $C = C_0 = k$ and $A = C_\bullet(X)$ the singular chain complex, taking into account the differentials go in the opposite direction expected. The cohomology sequence arises from taking $C = k$ and $A = \operatorname{Hom}_{\mathbb{Z}}\big(C_\bullet(X), \mathbb{Z}\big)$ the singular cochain complex, noting $A \underset{\mathbb{Z}}{\otimes} k \cong \operatorname{Hom}_{\mathbb{Z}}\big(C_\bullet(X), k\big)$. □

Theorem B.1.2 (Topological Künneth [Hat02, Thm. 3B.6, 3.21][Mas91, Thm. 11.2, p. 346])**.** *Let X and Y be topological spaces and k an abelian group. Suppose $H^*(X)$ is of finite type. Then for each $n \in \mathbb{N}$ one has the following split short exact sequences of abelian groups:*

$$0 \to \bigoplus_{0 \leqslant j \leqslant n} \big(H_j(X) \otimes H_{n-j}(Y)\big) \longrightarrow H_n(X \times Y; k) \longrightarrow \bigoplus_{0 \leqslant j \leqslant n} \operatorname{Tor}_1^k\big(H_j(X; k), H_{n-j-1}(Y; k)\big) \to 0;$$
$$0 \to \bigoplus_{0 \leqslant j \leqslant n} \big(H^j(X; Z) \otimes H^n(Y; k)\big) \longrightarrow H^n(X \times Y; k) \longrightarrow \bigoplus_{0 \leqslant j \leqslant n+1} \operatorname{Tor}_1^{\mathbb{Z}}\big(H^j(X; \mathbb{Z}), H^{n+1-j}(Y; k)\big) \to 0.$$

When one of the rings $H^*(X; k)$ or $H^*(Y; k)$ is free as a k-module, the Ext and Tor terms disappear and these isomorphisms assume a product form

$$H^*(X \times Y) \cong H^*X \otimes H^*Y.$$

One also obtains the following relation between integral homology and cohomology.

Proposition B.1.3. *Let X be a topological space. The torsion subgroups and torsion-free quotients of the singular homology and cohomology groups $H_*(X;\mathbb{Z})$ and $H^*(X;\mathbb{Z})$ satisfy*

$$H^n(X;\mathbb{Z}) \cong H_n(X;\mathbb{Z})_{\text{free}} \oplus H_{n-1}(X;\mathbb{Z})_{\text{tors}}$$

We will use fiber bundles frequently, and need a criterion for determining when the fundamental groups of their base spaces are trivial.

Theorem B.1.4 ([Hat02, Thm. 4.3])**.** *Let $F \to E \to B$ be a fiber bundle. Then there is associated a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_2(F) \longrightarrow \pi_2(E) \longrightarrow \pi_2(B) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \longrightarrow \pi_0(F) \longrightarrow \pi_0(E) \to 0.$$

There are important but subtle relations between the homology and homotopy groups.

Proposition B.1.5. *The first singular homology group of a space X is the abelianization of its fundamental group: $H_1(X;\mathbb{Z}) \cong \pi_1(X)^{\text{ab}}$.*

Theorem B.1.1. *Let X be a simply-connected topological space and let $n > 1$ be the least natural number such that $\pi_n X$ is nontrivial. Then the same n is also minimal such that $H_n X$ is nontrivial, and the natural* ***Hurewicz map***

$$\begin{aligned} \pi_n X &\longrightarrow H_n X, \\ [\sigma\colon S^n \longrightarrow X] &\longmapsto \sigma_*[S^n], \end{aligned}$$

taking the homotopy class of a map from a sphere to the pushforward of the fundamental class, is an isomorphism.

The homotopy groups completely determine homotopy type in the following sense.

Theorem B.1.6 (Whitehead [Hat02, Thm. 4.5, p. 346])**.** *Let $f\colon X \longrightarrow Y$ be a map of CW complexes such that $\pi_n f\colon \pi_n X \xrightarrow{\sim} \pi_n Y$ is an isomorphism for all $n \geqslant 0$ (a* ***weak homotopy equivalence****). Then f is a homotopy equivalence.*

Theorem B.1.7 (Whitehead [Hat02, Thm. 4.21, p. 356])**.** *Let $f\colon X \longrightarrow Y$ be a weak homotopy equivalence of topological spaces. Then $H^n f\colon H^n Y \xrightarrow{\sim} H^n X$ is an isomorphism for all n.*

We will also need the Lefschetz fixed point theorem. Note that if X is of finite type, the natural maps $H^n(X;\mathbb{Z}) \twoheadrightarrow H^n(X;\mathbb{Z})_{\text{free}} \rightarrowtail H^n(X;\mathbb{Q})$ carry a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module $H^n(X;\mathbb{Z})_{\text{free}}$ to a $\mathbb{Q}$-basis of $H^n(X;\mathbb{Q})$.

Definition B.1.8. Let $f\colon X \to X$ be a continuous self-map of a topological space X of finite type. Then associated endomorphisms $H^n(f) \in \operatorname{Aut}_{\mathbb{Q}} H^n(X;\mathbb{Q})$ are defined for each $n \geqslant 0$. The ***Lefschetz number***

$$\chi(f) := \sum_{n \geqslant 0} (-1)^n \operatorname{tr} H^n(f)$$

is the alternating sum of these traces, where each trace is taken with respect to a basis of $H^n(X;\mathbb{Q})$ inherited from $H^n(X;\mathbb{Z})_{\text{free}}$.

Since the trace of the identity map of a vector space is just the dimension of that space and $H^n(\mathrm{id}_X) = \mathrm{id}_{H^n(X;\mathbb{Q})}$ one immediately has the following.

Proposition B.1.9. *Let $f\colon X \longrightarrow X$ be a continuous self-map of a topological space of finite type. Then the Lefschetz number of the identity map id_X is the Euler characteristic of X:*

$$\chi(X) = \chi(\mathrm{id}_X).$$

The more interesting fact about the Lefschetz number is the Lefschetz fixed point theorem.

Theorem B.1.10 (Lefschetz, [Hat02, Thm. 2C.3, p. 179])**.** *Let X be a topological space which is a deformation retract of a simplicial complex and $f\colon X \longrightarrow X$ a continuous map without fixed points. Then the Lefschetz number $\chi(f)$ is 0.*

B.2. Covers and transfer isomorphisms

In this section, we leverage a standard result on the cohomology of covers to a statement we use later about the cohomology of homogeneous spaces.

Proposition B.2.1 ([Hat02, Prop. 3G.1])**.** *Let F be a finite group acting by homeomorphisms on a space X, so that $p\colon X \longrightarrow X/F$ is a finite covering. Suppose $|F|$ is invertible in k. Then the map*

$$p^*\colon H^*(X/F;k) \longrightarrow H^*(X;k)$$

is an injection with image the invariant subring $H^*(X;k)^F$.

Proof. Since simplices Δ^n are simply-connected, each singular simplex $\sigma\colon \Delta^n \longrightarrow X/F$ lifts to a singular simplex $\tilde{\sigma}\colon \Delta^n \longrightarrow X$. The map $\tau\colon \sigma \longmapsto \sum_{f\in F} f\circ\tilde{\sigma}$ summing over all such lifts then induces a ***transfer map*** $\tau\colon C_n(X/F) \longrightarrow C_n(X)$ of singular chain groups. For each lift $f\tilde{\sigma}$ we have $p(f\tilde{\sigma}) = \sigma$, so $p\circ\tau = |F|\cdot\mathrm{id}$ on $C_n(X/F)$. Dualizing yields a cochain map $\tau^*\colon C^n(X;k) \longrightarrow C^n(X/F;k)$ such that $\tau^*\circ p^* = |F|\cdot\mathrm{id}$ on $C^n(X;k)$, so the same holds in $H^*(X;k)$.

