Koszul duality for tori

Dissertation

zur Erlangung des akademischen Grades des
Doktors der Naturwissenschaften an der
Universität Konstanz, Fachbereich Mathematik
und Statistik, vorgelegt von

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Tag der mündlichen Prüfung: 25. Oktober 2001

Erratum

Proposition 1.8.2 is false because in general the map (1.35) does *not* commute with the differential. The error in the proof is that $\mathbf{h}M$ is not a module over $\mathbf{\Lambda}^*$. The calculations in Appendix 5 are correct, but they only show that $\mathbf{h}M$ is also a *left* module over M, which is not really interesting because in cohomology mutiplication is commutative. It could be of interest in the context of intersection homology, see Section 5 of my (correct) article "Koszul duality and equivariant cohomology for tori".

As a consequence, all statements about products on complexes of the form $\mathbf{h}M$ lack proof. In particular, in Theorem 2.12.3 and Corollary 2.12.4 (comparison of simplicial and algebraic Koszul functors) the word "$\mathbf{\Lambda}$-algebra" must be replaced by "$\mathbf{\Lambda}$-module". The same applies to Theorem 3.3.2 (toric varieties). Hence the complex $\mathbf{\Lambda}^* \tilde{\otimes} R[\Sigma]$ computes $H(X_\Sigma)$ as $\mathbf{\Lambda}$-module and as algebra (Buchstaber–Panov). I guess that by combining my techniques with those of Buchstaber–Panov one can show that there is an isomorphism between $H(\mathbf{\Lambda}^* \tilde{\otimes} R[\Sigma])$ and $H(X_\Sigma)$ which is compatible with both structures at the same time.

I doubt that there always exists a product on $\mathbf{h}M$ extending the one on M, even for $M = C^*(Y)$. The reason is that already for $r = 1$ the fact $\xi' \cup_1 \xi' \neq 0$ is an obstruction to the use of the higher products of the homotopy Gerstenhaber algebra $C^*(Y)$.

Zusammenfassung

Zentrales Thema dieser Arbeit ist der Zusammenhang zwischen der gewöhnlichen und der äquivarianten (singulären) Kohomologie von Räumen mit Torusoperationen. Dabei ist die äquivariante Kohomologie eines T-Raumes X definiert als die Kohomologie der Borelkonstruktion X_T. Sie trägt eine Modulstruktur über der Kohomologie $\mathbf{S}^* = H^*(BT)$ des klassifizierenden Raumes, welche eine symmetrische Algebra ist. Die gewöhnliche Kohomologie $H^*(X)$ ist vermöge der Torusoperation ein Modul über der äußeren Algebra $\mathbf{\Lambda} = H(T)$.

Man kann nicht erwarten, daß eine Kohomologie die andere eindeutig bestimmt – sonst gäbe es keinen Grund, äquivariante Kohomologie einzuführen. Schon seit langem ist aber bekannt, daß im Falle einer T-Mannigfaltigkeit X das Cartanmodell

$$\mathbf{S}^* \otimes \Omega^T(X)$$

mit Differential

$$d(\sigma \otimes \alpha) = \sigma \otimes d\alpha - \sum_{i=1}^{r} \xi_i \sigma \otimes x_i \alpha$$

die äquivariante Kohomologie von X mit reellen Koeffizienten aus dem Komplex der T-invarianten Differentialformen auf X berechnet. Hierbei bezeichnet (x_i) die durch eine Zerlegung $T \cong (S^1)^r$ gegebene Basis von $\mathbf{\Lambda}_1 = H_1(T)$ und (ξ_i) die dazu „duale“ Basis von $\mathbf{S}^2 = H^2(BT)$. Auf Differentialformen wirkt $\mathbf{\Lambda}$ durch Kontraktion mit erzeugenden Vektorfeldern.