If we demand $|F|$ be a unit in k, then $\tau^*\circ p^*$ is an isomorphism, so p^* is injective. Since $p\circ f = p$ for all $f \in F$, it follows $\operatorname{im} p^*$ is contained in the invariant subring $H^*(X;k)^F$. On the other hand, since $\tau\circ p$ sends $\tilde{\sigma} \longmapsto \sum_{f\in F} f\circ\tilde{\sigma}$, it follows $p^*\tau^*\alpha = \sum_{f\in F} f^*\alpha$ for all $\alpha \in H^*(X;k)$. In particular, if $\alpha \in H^*(X;k)$ is F-invariant, then $p^*\tau^*\alpha = |F|\alpha$, so p^* surjects onto $H(X;k)^F$. □

Corollary B.2.2. *In the situation of Proposition B.2.1, suppose the action of F on X is the restriction of a continuous action of a path-connected group Γ on X. Then*

$$H^*(X/F;k) \cong H^*(X;k)$$

Proof. Let $f \in F$. Since Γ is path-connected, the left translation $\gamma \longmapsto f\gamma$ on Γ is homotopic to the identity. It follows f acts trivially on $H^*(X;k)$. Thus $H^*(X;k)^F \cong H^*(X;k)$. □

Proposition B.2.3. *Let Γ be a path-connected group, H_0 a connected subgroup, and F a finite central subgroup of Γ. Write $F_0 = F \cap H_0$ and suppose $|F/F_0|$ is invertible in k. Then*

$$H^*(\Gamma/FH_0) \cong H^*(\Gamma/H_0).$$

Proof. The space Γ/FH_0 is the quotient of Γ/H_0 by the left action of F/F_0 given by $fF_0\cdot\gamma H_0 = \gamma f H_0$, which is well defined because F is central in Γ. But F/F_0 is a subgroup of the path-connected group Γ/F_0, so the result follows from Corollary B.2.2. □

Proposition B.2.4. *Let G be a compact, connected Lie group, K a closed, connected subgroup, $\widetilde{G} \to G$ a finite cover, $\widetilde{K}$ the preimage of K, and $\widetilde{K}_0$ the identity component of $\widetilde{K}$. Then*

$$H^*(G/K) \xrightarrow{\sim} H^*(\widetilde{G}/\widetilde{K}) \xrightarrow{\sim} H^*(\widetilde{G}/\widetilde{K}_0).$$

Proof. By Theorem B.4.5, F is central, so taking $\widetilde{G} = \Gamma$ and $H_0 = \widetilde{K}$ in the statement of Proposition B.2.3 we have $\Gamma/H_0 = \widetilde{G}/\widetilde{K}$ and $\Gamma/FH_0 \approx G/p(\widetilde{K}) = G/K$ and the result follows. □

The preceding two results are too simple not to have been known, yet the author knows no reference.

Proposition B.2.5. *Let $F \to X \to B$ be a finite-sheeted covering. If either of the Euler characteristics $\chi(X), \chi(B)$ is finite, then so is the other, and $\chi(X) = \chi(B) \cdot |F|$.*

Proof sketch. Taking a CW approximation, we may assume X and B to be CW complexes and $X \to B$ cellular. Each cell of B is covered by $|F|$ cells in X, so the result follows from cellular homology. □

B.3. Fiber bundles

A ***fiber space*** with is a continuous surjection $E \longrightarrow B$ such that for each $b \in B$, we have $h^{-1}\{b\} \approx F$ for some fixed space F, the ***fiber***. Each $h^{-1}\{b\}$ is also called a fiber, E is the ***total space***, and B the ***base***. We abbreviate this assemblage as $F \to E \to B$. Given two fiber spaces $p\colon E \to B$ and $p'\colon E' \to B'$, a map $h\colon E \longrightarrow E'$ of total spaces is ***fiber-preserving*** if it sends fibers into fibers. Equivalently, there is a map $\bar{h}$ of bases making the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\bar{h}} & B'. \end{array}$$

Then $hp^{-1}\{b\} \subseteq (p')^{-1}\{\bar{h}(b)\}$ for all $b \in B$. Fiber spaces with fiber F and fiber-preserving maps form a category whose ***isomorphisms*** are fiber-preserving homeomorphisms.

A fiber space $p\colon E \to B$ with fiber F is a ***fiber bundle***, or an ***F-bundle*** (or *locally trivial*), if

- the base B admits an open cover of sets U such that there are fiber space isomorphisms $\phi_U\colon p^{-1}(U) \xrightarrow{\approx} U \times F$, called *(local)* ***trivializations***, and
- these trivializations are compatible in the sense that given two overlapping trivializing opens U and V, the ***transition functions*** $g_{U,V}$ defined by the composite homeomorphism

$$\begin{aligned} \phi_{U,V}\colon (U \cap V) \times F \xrightarrow{\phi_V^{-1}} p^{-1}(U \cap V) &\xrightarrow{\phi_U} (U \cap V) \times F \\ (x, f) &\longmapsto \big(x, g_{U,V}(x)(f)\big), \end{aligned}$$

  are continuous maps $U \cap V \longrightarrow \operatorname{Homeo} F$. Morally, different coordinatizations of the same trivial subbundle should differ continuously.

Given a fiber space $F \to E \xrightarrow{p} B$ and an subset $U \subseteq B$, the ***restriction*** $E|_U$ is the F-bundle $(p|_U)\colon p^{-1}(U) \to U$. This generalizes to the following construction. a continuous map $h\colon X \longrightarrow B$ (for restrictions, an inclusion), we can construct a ***pullback*** space $h^*E \to X$ with fiber F fitting into the commutative square

$$\begin{array}{ccc} h^*E & \xrightarrow{\tilde{h} = \mathrm{pr}_2} & E \\ {\scriptstyle h^*p := \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{h} & B, \end{array}$$

where the new total space is

$$f^*E = X \underset{B}{\times} E := \big\{(x,e) \in X \times E : h(x) = p(e)\big\} \subsetneq X \times E$$

and the new maps the restrictions of the factor projections from $X \times E$. This total space is called the ***fiber product***, and (with the maps), it is the pullback of the diagram $X \to B \leftarrow E$ in Top.[1] If $E \to B$ was an F-bundle, so also is $h^*E \to X$: given a local trivialization

$$\phi = (p,\rho)\colon p^{-1}U \xrightarrow{\approx} U \times F,$$

a trivialization of the pullback $(h^*E)|_{h^{-1}(U)}$ is given by

$$\mathrm{id}_X \times \rho\colon (x,e) \longmapsto \big(x, \rho(e)\big),$$

and such sets $h^{-1}(U)$ cover X. The resulting bundle is a ***pullback bundle***.

If F, E, B are all smooth manifolds and the fiber inclusion, projection, and transition functions are all C^∞, we say $F \to E \to B$ is a ***smooth bundle***. One can similarly define holomorphic and algebraic bundles, but smooth and merely continuous bundles are all we shall work with.

B.3.1. Principal bundles

Now suppose we are given a fiber bundle $F \to E \to B$ admitting trivializations $(\phi_U)_{U\in\mathscr{U}}$, such that each transition function $g_{U,V}$ takes values in some subgroup G of the group $\operatorname{Homeo} F$ of self-homeomorphisms of the fiber. As G is a topological group, its multiplication is continuous, and left multiplication ℓ_g by any element of $g \in G$ is a self-homeomorphism of G. In this way the transition function values $g_{U,V}(x) \in G$ can be viewed as elements of $\operatorname{Homeo} G$, and we can form a G-bundle $G \to P \to B$ by starting with the disjoint union $\coprod_{U\in\mathscr{U}} U \times G$ and gluing the pieces by the relations

$$(x,g) \sim \big(x, g_{U,V}(x)\cdot g\big)$$

for all nonempty intersections $U \cap V$ of sets in $\mathscr{U}$ and all $x \in U \cap V$ and $g \in G$.

The disjoint union we started with admits a global right G-action $(u,g)\cdot g' = (u, gg')$, which descends to a right G-action on P since the transition functions act on the *left* of the fibers G. This right action is simply transitive on each fiber. We call a G-bundle admitting a right G-action acting simply transitively on each fiber a ***principal G-bundle***; this motivating bundle $G \to P \to B$ is one such.

[1] This notation $X \underset{B}{\times} E$, now universal, is due to Paul Baum [Smi67, p. 68].

We can recover the original $F \to E \to B$ from $G \to P \to B$ and the map $\psi\colon G \longrightarrow \operatorname{Homeo} F$ by a pushout construction:

$$E \approx P \bigotimes_{G} F := \frac{P \times F}{\big([x,g],f\big) \sim \big([x,1]\psi(g)f\big)} \approx \frac{\coprod_{U \in \mathscr{U}} U \times G \times F}{\big(x, g_{U,V}(x)g, f\big) \sim (x,g,f) \sim \big(x,1,\psi(g)f\big)} \qquad \text{(B.3.1)}$$

Verbally, this can be seen as extracting the G-valued transition functions from a principal G-bundle and applying them to F instead of G. For this reason, the bundles $E \to B$ and $P \to B$ are said to be ***associated***. Because this correspondence is reversible, principal bundles carry essentially all information about fiber bundles.

Every homogeneous space G/K is the base space of a principal K-bundle $K \to G \to G/K$ by Theorem B.4.4. Further, in Chapter 5, we construct a *universal principal G-bundle $EG \to BG$* that every principal G-bundle is a pullback of. Given such a bundle, a space F, and a homomorphism $\psi\colon G \longrightarrow \operatorname{Homeo} F$, it follows the the associated F-bundle $EG \times_G F \to BG$ is universal for F-bundles with transition functions in $\psi(G)$; for example, $E\mathrm{GL}(n,\mathbb{R}) \otimes_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n \to B\mathrm{GL}(n,\mathbb{R})$ is a universal vector bundle. Once we have done this, we will be able to associate to each G/K a homotopy-equivalent space G_K fitting into a principal G-bundle $G \to G_K \to BK$.

B.3.2. Fibrations

[To be written...]

B.4. The structure of Lie groups

In this section, we record—without much in the way of explanation or interstitial verbiage—the background we require on compact Lie groups. Dwyer and Wilkinson [DW98] develop this material in an atypical algebro-topological manner concordant with the approach adopted here. Bröcker and tom Dieck [BtD85] is another standard reference.

Let G be connected Lie group and H a closed, connected subgroup. By the Cartan–Iwasawa–Malcev theorem, there exists a maximal compact subgroup K_H of H, unique up to conjugacy [HM07, Cor. 12.77], which is necessarily connected, such that there is a homeomorphism $H \approx K_H \times \mathbb{R}^n$ for some n [HM07, Cor. 12.82]. Likewise G contains a maximal compact subgroup K_G, which after conjugation can be chosen to contain K_H. This yields a reduction result.

Proposition B.4.1. *Suppose G is a connected Lie group and H a connected, closed subgroup, with respective compact, connected subgroups K_G and K_H, the one containing the other. Then G/H is homotopy-equivalent to K_G/K_H.*

Proof. A left–K_G-equivariant deformation retraction of G to K_G induces a deformation retraction from G/K_H to K_G/K_H. The fiber of $G/K_H \longrightarrow G/H$ is H/K_H, which is homeomorphic to Euclidean space, and G/K_H and G/H each have the homotopy type of a CW complex so the long exact sequence of homotopy groups and Whitehead's theorem shows the maps is a homotopy equivalence. □

Proposition B.4.2. *There exists a smooth map $\exp\colon \mathfrak{g} \longrightarrow G$, the **exponential**, which is surjective if G is compact and connected, which restricts to a homomorphism on the preimage of any connected abelian subgroup (in particular, on any line), and whose inverse in a neighborhood of $1 \in G$ serves as a smooth chart.*

Proposition B.4.3 ([Wik14]). *The fundamental group of a topological group is abelian.*

Theorem B.4.4 ([War71, Thm. 3.58, p. 120][GGK02, Prop. B.18, p. 179]). *Let G be a Lie group and K a closed subgroup. Then G/K is a manifold and $K \to G \to G/K$ a principal K-bundle.*

One of the main structure theorems for compact Lie groups is the following.

Theorem B.4.5 ([HM06, Thm. 2.19, p. 207]). *Every compact, connected Lie group G admits a finite central extension*

$$0 \to F \longrightarrow \widetilde{G} \longrightarrow G \to 0$$

such that $\widetilde{G}$ is the direct product of a compact, simply-connected Lie group K and a torus A. Thus

$$G \;\cong\; {}^{A \times K}\big/_{F}.$$

We call $\widetilde{G}$ the ***universal compact cover*** of G; it is uniquely determined up to isomorphism.[2]

Proposition B.4.6 (Élie Cartan–Wilhelm Killing). *Every simply-connected Lie group K is a direct product of finitely many **simple groups**, groups whose proper normal subgroups are finite. A simply-connected simple group is one of the following:*

$$\mathrm{SU}(n), \quad \mathrm{Sp}(n), \quad \mathrm{Spin}(n), \quad G_2, \quad F_4, \quad \widetilde{E}_6, \quad \widetilde{E}_7, \quad E_8,$$

with the exception of $\mathrm{Spin}(1) = \mathrm{O}(1)$ *and* $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$; *the three infinite families comprise the simply-connected **classical groups** and the last five the **exceptional groups**.*

We will not explain the exceptional groups, but the groups $\mathrm{Spin}(n)$ are double covers of $\mathrm{SO}(n)$ for $n \geqslant 3$ (when $\pi_1\mathrm{SO}(n) \cong \mathbb{Z}/2$) and $\mathrm{Spin}(2) = \mathrm{SO}(2) \cong S^1$. A compact group whose universal cover is a direct product of simple groups is called ***semisimple***.[3]

Proposition B.4.7. *Let G be a compact, semisimple Lie group. Then $H^1(G;\mathbb{Q}) = 0$.*

Proof. By our definition, G admits a simply-connected finite cover $\widetilde{G}$. By the universal coefficient theorem B.1.1, we have $H^1(\widetilde{G};\mathbb{Q}) \cong H_1(\widetilde{G};\mathbb{Q}) \cong H_1(\widetilde{G};\mathbb{Z}) \otimes \mathbb{Q}$, and by Proposition B.4.3 and Proposition B.1.5 we know $H_1(\widetilde{G};\mathbb{Z}) \cong \pi_1\widetilde{G}$, which we have assumed to be a finite group. □

A classification-type result in the opposite direction is that all compact Lie groups can be seen as closed subgroups of $\mathrm{U}(n)$.

Theorem B.4.8 (Fritz Peter–Hermann Weyl [BtD85, Thm. III.4.1, p. 136]). *Every compact Lie group G admits a faithful representation.*

This is a corollary of the Peter–Weyl theorem, and implies in particular [SAY WHY] that every compact Lie group embeds as a closed subgroup of $\mathrm{U}(n)$ for n sufficiently large.

[2] This is arguably a misnomer; this object cannot be initial in that we can always cover the toral factor A with another torus, and in particular the fiber F is not uniquely determined by this characterization.

[3] It is much more usual to equivalently demand that the Lie algebra $\mathfrak{g}$ be a direct sum of simple Lie algebras, but our focus is away from the Lie algebra level.