Goresky, Kottwitz und MacPherson [**GKM**] haben dieses Ergebnis auf subanalytische T-Räume ausgeweitet und in einen größeren algebraischen Zusammenhang gestellt: Zu jedem solchen Raum X gibt es einen Kokettenkomplex $C^*(X)$, der durch die T-Wirkung zu einem differentiellen $\mathbf{\Lambda}$-Modul wird. Der Koszulfunktor $\mathbf{t}$ ordnet jedem differentiellen $\mathbf{\Lambda}$-Modul N einen $\mathbf{S}^*$-Modul $\mathbf{t}N = \mathbf{S}^* \otimes N$ mit Differential

$$d(\sigma \otimes n) = \sigma \otimes dn - \sum_{i=1}^{r} \xi_i \sigma \otimes x_i n$$

zu. Goreskys, Kottwitz’ und MacPhersons Verallgemeinerung des Cartanmodells ist nun ein Isomorphismus von $\mathbf{S}^*$-Moduln

$$H(\mathbf{t}C^*(X)) \cong H_T^*(X).$$

Der Funktor $\mathbf{t}$ besitzt ein Gegenstück, nämlich den Koszulfunktor $\mathbf{h}$, der jeden differentiellen $\mathbf{S}^*$-Modul M auf einen $\mathbf{\Lambda}$-Modul $\mathbf{h}M = \mathbf{\Lambda}^* \otimes M$ mit Differential

$$(*) \qquad d(\alpha \otimes m) = (-1)^{|\alpha|} \alpha \otimes dm + \sum_{i=1}^{r} x_i \alpha \otimes \xi_i m$$

abbildet. Koszuldualität bezeichnet die Tatsache, daß beide Funktoren quasiinvers zueinander sind. In [**GKM**] wird nun weiter ein Kokettenkomplex $C^*(X_T)$ der Borelkonstruktion mit einer differentiellen Modulstruktur über $\mathbf{S}^*$ konstruiert und gezeigt, daß man aus ihm die gewöhnliche Kohomologie von X gewinnen kann:

$$H(\mathbf{h}C^*(X_T)) \cong H^*(X),$$

und dieser Isomorphismus ist einer von $\mathbf{\Lambda}$-Moduln. Diese Ergebnisse gelten allgemeiner für beliebige zusammenhängende, kompakte Liegruppen (wie auch das Cartanmodell) und zusätzlich beispielsweise für Schnitthomologie.

Allday und Puppe [**AP′**] wiesen kurze Zeit später darauf hin, daß die Koszuldualität hier eine zugrundeliegende topologische Dualität widerspiegelt: Die Borelkonstruktion ist ein Funktor $\boldsymbol{t}$ von der Kategorie der T-Räume zu der der Räume über BT. Ein dazu quasiinverser Funktor $\boldsymbol{h}$ ordnet jedem Raum Y über BT das Faserprodukt $Y \times^{BT} ET$ zu. Das obengenannte Ergebnis von [**GKM**] kann man daher im wesentlichen so formulieren, daß für subanalytische Räume und reelle Koeffizienten die Funktoren $C^* \circ \boldsymbol{t}$ und $\mathbf{t} \circ C^*$ von der Kategorie der T-Räume zu der der differentiellen $\mathbf{\Lambda}$-Moduln quasiisomorph sind, und ebenso die Funktoren $C^* \circ \boldsymbol{h}$ und $\mathbf{h} \circ C^*$ von der Kategorie der Räume über BT zu der der differentiellen $\mathbf{S}^*$-Moduln.

In der vorliegenden Arbeit wird der Frage nachgegangen, inwieweit sich dieses Ergebnis von reellen Koeffizienten auf beliebige Ringe R überträgt, insbesondere natürlich auf die ganzen Zahlen. Als größtes Hindernis bei diesem Unterfangen stellt sich die Tatsache heraus, daß der singuläre Kokettenkomplex eines Raumes über BT im allgemeinen *kein* Modul über $\mathbf{S}^* = H^*(BT; R)$ ist, da das Cup-Produkt singulärer Koketten nicht (graduiert) kommutativ ist. Dies ist ein fundamentaler Unterschied zu den in [**GKM**] verwandten Differentialformen.

Einen Ausweg bietet das Cup_1-Produkt, das eine Homotopie zwischen dem gewöhnlichen Cup-Produkt und dem mit vertauschen Faktoren darstellt und zudem eine Rechtsderivation des Cup-Produktes ist. Mit seiner Hilfe kann man auf dem graduierten R-Modul $\mathbf{\Lambda}^* \otimes C^*(Y; R)$ ein $\mathbf{\Lambda}$-äquivariantes Differential einführen, das eine Deformation des Differentiales $(*)$ ist. Komplexe, die eine solche Konstruktion zulassen, nenne ich schwache $\mathbf{S}^*$-Moduln. Der Koszulfunktor $\mathbf{h}$ wird auf sie erweitert vermöge der Zuordnung $M \mapsto \mathbf{\Lambda}^* \otimes M$ zusammen mit dem neu konstruierten Differential.