Proof, assuming Peter–Weyl. The Peter–Weyl theorem states, in one version, that the span of the set of continuous functions $G \longrightarrow \mathbb{C}$ that appear as coefficient functions $\rho_{i,j}$ in irreducible unitary representations $\rho\colon G \longrightarrow \mathrm{U}(n) < \mathbb{C}^{n\times n}$ is dense in $L^2(G;\mathbb{C})$. Particularly, the matrix coefficients must *separate points* of G, meaning that for any $g, h \in G$ there is some irreducible unitary representation ρ such that some coefficient $\rho_{i,j}(g) \neq \rho_{i,j}(h)$. Particularly, taking $h = 1$, it follows $\rho(g) \neq \rho(1) = \mathrm{id}$; in words, no nontrivial element is in the kernel of every irreducible unitary representation ρ, or $\bigcap \ker\rho = \{1\}$. Each intersection of finitely many $\ker\rho$ is a closed submanifold of G, so in fact one can select finitely many ρ_n such that $\bigcap \ker\rho_n = \{1\}$ and hence $\bigoplus \rho_n$ is faithful. □

Exercise B.4.9. Why does one only need to take a finite intersection in the preceding proof?

B.4.1. The maximal torus

A *real* ***torus*** is a Lie group smoothly isomorphic to the direct product of finitely many copies of the complex circle group $S^1 \cong \mathrm{U}(1)$; for us tori are always considered as Lie groups. A one-dimensional torus is a ***circle***. Much of the structure of the structure of compact, connected Lie groups arises due to tori they contain. The ***centralizer*** $Z_G(K)$ of a subgroup K of a group G is the set of $g \in G$ such that $gkg^{-1} = k$ for each $k \in K$.

Lemma B.4.10. *Any torus T contains a* ***topological generator****, an element generating a dense subgroup.*

Sketch of proof. Any element of $\mathbb{R}^\ell$ none of whose coordinates is a rational multiple of any other will project to such an element in $(\mathbb{R}/\mathbb{Z})^\ell \cong T$. □

Theorem B.4.11. *Let G be a compact, connected Lie group. Every torus S of G is contained in a torus T which is properly contained in no other torus; such a T is called a* ***maximal torus*** *of G. Every element lies in some maximal torus, each maximal torus is its own centralizer in G, and all maximal tori are conjugate in G. The centralizer $Z_G(T)$ of a maximal torus is T itself.*

Given a group G, and a subgroup K of G, we write $N_G(K)$ for the ***normalizer*** of K in G, the set of elements $g \in G$ such that $gKg^{-1} = K$. The ***Weyl group*** $W(G)$ of G is defined to be the quotient group $N_G(T)/T$, the collection of nontrivial symmetries of T induced by conjugation. It is always a finite reflection group.

Proposition B.4.12. *Let G be a connected, compact Lie group. Then the center $Z(G)$ is the intersection of all maximal tori in G.*

Proof [DW98, Prop. 7.1]. Any element of $Z(G)$ must lie in $Z_G(T) = T$ for any maximal torus T. Conversely, any element $x \in G$ has some conjugate gxg^{-1} in any given maximal torus. If x itself does not lie in that torus, then $x \neq gxg^{-1}$, so x is not central. □

Proposition B.4.13 ([BtD85, Prop. V.(5.13), p. 214])**.** *On the Lie algebra $\mathfrak{g}$ of a compact, connected Lie group G, there exists a symmetric bilinear form $B(-,-)$, the* ***Killing form****, which is invariant under the adjoint action of G. This form is negative definite if G is compact.*

[

Sketch construction. Conjugation $c_g \colon h \longmapsto ghg^{-1}$ on G is smooth, so induces a derivative $\mathrm{Ad}(g) := (c_g)_* \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ on the tangent space $\mathfrak{g} := T_1 G$. The map $\mathrm{Ad} \colon G \longrightarrow \mathrm{Aut}_{\mathbb{R}}\, \mathfrak{g}$ is itself smooth when $\mathrm{Aut}_{\mathbb{R}}\, \mathfrak{g} \cong \mathrm{Aut}_{\mathbb{R}}\, \mathbb{R}^n$ is topologized as an open subset of the space $\mathbb{R}^{n \times n}$ of matrices, thus inducing a derivative $\mathrm{ad} \colon \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{R}}\, \mathfrak{g}$. This is the multiplication in the Lie algebra $\mathfrak{g}$: one sets $[x, y] := \mathrm{ad}(x) y$. Once a basis of $\mathfrak{g}$ is selected, a trace is well defined, and one sets $B(x, y) := \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$. This is clearly bilinear. To see it is $\mathrm{Ad}(G)$-invariant, one notes that if γ_x is a curve in G with $\gamma_x(0) = 1$ and $\gamma_x'(0) = x$, then $c \colon t \longmapsto g\gamma_x(t) g^{-1}$ satisfies $c(0) = 1$ and $c'(1) = \mathrm{Ad}(g)x$, so that

$$\mathrm{ad}\left(\mathrm{Ad}(g)x\right) = \frac{d}{dt} \mathrm{Ad}\left(g\gamma_x(t)g^{-1}\right)\big|_{t=0} = \mathrm{Ad}(g) \frac{d}{dt} \mathrm{Ad}\left(\gamma_x(t)\right)\big|_{t=0} \mathrm{Ad}(g^{-1}) = \mathrm{Ad}(g)\, \mathrm{ad}(x)\, \mathrm{Ad}(g)^{-1}$$

and recalls the trace of a matrix is invariant under conjugation. [NEGATIVE-DEF?] □

]

Appendix C

Borel's proof of Chevalley's theorem

Borel's proof of Cartan's theorem in his thesis does not use the Serre spectral sequence, but the Leray spectral sequence, which was at the time phrased in a vocabulary no longer in use. This appendix rephrases his original proof, hopefully without too much violence, in still-current terminology. The translation effort was not entirely trivial; needless to say, any errors belong to the author, not to Borel.

C.1. Sheaf cohomology

We will require standard material on sheaves and sheaf cohomology to proceed [War71, Ch. 5][ET14, Sec. 2]. The development to follow is no longer standard; this is what things looked like circa 1950.

We take as known the concepts of sheaf, presheaf, constant sheaf, and stalk. Let k be a principal ideal domain and $\underline{k}$ the constant sheaf in the rest of this subsection.

Definition C.1.1. Let $\mathscr{A}$ be a sheaf of k-modules over a topological space X. A ***resolution*** $\mathscr{C}^\bullet$ of $\mathscr{A}$ is a sequence

$$0 \to \mathscr{A} \longrightarrow \mathscr{C}^0 \longrightarrow \mathscr{C}^1 \longrightarrow \mathscr{C}^2 \longrightarrow \cdots$$

of sheaf homomorphisms such that the induced sequence of stalks over each point of X is exact. We say $\mathscr{A}$ is ***acyclic*** if the induced sequence

$$0 \to \mathscr{A}(X) \longrightarrow \mathscr{C}^0(X) \longrightarrow \mathscr{C}^1(X) \longrightarrow \mathscr{C}^2(X) \longrightarrow \cdots$$

of global sections is exact. If $\mathscr{C}^\bullet$ is a resolution of $\mathscr{A}$ by acyclic sheaves, then the ***sheaf cohomology*** $H^*(X;\mathscr{A})$ of $\mathscr{A}$ is the cohomology of the complex

$$0 \to \mathscr{C}^0(X) \longrightarrow \mathscr{C}^1(X) \longrightarrow \mathscr{C}^2(X) \longrightarrow \cdots .$$

If space X is paracompact, we say $\mathscr{A}$ is ***fine*** if for every open cover (U_α) of X, there is a family (φ_α) of k-linear sheaf endomorphisms of $\mathscr{A}$ such that $\sum \varphi_\alpha = \mathrm{id}$ and the closure of each set $\{x \in X : \varphi_\alpha(x) \neq 0\}$ (the *support*) lies in U_α.

Note what is implicit in this definition, that the choice of acyclic resolution of $\mathscr{A}$ does not affect the end result. Our interest in sheaf cohomology is the following:

Proposition C.1.2. *Let X be a topological space homotopy equivalent to a finite CW complex. Then the singular cohomology and sheaf cohomology rings*

$$H^*(X;k) \cong H^*(X;\underline{k})$$

are isomorphic.

By the following proposition, then, to compute singular cohomology, we can resolve $\underline{k}$ by fine sheaves and take the cohomology of the sequence of global sections.

Proposition C.1.3. *Fine sheaves are acyclic.*

Example C.1.4. The canonical fine sheaves are the sheaves Ω^p of differential forms on a smooth manifold M. The Poincaré lemma is that the sequence $\Omega^\bullet$ of sheaves resolves the constant sheaf $\underline{\mathbb{R}}$ on M. In this guise, de Rham's theorem that the cohomology of the de Rham complex is $H^*(M;\mathbb{R})$ becomes a consequence of Proposition C.1.2. In fact, it is still a little stronger in that the sequence of sheaves $\Omega^\bullet$ has its own internal multiplication (we say it is a ***sheaf of*** $\mathbb{R}$-CDGAS) and this multiplication of global sections induces a multiplicative structure on $H^*(X;\underline{\mathbb{R}})$ which corresponds to the cup product on $H^*(X;\mathbb{R})$.

This situation is important enough that we codify it.

Definition C.1.5. Let X be a paracompact space and $\mathscr{F}^\bullet$ a sheaf valued in k-CDGAS. This can be seen, forgetting the multiplication, as a complex $\mathscr{F}^0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \cdots$ of sheaves of $\mathbb{R}$-modules, and the inclusion of locally constant functions via $c \longmapsto c \cdot 1$ is a sheaf homomorphism $\underline{k} \longrightarrow \mathscr{F}^0$. If the resulting sequence $0 \to \underline{k} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots$ is exact, such a sheaf of k-DGAS can be seen as an acyclic resolution as required in Definition C.1.1. If additionally the values $\mathscr{F}^\bullet(U)$ are free k-modules, we will say $\mathscr{F}^\bullet$ is a CDGA-***resolution of*** $\underline{k}$. In general, if $\mathscr{F}^\bullet$ is sheaf of chain complexes on a space X, write $\mathscr{H}^p(\mathscr{F}^\bullet)$ for its p^{th} ***cohomology sheaf***, whose stalk at $x \in X$ is $H^p(\mathscr{F}^\bullet|_x)$.

These exist in the cases we are interested in, via a clever trick.

Proposition C.1.6 (Cartan (unpublished))**.** *Let X be a compact metrizable space. Then there exists a fine sheaf of* $\mathbb{R}$-CDGAS *on X resolving the constant sheaf $\underline{\mathbb{R}}$.*

Proof. By the Menger–Nöbeling theorem, X with any compatible metric embeds isometrically into a Euclidean space (specifically $\mathbb{R}^{1+2\dim X}$, where $\dim X$ is the Lebesgue covering dimension). The de Rham sheaf $U \longmapsto \Omega^\bullet(U)$ is a fine sheaf of $\mathbb{R}$-CDGAS resolving $\underline{\mathbb{R}}$ on Euclidean space, and so, by restriction, induces such a sheaf on X. □

We will need to compare and combine sheaves to prove Leray's theorem.

Definition C.1.7. By a ***tensor product*** of sheaves of k-DGAS $\mathscr{G}^\bullet \otimes \mathscr{F}^\bullet$ we mean the sheaf whose stalks are the rings $\mathscr{G}^\bullet|_x \otimes \mathscr{F}^\bullet|_x$, singly graded by $(\mathscr{G}^\bullet \otimes \mathscr{F}^\bullet)^n := \bigoplus_{p+q=n} \mathscr{G}^p \otimes \mathscr{F}^q$ equipped with the unique differential restricting to the original differentials on $\mathscr{G}^\bullet \otimes \underline{k}$ and $\underline{k} \otimes \mathscr{F}^\bullet$. Let $f\colon X \longrightarrow Y$ be a continuous map, $\mathscr{F}$ a sheaf on X, and $\mathscr{G}$ a sheaf on Y. Then the ***direct image sheaf*** $f_*\mathscr{F}$ on Y and ***inverse image sheaf*** $f^{-1}\mathscr{G}$ on X are given respectively by

$$V \longmapsto \mathscr{F}\big(f^{-1}(V)\big),$$
$$U \longmapsto \varinjlim_{V \supseteq \pi(U)} \mathscr{G}(V).$$

The following is unsurprising, since we can reuse the same partition of unity:

Proposition C.1.8. *If $\mathscr{F}$ is a sheaf and $\mathscr{G}$ a fine sheaf of k-modules on a paracompact topological space X, then the sheaf tensor product $\mathscr{F} \otimes \mathscr{G}$ is again fine.*

A CDGA-resolution of $\underline{k}$ allows us to find resolutions of sheaves in a canonical way.

Proposition C.1.9. *Let $\mathscr{F}$ be a sheaf on a paracompact space X and $\mathscr{G}^\bullet$ a fine CDGA-resolution of $\underline{k}$. Then $\mathscr{G}^\bullet \otimes \mathscr{F}$ is a fine resolution of $\mathscr{F}$, so that $H^*(X;\mathscr{F})$ can be calculated as the cohomology of the complex $(\mathscr{G}^\bullet \otimes \mathscr{F})(X)$.*

Proof. Because the stalks $\mathscr{G}^\bullet(x)$ are free k-modules, $\mathscr{F} \to \mathscr{G}^0 \otimes \mathscr{F} \to \mathscr{G}^1 \otimes \mathscr{F} \to \cdots$ is a resolution of $\mathscr{F}$. Because each $\mathscr{G}^p$ is fine, so is $\mathscr{G}^p \otimes \mathscr{F}$, by Proposition C.1.8. □

Remark C.1.10. Borel actually takes this as his definition of sheaf cohomology.

Proposition C.1.11. *Let $f\colon X \longrightarrow Y$ be a continuous map of Hausdorff spaces, $\mathscr{F}$ a fine sheaf on X, and $\mathscr{G}$ a sheaf on Y. Then pullback along f induces an isomorphism*

$$\mathscr{G} \otimes f_*\mathscr{F} \cong f_*(f^*\mathscr{G} \otimes \mathscr{F}).$$

Proof. The stalk of the former sheaf over $y \in Y$ is

$$(\mathscr{G} \otimes f_*\mathscr{F})(y) = \mathscr{G}(y) \otimes (f_*\mathscr{F})(y) = \mathscr{G}(y) \otimes \varinjlim_{V \ni y} \mathscr{F}\big(f^{-1}(V)\big) \cong \mathscr{G}(y) \otimes \mathscr{F}\big(f^{-1}(y)\big)$$

since $\{y\}$ is closed and $\mathscr{F}$ is fine. On the other hand

$$\big(f_*(f^*\mathscr{G} \otimes \mathscr{F})\big)(y) = \varinjlim_{V \ni y} \big(f_*(f^*\mathscr{G} \otimes \mathscr{F})\big)(V) = \varinjlim_{V \ni y} (f^*\mathscr{G} \otimes \mathscr{F})\big(f^{-1}(V)\big) = (f^*\mathscr{G} \otimes \mathscr{F})\big(f^{-1}(y)\big)$$

for the same reason. This last is the module of continuous sections over $f^{-1}(y)$ of an étalé space whose stalk at $x \in X$ is

$$(f^*\mathscr{G} \otimes \mathscr{F})(x) = \varinjlim_{U \ni x} \varinjlim_{V \supseteq f(U)} \mathscr{G}(V) \otimes \mathscr{F}(x) \cong \mathscr{G}\big(f(x)\big) \otimes \mathscr{F}(x).$$

But then a continuous section is precisely an element of $\mathscr{G}(y) \otimes \mathscr{F}\big(f^{-1}(y)\big)$. □

Corollary C.1.12. *Let $f\colon X \longrightarrow Y$ be a continuous map of Hausdorff spaces, $\mathscr{F}$ a fine sheaf on X, and $\mathscr{G}$ a sheaf on Y. Then pullback along f induces an isomorphism*

$$(\mathscr{G} \otimes f_*\mathscr{F})(Y) \cong (f^*\mathscr{G} \otimes \mathscr{F})(X).$$