Das Hauptergebnis dieser Arbeit lautet wie folgt:

Theorem Bezeichne R einen kommutativen Ring mit Einselement.

1. Sei X ein T-Raum. Dann ist der singuläre Kokettenkomplex $C^*(X; R)$ ein differentieller $\mathbf{\Lambda}$-Modul, und $H(\mathbf{t}C^*(X; R)) \cong H^*(\boldsymbol{t}X; R)$ als $\mathbf{S}^*$-Moduln.
2. Sei Y ein Raum über BT. Dann ist der singuläre Kokettenkomplex $C^*(X; R)$ ein schwacher $\mathbf{S}^*$-Modul, und $H(\mathbf{h}C^*(Y; R)) \cong H^*(\boldsymbol{h}Y; R)$ als $\mathbf{\Lambda}$-Moduln.

Der Beweis geschieht durch Konstruktion verbindender äquivarianter Abbildungen zwischen den betreffenden Komplexen. Hierbei erweist es sich als entscheidend, statt in der Kategorie der topologischen Räume in der der simplizialen Mengen zu arbeiten. Wie in der Arbeit gezeigt wird, hat dieses keinen Einfluß auf die Kohomologien. Wichtig ist außerdem, stets normalisierte Kokettenkomplexe zu verwenden.

Weil die Kohomologie eines jeden Raumes ein Produkt besitzt, liegt die Frage nahe, ob auch die Multiplikation in einem Kokettenkomplex die Multiplikation in der dualen Kohomologie bestimmt. Als Antwort definiere ich explizite Produkte auf $\mathbf{t}C^*(X)$ und $\mathbf{h}C^*(Y)$ und zeige:

Addendum Die im Theorem auftretenden Isomorphismen sind multiplikativ.

Als eine Anwendung bestimme ich die gewöhnliche Kohomologie einer durch einen Fächer Σ beschriebenen, (hinreichend) glatten torischen Varietät X_Σ über den Umweg der äquivarianten Kohomologie. Letztere ist gleich dem Stanley-Reisner-Ring $R[\Sigma]$, der sich als quasiisomorph zum singulären Kokettenkomplex der Borelkonstruktion von X_Σ erweist. Mit Hilfe obigen Theorems und Addendums folgt für die Kohomologie von X_Σ der multiplikative Isomorphismus von $\mathbf{\Lambda}$-Moduln

$$H^*(X_\Sigma) \cong H(\mathbf{h}R[\Sigma]).$$

Dieses Ergebnis verfeinert ein Resultat von Buchstaber und Panow [**BP**] insofern, als es die Modulstruktur miteinbezieht.

Danksagung

An erster Stelle danke ich meinem Betreuer Volker Puppe – dafür, mir ein Thema vorgeschlagen zu haben, das sich als so fruchtbar erweisen sollte, und dafür, jederzeit ein offenes Ohr für meine hin und wieder recht abenteuerlichen Ideen gehabt zu haben und auch mit Erklärungen und Anregungen nicht sparsam umgegangen zu sein. Dabei schätzte ich sehr, daß er mir immer die Freiheit ließ, diejenige Richtung zu verfolgen, die mir selbst am besten erschien. Auch das Interesse, das Gottfried Barthel und Ludger Kaup meiner Arbeit entgegenbrachten, war mir ein Ansporn. Volker Strassen hat meinen bisherigen mathematischen Werdegang in vielerlei Hinsicht entscheidend geprägt, und ich danke ihm herzlich für alles, was er mich gelehrt und für mich getan hat. Allen bisher Genannten sowie meinen Eltern und Freunden bin ich dankbar für den Rückhalt, den sie mir während dieses nicht immer einfachen Weges zuteil werden ließen.

Schließlich gebührt John McCleary dafür Dank, daß er mir vor Drucklegung Zugang zu der neuen Auflage seines Buches [**MC**] verschaffte, und allen meinen Mitbürgern für die wenn auch schwindende, so doch nicht völlig erloschene Bereitschaft, mathematische Forschung zu bezahlen. Letztere verzeihen, so hoffe ich, daß ich mit dieser Arbeit Hand dazu gereicht habe, der deutschen Sprache ein Grab zu schaufeln.