Proof. Now $(\mathscr{G} \otimes f_*\mathscr{F})(Y) \cong \big(f_*(f^*\mathscr{G} \otimes \mathscr{F})\big)(Y)$, but this is $(f^*\mathscr{G} \otimes \mathscr{F})(X)$ by definition. □

Corollary C.1.13. *Let $f\colon X \longrightarrow Y$ be a continuous map of Hausdorff spaces, $\mathscr{F}^\bullet$ a fine CDGA-resolution of $\underline{k}$ on X, and $\mathscr{G}^\bullet$ a fine CDGA-resolution of $\underline{k}$ on Y. Then $f^*\mathscr{G}^\bullet \otimes \mathscr{F}^\bullet$ is also a fine CDGA-resolution of $\underline{k}$ on X.*

Proof. By Proposition C.1.8, $f^*\mathscr{G}^\bullet \otimes \mathscr{F}^\bullet$ is fine, and we saw in the proof of Proposition C.1.11 that its stalk at $x \in X$ is $\mathscr{G}^\bullet\big(f(x)\big) \otimes \mathscr{F}^\bullet(x)$. This stalk is a free k-module since the tensor factors are, and an acyclic CDGA by the Künneth theorem Corollary A.3.3 since the tensor factors are acyclic and free over k. □

Example C.1.14. Let $\pi\colon E \longrightarrow B$ be a smooth fiber bundle with compact total space. Let $\Omega^\bullet_E$ be the sheaf of de Rham algebras over E and $\Omega^\bullet_B$ that over B. The tensor sheaf $\mathscr{C} := \pi^*\Omega^\bullet_B \otimes \Omega^\bullet_E$, which is another CDGA-resolution of $\underline{\mathbb{R}}$ on E by Corollary C.1.13. Thus the de Rham cohomology $H^*(E;\mathbb{R})$ is the cohomology of the complex

$$A = \mathscr{C}(E) = \big(\pi^*\Omega^\bullet_B \otimes \Omega^\bullet_E\big)(E).$$

This looks at first as if it will violate the Künneth theorem Corollary A.3.3, since $\pi^*\Omega^\bullet_B(E) = \Omega^\bullet(B)$, but $\mathscr{C}$ is the *sheaf associated* to the presheaf $U \longmapsto \Omega^\bullet\big(\pi(U)\big) \otimes \Omega^\bullet(U)$, which is very different from the presheaf itself.

C.2. The Leray spectral sequence

We paraphrase Borel's 1951 ETH exposition of the Leray spectral sequence [Bor51, Exposé VII-3]. Let $f\colon X \longrightarrow Y$ be a map of Hausdorff spaces, with Y paracompact, $\mathscr{F}^\bullet$ a fine CDGA-resolution of $\underline{k}$ on X, and $\mathscr{G}^\bullet$ a fine CDGA-resolution of $\underline{k}$ on Y.

Now $f^*\mathscr{G}^\bullet \otimes \mathscr{F}^\bullet$ is again a fine CDGA-resolution of $\underline{k}$ on X by Corollary C.1.13. Thus the complex $(f^*\mathscr{G}^\bullet \otimes \mathscr{F}^\bullet)(X)$ of global sections computes $H^*(X;\underline{k})$. Note from Corollary C.1.12 that this complex of global sections can equally be viewed as $(\mathscr{G}^\bullet \otimes f_*\mathscr{F}^\bullet)(X)$.

Let us filter this by base degree, taking p^{th} filtrand

$$(\mathscr{G}^{\geqslant p} \otimes f_*\mathscr{F}^\bullet)(X),$$

and consider the associated filtration spectral sequence as described in Corollary 2.6.8. We know already that it converges to $H^*(X;\underline{k})$, and we seek to identify the first two terms. The zero term, the associated graded algebra of the p-filtration, is just $\bigoplus(\mathscr{G}^p \otimes f_*\mathscr{F})(X) = (\mathscr{G}^\bullet \otimes f_*\mathscr{F})(X)$ again. The complex $(\mathscr{G}^\bullet \otimes f_*\mathscr{F}^\bullet)(Y)$ is actually bigraded and by definition its differential is the sum of two components, one of bidegree $(1,0)$ and extending the differential $d_{\mathscr{G}^\bullet}$ and the other of bidegree $(0,1)$ and extending the differential $d_{f_*\mathscr{F}^\bullet}$. The former increases the filtration degree so differential induced on the associated graded E_0 is $d_0 = (\mathrm{id} \otimes d_{f_*\mathscr{F}^\bullet})(X)$.

We claim the cohomology E_1 of this complex can be identified with the space of global sections $\big(\mathscr{G}^\bullet \otimes \mathscr{H}^\bullet(f_*\mathscr{F}^\bullet)\big)(X)$. It is easiest to see this first at the stalk level, where clearly an element of $\mathscr{G}^\bullet(x) \otimes \ker\big(d_{f_*\mathscr{F}^\bullet}(x)\big) \leqslant \mathscr{G}^\bullet(x) \otimes f_*\mathscr{F}^\bullet(x)$ is the same as an element of $\ker\big(\mathrm{id} \otimes d_{f_*\mathscr{F}^\bullet})(x)\big)$ since $\mathscr{G}^\bullet(x)$ is a free k-module, and similarly an element of $\mathrm{im}(\mathrm{id}_{\mathscr{G}^\bullet(x)} \otimes d_{f_*\mathscr{F}^\bullet})(x)$ is the same as an element of $\mathscr{G}^\bullet(x) \otimes \mathrm{im}\big(d_{f_*\mathscr{F}^\bullet}(x)\big)$.

The differential d_1 on E_1 takes elements in the kernel of d_0 one level forward in the filtration, and hence iis nduced by $d_{\mathscr{G}^\bullet}$, so it is given under our identification by $(d_{\mathscr{G}^\bullet} \otimes \mathrm{id})(Y)$. Recall from Proposition C.1.9 that since $\mathscr{G}$ is acyclic, sheaf cohomology on Y with coefficients in any sheaf $\mathscr{A}$ is given by the cohomology of the module of global sections of $\mathscr{G}^\bullet \otimes \mathscr{A})$. In particular fixing $\mathscr{A} = \mathscr{H}^q(f_*\mathscr{F})$, one finds

$$E_2^{p,q} \cong H^p\big(Y;\mathscr{H}^q(f_*\mathscr{F})\big).$$

To conceptualize this, recall that the pushforward $f_*\mathscr{F}$ is the sheaf whose stalk at $y \in Y$ is the direct limit of $\mathscr{F}(U)$ over neighborhoods U of $f^{-1}\{y\}$, so

$$(f_*\mathscr{F})(y) = H^*\big(f^{-1}\{y\};k\big).$$

Thus the E_2 page is the cohomology of Y with coefficients varying over the cohomology of the fibers. This spectral sequence (E_r, d_r) is the ***Leray spectral sequence*** of the map $f\colon X \longrightarrow Y$.

Theorem C.2.1 (Leray). *Let $f\colon X \longrightarrow Y$ be a map of spaces, with Y paracompact. Let $\mathscr{F}$ be a fine* CDGA*-resolution of $\underline{k}$ on X and $\mathscr{G}$ a fine* CDGA*-resolution of $\underline{k}$ on Y. Then the filtration spectral sequence of the sheaf $\mathscr{G}^\bullet \otimes f_*\mathscr{F}^\bullet$ with the horizontal filtration induced by the grading of $\mathscr{G}^\bullet$ is a spectral sequence of k-*DGAs *satisfying*

- $E_0^{p,q} \cong (\mathscr{G}^p \otimes f_*\mathscr{F}^q)(Y)$,
- $E_2^{p,q} \cong H^p\big(Y; \mathscr{H}^q(f_*\mathscr{F})\big)$,
- $E_\infty^{p,q} \cong \mathrm{gr}_p\, H^{p+q}(X;k)$.

Corollary C.2.2 (Vietoris–Begle [Vie27, Beg50]). *Let $f\colon X \longrightarrow Y$ be a map of Hausdorff spaces, Y paracompact, and suppose each for each $y \in Y$ that $\widetilde{H}^{\leqslant n}\big(f^{-1}(y);k\big) = 0$. Then $f^*\colon H^j(Y;k) \longrightarrow H^j(X;k)$ is an isomorphism for $0 \leqslant j \leqslant n$ and an injection for $j = n+1$.*

Proof. This is immediate from the E_2 term of the Leray spectral sequence, where rows 1 through $n+1$ are empty, so that no differential can hit the first segment $E_\bullet^{\leqslant n+1,\bullet}$ of the bottom row, which hence survives to E_∞. As the diagonals of total degree $\leqslant n$ are only inhabited by these elements, there is no extension problem. □

Remark C.2.3. Borel states this a bit more generally. Without complicating the proof, one can replace $\mathscr{F}^\bullet$ with the extended sheaf $(\mathscr{F}^\bullet \otimes M)(U) := \mathscr{F}^\bullet(U) \otimes M$ for any k-module M to get a Leray spectral sequence with coefficients in M. More generally still, although his exposition does not do this, one can replace M with another sheaf $\mathscr{A}$ on X to arrive at a spectral sequence $H^*\big(Y; \mathscr{H}^\bullet(f_*\mathscr{A})\big) \implies H^*(X;\mathscr{A})$.

Another important difference is that Borel works with compactly supported cohomology on locally compact Hausdorff spaces. This makes no difference for compact total spaces, but the compactness necessary to construct the CDGA-resolution of $\underline{\mathbb{R}}$ an important reason why Serre had to reformulate the theory in his thesis, which deals extensively, for example, with the path fibration $\Omega X \to PX \to X$.

Now suppose $f\colon X \longrightarrow Y$ is a bundle with fiber F. Then preimages of small enough neighborhoods $V \subseteq Y$ are of the form $V \times F$, so $f_*\mathscr{F}^\bullet\colon V \longmapsto \mathscr{F}^\bullet(V \times F)$ and $\mathscr{H}^q(f_*\mathscr{F}^\bullet)(y) = H^q\big(F_y;k\big)$ is a locally constant sheaf, so the cohomology groups $H^*(F)$ of individual fibers are isomorphic. They are related to one another by isomorphisms $\gamma_*\colon H^*(E|_{\gamma(0)}) \longrightarrow H^*(E|_{\gamma(1)})$ induced by lifting paths $\gamma\colon [0,1] \longrightarrow Y$ in the base to homeomorphisms between fibers, and it is possible to convert these sheaves into a local coefficient system. Thus $E_2^{\bullet,q}$ can be shown to isomorphic to the cohomology of the complex $\mathrm{Hom}_{\pi_1(Y)}\big(C^\bullet(Y), H^q(F;k)\big)$, where $H^q(F;k)$ is viewed as a $\pi_1(Y)$-module through the conversion just hinted at, and in fact the Leray spectral sequence of a bundle agrees with the Serre spectral sequence from E_2 onward.

C.3. Borel's proof

In this section, we provide a proof of Chevalley's theorem close to Borel's original. Most of it is in the setup; once the relevant DGAs are defined, the quasi-isomorphisms are nearly immediate.

Let $k = \mathbb{R}$, let G be a compact, connected Lie group, and let $G \to E \xrightarrow{\pi} B$ be a smooth principal G-bundle. Write $P = PG$ for the space of primitives of $H^*(G) = H^*(G;\mathbb{R})$, so that $H^*(G) \cong \Lambda P$. Fix a transgression

$$\tau\colon P \xrightarrow{\sim} QH^*(BG) \rightarrowtail H^*(BG).$$

As $\pi\colon E \to B$ is a principal G-bundle, there is a classifying map $\chi\colon B \longrightarrow BG$. Let $[x_j]$ be a basis of P and $[b_j] = \chi^*\tau(x_j) \in H^*(B)$ for each j.

Let $\mathscr{B}$ be an fine sheaf of $\mathbb{R}$-cdgas resolving the constant sheaf $\underline{\mathbb{R}}$ on B, as guaranteed by Proposition C.1.6 and likewise $\mathscr{E}$ be a fine sheaf of $\mathbb{R}$-cdgas resolving the constant sheaf $\underline{\mathbb{R}}$ on E, so that by Definition C.1.1 and Proposition C.1.2,

$$H^*(B) \cong H^*\big(\mathscr{B}(B)\big); \qquad H^*(E) \cong H^*\big(\mathscr{E}(E)\big).$$

We can pull $\mathscr{B}$ back to a sheaf $\pi^*\mathscr{B}$ on E and then the tensor product $\pi^*\mathscr{B}\otimes\mathscr{E}$ is another fine sheaf on E. If we set $C = (\pi^*\mathscr{B}\otimes\mathscr{E})(E)$ with the expected differential, then by Corollary C.1.13,

$$H^*(C) = H^*\big((\pi^*\mathscr{B}\otimes\mathscr{E})(E)\big) \cong H^*(E)$$

as well. This C can be seen as the quotient of $\mathscr{B}(B)\otimes\mathscr{E}(E)$ by the ideal $\mathfrak{n}$ spanned by elements of empty support.

By Theorem 7.4.5, the classes $[x_j] \in PG$ are universally transgressive, which in particular means in this instance they transgress in the filtration spectral sequence (E_r, d_r) of C as filtered by

$$C^p := \bigoplus_{i \geqslant p} (\pi^*\mathscr{B}^i \otimes \mathscr{E})(E).$$

By Theorem C.2.1, this is a version of the Leray spectral sequence of $\pi\colon E \longrightarrow B$, which from $E_2 \cong H^*(B)\otimes H^*(G)$ on, is isomorphic to the Serre spectral sequence of this bundle. Thus, as discussed in Proposition 2.2.21, the transgression of the primitive classes $[x_j] \in PG$ means there exist elements $c_j \in C$ such that $d_C c_j = \pi^* b_j \otimes 1 \pmod{\mathfrak{n}}$,

These transgressive cochains allow us to compile a simpler model of $H^*(E)$ as in the previously cited version Theorem 8.1.5 of Theorem C.3.1. As ΛP is a free CGA, we can lift it to a subalgebra $\Lambda[x_j]$ of $(\pi^*\mathscr{B}\otimes\mathscr{E})(E)$ generated by global sections x_j of $\pi^*\mathscr{B}\otimes\mathscr{E}$. Let $C' = \mathscr{B}(B)\otimes\Lambda[x_j]$, with differential the unique antiderivation $d_{C'}$ satisfying

$$d_{C'}(b\otimes 1) = d_{\mathscr{B}} b \otimes 1, \qquad d_{C'}(1\otimes x_j) = b_j \otimes 1$$

and filtration

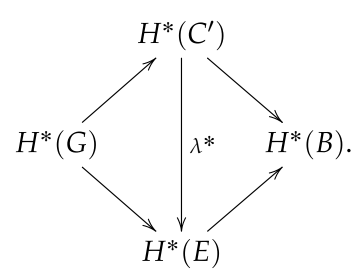
$$(C')^p := \bigoplus_{i \geqslant p} \mathscr{B}^i(B)\otimes H^*(G).$$

Then the map

$$\begin{aligned} \lambda\colon C' &\longrightarrow C : \\ b\otimes 1 &\longmapsto \pi^* b \otimes 1, \\ 1\otimes[x_j] &\longmapsto c_j \end{aligned}$$

is a filtration-preserving dga homomorphism, which we hope to show is a quasi-isomorphism.