Konstanz, im Juli 2001 Matthias Franz

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Introduction

Let G be a topological group, X a G-space, and R a commutative ring with unit element. The equivariant cohomology $H_G^*(X) = H_G^*(X;R)$ of X is usually defined as the singular cohomology of the total space of the bundle $X_G \to BG$ with fibre X, associated with the universal G-bundle $EG \to BG$. This construction is natural in G and X. In particular, $H_G^*(X)$ is a module over the cohomology of the classifying space BG.

Since its introduction by Borel some 40 years ago, several methods have been studied to substitute the equivariant cochain complex $C^*(X_G)$, for theoretical and practical reasons, by a simpler model. In the case of a Lie group G acting smoothly upon a manifold X and real coefficients one was found to be the Cartan model

$$\big(S(\mathfrak{g}^*) \otimes \Omega(X)\big)^G \tag{0.1}$$

(which actually predates the Borel construction). Here $\Omega(X)$ denotes the de Rham complex of X and $S(\mathfrak{g}^*)$ the algebra of polynomials on the Lie algebra $\mathfrak{g}$ of G with twice the usual degree. The subspace of G-invariants is a complex with differential

$$d(\sigma \otimes \alpha) = \sigma \otimes d\alpha - \sum_{i=1}^{m} \xi_i \sigma \otimes \mathbf{i}_{X_i}\, \alpha, \tag{0.2}$$

where (ξ_i) is a basis of $\mathfrak{g}^*$ and (X_i) the family of generating vector fields on X corresponding to the basis (x_i) of $\mathfrak{g}$ dual to (ξ_i). Then the homology of the complex (0.1) is isomorphic to $H_G^*(X)$ as $H^*(BG)$-algebra, cf. [**GS**, Sec. 4.2].

In the general topological situation progress was made by Moore, who established for bundles associated with arbitrary principal bundles $E \to B$ an isomorphism

$$H(X \underset{G}{\times} E) \cong \operatorname{Tor}^{C(G)}\big(C(X), C(E)\big). \tag{0.3}$$

The Tor functor on the right is a generalisation of the Tor functor for modules without differential or grading. This isomorphism would not be very useful if one could not replace the complexes on right by simpler objects. Fortunately, an important feature of the Tor functor is that it only "sees" the homology of the complexes involved in the sense that one may replace all complexes above by others provided that they are related to the original ones by c-equivalences (i. e., by chain maps inducing isomorphisms in homology) which are compatible with each other.

Taking $G = T$ a torus for instance, it is not hard to construct a c-equivalence of algebras $H(T) \to C(T)$. Replacing $C(T)$ by the exterior algebra $\mathbf{\Lambda} = H(T)$ and then $C(ET)$ by R via the equivariant projection $ET \to *$, one ends up with

$$\operatorname{Tor}^{\mathbf{\Lambda}}(C(X), R) = H(C(X) \underset{\mathbf{\Lambda}}{\otimes} K), \tag{0.4}$$

where K is the total complex of any projective resolution of the trivial $\mathbf{\Lambda}$-module R. For example, one may choose the Koszul complex, which is the tensor product

$$K = \mathbf{\Lambda} \otimes \mathbf{S} \tag{0.5}$$

of $\mathbf{\Lambda}$ with the symmetric algebra $\mathbf{S} = S(\mathfrak{t})$ with degrees doubled as above. This complex carries the differential

$$d(a \otimes s) = \sum_i x_i \wedge a \otimes s \cap \xi_i.$$

The cap product $s \cap \xi_i$ means reducing by one the exponent of x_i in each monomial of s (or dropping it if x_i does not appear). Substituting (0.5) into (0.4) gives after dualisation an isomorphism between $H_T^*(X)$ and the homology of the complex

$$\mathbf{t}C^*(X) = \mathbf{S}^* \otimes C^*(X) \tag{0.6}$$

with differential

$$d(\sigma \otimes \alpha) = -\sum_i \xi_i \sigma \otimes x_i \cdot \alpha + \sigma \otimes d\alpha,$$

where $x_i \cdot \alpha$ denotes the action of the chosen representative in $C(T)$ of $x_i \in \mathbf{\Lambda}$ on α, dual to the "sweep action" on the chains of X. Note that this differential is "twisted" in a way similar to the one appearing in the Cartan model (0.2).