Theorem C.3.1 (Chevalley)**.** *This map λ is a quasi-isomorphism completing a commutative diagram*



Proof (Borel). Apply the filtration spectral sequence of (Corollary 2.6.8) to both DGAs and the map λ^*. As discussed above, the spectral sequence (E_r, d_r) of C is the Leray spectral sequence of $\pi\colon E \longrightarrow B$. Write $('E_r, 'd_r)$ for the spectral sequence of C'. The 0^{th} page is the associated graded algebra of the filtration:

$$'E_0^{p,\bullet} = \mathscr{B}^p(B) \otimes H^*(G).$$

Since $\deg x_j \geqslant 1$, we have $\deg b_j \geqslant 2$, so $d_{C'}$ increases the filtration degree of each element of $H^*(G)$ by at least 2, and the filtration degree of elements of $\mathscr{B}(B)$ by 1. Thus no image of $d_{C'}$ survives the "associated graded" procedure, so $'d_0 = 0$ and

$$'E_1 = 'E_0 = \mathscr{B}(B) \otimes H^*(G).$$

The differential $'d_1$ still sends generators of $H^*(G)$ at least two filtration degrees forward, but acts as $d_{\mathscr{B}}$ on $\mathscr{B}(B)$, so that $'d_1 = \delta_{\mathscr{B}} \otimes \mathrm{id}_{H^*(G)}$ and

$$'E_2 \cong H^*(B) \otimes H^*(G).$$

Thus $'E_2 \cong E_2$; it just remains to see the map $\lambda_2\colon 'E_2 \longrightarrow E_2$ itself is such an isomorphism in a manner making the diagram commute. But $1 \otimes [x_j] \in C'$ and $1 \otimes x_j \pmod{\mathfrak{n}} \in C$ both become $1 \otimes [x_j]$ in $H^*(B) \otimes H^*(G)$, and $b \otimes 1 \in C'$ and $b \otimes 1 \pmod{\mathfrak{n}} \in C$ both become $[b] \otimes 1$ in $H^*(B) \otimes H^*(G)$. □

Historical remarks C.3.2. The proof presented above is in terms of a historically late formulation of Leray's technology; there were several such accounts, of gradually improving comprehensibility. The entirety of the recounting that follows is derived from work of Borel expositing Leray's approach, both in 1951 and 1997 [Bor51, Bor98].

Leray's original motivation for the topological edifice he erected seems to have been the de Rham complex. This is an $\mathbb{R}$-CGA of forms supported on various subsets, yielding a complex which Poincaré already had shown to be trivial on Euclidean subsets, but which collate together nonetheless to contain global information about a manifold, as conjectured by Élie Cartan and proven by Georges de Rham in his thesis. Recall that if ω, τ are forms on a manifold M and f a smooth function, the support satisfies these axioms:

$$\operatorname{supp}(\tau + \omega) \subseteq \operatorname{supp}\tau \cup \operatorname{supp}\omega; \qquad \operatorname{supp} 0 = \varnothing; \qquad \operatorname{supp}(f \cdot \omega) \subseteq \operatorname{supp}\omega \quad;$$
$$\operatorname{supp}(\tau \wedge \omega) \subseteq \operatorname{supp}\tau \cap \operatorname{supp}\omega; \qquad \operatorname{supp} d\omega \subseteq \operatorname{supp}(\omega).$$

Leray's idea, beautiful in its audacity, is to equip a topological space X with a complex (*complexe concrete*) K of "forms on a space," equipped with a *support function* $k \longmapsto |k|$, valued in closed subsets of X, satisfying the first three axioms above despite the absence of any native notion of smoothness (or notion of what "the germ of k at a point" would mean, k not being a function in any real sense). As a purely algebraic object, a complex is a module over a commutative coefficient ring (which we will write as A to allow us to write $k \in K$); only the support function imparts any topological content. If the complex is a DGA, we ask the last two axioms be satisfied as well.

With this setup, and some further definitions, Leray is able to reprove a good amount of existing algebraic topology as of 1945, proving that the cohomology of certain types of complexes recovers Hopf's and Samelson's theorems on Lie groups, the Lefschetz fixed-point theorem, the Brouwer fixed-point theorem, invariance of domain, Poincaré duality, and Alexander duality.

Building up *couvertures* (defined below) associated to nerves of a cofinal sequence of closed covers of a topological space, Leray can show his cohomology is isomorphic to Čech cohomology on compact Hausdorff spaces X.

Here are some of those further definitions. Given a function $f\colon X \longrightarrow Y$ and a complex K on X, one defines the complex fK on Y to have the same underlying module and new support function $|k|_Y = f(|k|_X)$. Given a complex L on Y, one defines, on the module level,

$$f^{-1}L := L/\{\ell \in K : f^{-1}|\ell| = \varnothing\}$$

with supports $\big|[\ell]\big| := f^{-1}|\ell|$. As a particular example, if $F \subseteq X$ is a closed subset, the ***intersection*** $F.K$ is defined to be the quotient

$$F.K := K/\{k \in K : |k| \cap F = \varnothing\}$$

with support function $k \longmapsto |k| \cap F$, and there is a natural restriction homomorphism $K \longrightarrow F.K$. If $F = \{x\}$ is a singleton, one writes xK; these are the germs of forms if $K = \Omega(M)$ is the de Rham complex. The system of such restrictions $F \longmapsto F.K$ is an example of a ***sheaf*** (*faisceau*) under Leray's later (1946) definition, which should be contrasted with the modern definition depending on an open cover; at this point, Leray was interested in cohomology with compact supports on a locally compact space. An element z of the tensor product $K \otimes K'$ of two complexes is assigned support

$$|z| := \{x \in X : \text{the image of } z \text{ is nonzero under } K \otimes K' \longrightarrow xK \otimes xK'\}.$$

The ***intersection*** $K \bigcirc K'$ is given by

$$K \bigcirc K' := {}^{K \otimes K'}\!\big/\{z \in K \otimes K' : |z| = \varnothing\}.$$

An A-complex is *fine* (*fin*) if every finite cover (U_j) of X admits a partition of unity, which is a set of A-endomorphisms $\varphi_j\colon K \longrightarrow K$ such that for each $k \in K$

$$\operatorname{supp} \varphi_j(k) \subseteq \overline{U_j} \cap \operatorname{supp} k \quad \text{and} \quad \sum \varphi_j(k) = k.$$

An A-complex which is a DGA is a ***couverture*** if is A-torsion free, its stalks are acyclic, i.e., if $H^*(xK) = H^0(xK) \cong A$ for every $x \in X$, and there exists $u \in K$ such that $xu \leftrightarrow 1$ under $H^0(xK) \cong A$ for all $x \in X$. This is the notion our "CDGA-resolution of $\underline{k}$" translates. Leray's original cohomology theory on a normal space was the cohomology of the union of all *couvertures*.

The intersection $K \bigcirc \mathscr{F}$ of a sheaf and a complex is defined in a way that winds up equivalent to taking the associated sheaf $\mathscr{K} : F \longmapsto F.K$ and forming the tensor sheaf $\mathscr{K} \otimes \mathscr{F}$ in the modern sense. One can also extend the coefficients of a complex K to an A-module M by considering M as a complex with $|m| = X$ for $m \neq 0$ and taking $K \bigcirc M$.

By the time of Borel's 1951 lectures on Leray's work [Bor51], a sheaf (Borel credits this definition to Lazard) has become the *espace étalé*, associated to a presheaf satisfying the gluing axioms, which is equivalent to the modern definition. The statement of the Leray spectral sequence in these lecture notes is as follows (translation due to the present author).

Theorem C.3.3 (Leray). *Let $f\colon X \longrightarrow Y$ be a continuous map, K and L fine A-*couvertures*, M an A-algebra, and $\underline{F}$ the sheaf associated to $f(K \otimes M)$. Then there exists a spectral sequence in which*

$$E_0 = \mathtt{G}\big(f^{-1}(L) \bigcirc K \otimes M\big), \qquad E_1 = L \bigcirc H(\underline{F}), \qquad E_2 = H\big(L \bigcirc H(\underline{F})\big),$$

(d_0 is the derivation with respect to K, d_1 the derivation with respect to L) and which terminates in the associated graded algebra of $H^*(X, M)$, suitably filtered.

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