A few years ago, Goresky, Kottwitz, and MacPherson [**GKM**] proved (among other things) that the complex (0.6) gives the equivariant cohomology of a subanalytic space X for arbitrary compact Lie groups G and real coefficients. The algebra $\mathbf{S}^*$ is again the cohomology of BG and $\mathbf{\Lambda}$ the homology of G (which are, respectively, symmetric and exterior algebras by the choice of real coefficients). Their result is actually much more general in that it also applies to intersection cohomology and other cohomology theories defined via "equivariant complexes of sheaves." In addition, they showed how to recover the ordinary cohomology from a suitable model M of the equivariant cochain complex of X. This complex is a module over $\mathbf{S}^*$, and a twisted differential on the tensor product

$$\mathbf{h}M = \mathbf{\Lambda}^* \otimes M, \tag{0.7}$$

similar to (0.6), gives the ordinary cohomology of X. This includes the structure as module over $\mathbf{\Lambda}$, as does (0.6) the $\mathbf{S}^*$-module structure.

This passing from $\mathbf{\Lambda}$-modules to $\mathbf{S}^*$-modules and back is purely algebraic and known as "Koszul duality." The essence is that the functors $\mathbf{t}$ and $\mathbf{h}$ defined above are inverse to each other in the sense that there are natural c-equivalences

$$N \to \mathbf{ht}N \qquad \text{and} \qquad \mathbf{th}M \to M. \tag{0.8}$$

(Hence, the compositions of these functors are isomorphic to the respective identities in the derived categories.)

As Allday and Puppe [**AP′**] have pointed out, the appearance of Koszul duality in equivariant cohomology reflects an underlying topological duality between G-spaces on the one hand and spaces over BG on the other. The Borel construction $X \to X_G$ is a functor from the former category to the latter. In order to get back one associates to each space Y over BG the total space of the pullback of the universal G-bundle, which is a G-space. (More precisely, one also has to switch between left and right G-operations somewhere.) Then the compositions of both functors are homotopy equivalent to the respective identities in the category of topological spaces, i. e., when forgetting any additional structure.

The Moore theorem referred to above also has a companion concerning pullbacks of bundles: By a theorem of Eilenberg and Moore the cohomology of the pullback $Y \times^B E$ of the bundle $E \to B$ along a map $Y \to B$ is isomorphic to

$$\operatorname{Tor}_{C^*(B)}\bigl(C^*(E), C^*(Y)\bigr). \tag{0.9}$$

Given a c-equivalence of algebras $\mathbf{S}^* = H^*(BG) \to C^*(BG)$, we may conclude as before that the cohomology of the pullback of the universal G-bundle is isomorphic to that of the complex

$$K^* \underset{\mathbf{S}^*}{\otimes} C^*(Y) = \mathbf{\Lambda}^* \otimes C^*(Y), \tag{0.10}$$

because the dual K^* of K is the total complex of a free resolution of R over the polynomial algebra $\mathbf{S}^*$. The differential in this case is

$$d(\alpha \otimes \gamma) = \sum_i x_i \cdot \alpha \otimes \xi_i \gamma + (-1)^{|\alpha|} \alpha \otimes d\gamma,$$

and we recover the functor $\mathbf{h}$ from (0.7).

Now the problem is that in general such a map $\mathbf{S}^* \to C^*(BG)$ does not exist – at least not for the singular cochain complex, which one is confined to if one wants to consider arbitrary rings R. The reason is that the cup product of cochains fails to be (graded) commutative. But some structure remains: The cup product is commutative up to homotopy (that's why it becomes commutative after passing to homology). An explicit chain homotopy is given by the cup_1 product introduced by Steenrod. From this one can develop a theory of strongly homotopy commutative maps to tackle the passage from (0.9) to smaller complexes involving the bar resolution of $\mathbf{S}^*$, cf. [**MC**, pp. 292-7]. But it turns out that these complexes are much too big.

The solution I am going to present can to some extent already be found in the work of Gugenheim and May [**GM**] (from which I have also borrowed the aforementioned generalisations of the original Eilenberg–Moore theorems): One can keep the complex (0.10), but has to enlarge the differential to accommodate for multiplications by higher order elements from $C^*(BT)$ compensating the lack of commutativity on the cochain level. This is done quite efficiently: If $T = S^1$, no higher order terms are introduced, corresponding to the existence of a multiplicative c-equivalence $H^*(BS^1) \to C^*(BS^1)$ in this case.

Our point of view towards the complexes (0.6) and (0.7) will differ slightly from that of Eilenberg and Moore: Instead of regarding K and K^* as (algebraic) resolutions of the ground ring over $\mathbf{\Lambda}$ and $\mathbf{S}^*$, respectively, we consider them as algebraic models of the chain and cochain complexes of ET. The latter is a contractible free T-space, hence some sort of topological resolution of the one-point space as T-space and space over BT. When evaluating the Tor terms in (0.3) and (0.9), one would therefore have no need for further resolutions because one factor is already free.

The actual approach of the present work does not rely on any of the theories outlined above. Instead, we will construct equivariant chain maps

$$\mathbf{S}^* \otimes C^*(X) \leftarrow C^*(X \underset{T}{\times} ET) \qquad \text{and} \qquad \mathbf{\Lambda}^* \otimes C^*(Y) \to C^*(Y \overset{BT}{\times} ET) \tag{0.11}$$

and show that they induce isomorphisms in homology. The key link between the topological and algebraic constructions will be the Serre spectral sequence, which also underpins the Eilenberg–Moore theorems cited above.

The equivariance of the maps (0.11) (which still has to be made precise for the first map) immediately gives the dual module structures, which have been lost in the naive application of the Eilenberg–Moore theorems above. We thus obtain a new proof of the results of [**GKM**], valid only for tori and singular cohomology, but for arbitrary rings R. In addition, we show how to recover the products in $H^*(X \times_T ET)$ and $H^*(Y \times^{BT} ET)$ from those of $C^*(X)$ and $C^*(Y)$, respectively. The ring structure of ordinary and equivariant cohomology is not considered in [**GKM**], and much less apparent when using the Eilenberg–Moore theorems.

In a first chapter we will lay the algebraic foundations by an exposition of Koszul duality between differential graded modules over exterior and symmetric algebras. This again goes through for arbitrary commutative rings. We do not only show that the maps (0.8) are c-equivalences, but even homotopy equivalences in the category of complexes. In contrast to the usual spectral sequence proofs, our proof works for arbitrary modules, not just bounded ones.

The next section covers the passage from modules over non-commutative algebras such as $C^*(BT)$ to “modules” over their homologies. We call these new objects “weak modules”, because they lack strict structures. As remarked above, a key role is played by the additional structure given by the cup_1 product, notably by the Hirsch formula. This part is related to the theory of operads, see for example [**Lo**, § 5.1].

Furthermore, we will define explicit products on the complexes (0.6) and (0.7), compatible with the module structures. This is simple in the first case, but rather involved in the second.

It turns out that the right setting for the construction of the maps (0.11) is the category of simplicial sets. We will therefore begin the second chapter by reviewing their basic theory, in particular the Eilenberg–Zilber maps relating the chain complex of a product to the tensor product of the respective chain complexes and the “Steenrod map” underlying the cup_1 product. We will study in great detail how all these maps interact because knowledge only ‘up to homotopy’ on the chain level is not sufficient for our purposes. It seems that some of our formulas have not appeared in the literature before.

We then recall the construction of universal bundles and classifying spaces in the simplicial category and introduce simplicial versions of the functors

$$X \mapsto X \underset{G}{\times} EG \qquad \text{and} \qquad Y \mapsto Y \overset{BG}{\times} EG.$$

I have decided to call them “simplicial Koszul functors” (as opposed to the “algebraic” ones) in order to stress the close relationship between them. We give proofs in the simplicial setting of some observations due to [**AP′**] and explain briefly why these constructions give the same results as their topological counterparts.

We define equivariant maps

$$K \to C(EG) \qquad \text{and} \qquad \mathbf{\Lambda}^* \otimes C^*(BG) \to C^*(EG),$$

which will easily give us the natural transformations connecting the algebraic and simplicial Koszul functors. Along the way, we give a new construction of a certain c-equivalence of algebras $C^*(BT) \to H^*(BT)$, whose existence is also due to Gugenheim and May.

Then we prove the main result of this work, namely that the maps (0.11) are c-equivalences. Moreover, we show that the product structures introduced on the complexes (0.6) and (0.7) induce the correct products in homology.

We finally turn to some applications to show how our theory works in practice. Before touching upon equivariantly formal spaces in the spirit of [**GKM**], we derive the Cartan model (0.1) from our results. A longer section shows how to compute the ordinary and equivariant cohomology of a (sufficiently) smooth toric variety. This has already been done by Buchstaber and Panov [**BP**] in the framework of so-called moment-angle complexes. The advantage of the present approach is that it includes the $\mathbf{\Lambda}$-module structure of the ordinary cohomology.

We have already stressed several times that most of our results are based on explicit constructions. The proofs are usually elementary and consist of verifying that some claimed identities do hold. As a consequence, they are sometimes quite lengthy. In order not to annoy the reader by pages of formal manipulations, many calculations have been banned to an appendix.

A concluding remark about references: I usually cite books and articles which I consider the most accessible and most useful ones for readers who want more information on a subject. As a consequence, references to the literature do not imply claims about originality.

CHAPTER 1

Algebraic Koszul duality

1. Signs and partitions

Throughout this work the letter R denotes a fixed commutative ring with unit element. Moreover, $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the natural numbers, the integers, the rationals, the reals and the complex numbers, respectively. We write $[r]$ for the set $\{1,\ldots,r\} \subset \mathbf{N}$. The degree $|\pi|$ of a finite set π is the number of its elements.

Generally, whenever an object x has a degree $|x| \in \mathbf{Z}$, we call

$$\{x\} = (-1)^{|x|} \in R$$

its **sign**. By abuse of language, we say that x is of even degree if $\{x\} = 1 \in R$, and of odd degree if $\{x\} = -1 \in R$. We also define the abbreviation

$$\{x, y\} = (-1)^{|x|\,|y|} \in R$$

for any pair x, y of objects with degree.

We write $(\mu, \nu) \vdash \pi$ to denote a partition of a finite set $\pi = \mu \mathbin{\dot\cup} \nu \subset \mathbf{N}$. If both subsets are required not to be empty, we write$(\mu, \nu) \vDash \pi$ instead. Moreover, the symbol $(\mu, \nu) \vdash (p, q)$ denotes a partition $\mu \mathbin{\dot\cup} \nu = \{0, \ldots, p+q-1\}$ with $|\mu| = p$ and $|\nu| = q$, i. e., a ***(p, q)*-shuffle**.

The sign $\{(\mu, \nu)\}$ of a partition $(\mu, \nu) \vdash \pi$ is that of the induced permutation of π sending its p smallest elements to those of μ and the rest to those of ν, both in ascending order. One has for any partition $(\mu, \nu) \vdash \pi$ the identity

$$\{(\nu, \mu)\} = \{\mu, \nu\}\{(\mu, \nu)\}. \tag{1.1}$$

We will sometimes consider partitions into more than two subsets. Our definitions carry over in the obvious way. Note that, for example, the sign of a partition $(\lambda, \mu, \nu) \vdash \pi$ satisfies

$$\{(\lambda, \mu, \nu)\} = \{(\lambda, \mu)\}\{(\lambda \cup \mu, \nu)\} = \{(\lambda, \mu \cup \nu)\}\{(\mu, \nu)\}. \tag{1.2}$$

2. Complexes

The purpose of this and the following sections is to fix notation and terminology for complexes, which are mostly as in [**D**]. A major difference is the pairing (1.5) between a complex C and its dual C^*, where we write the functional on the *right*. This also implies a different definition of the differential on C^*. The reason for this deviation is that later on we want to end up with left module structures on cochain complexes.

Following [**ML**], we define a **graded *R*-module** C as a $\mathbf{Z}$-indexed family (C_n) of R-modules, not as their direct sum. Consequently, any element $c \in C$ has a degree $|c| \in \mathbf{Z}$. A **complex** is a graded R-module C together with a module homomorphism d of degree -1, called the **differential**, such that $d \circ d = 0$. Any

$\mathbf{Z}$-graded R-module M can be looked at as a complex with trivial differential, and R as a complex concentrated in degree 0.

A **map** of complexes is a morphism f (of arbitrary degree) of the underlying graded R-modules. A map of degree 0 commuting with differentials is called a **chain map**. Let f, $f'\colon C \to B$ be chain maps. A (chain) **homotopy** from f to f' is a map $h\colon C \to B$ of degree 1 such that $f' - f = d \circ h + h \circ d$.

The **dual** C^* of a complex C is defined by setting

$$(C^*)_{-n} = \operatorname{Hom}_R(C_n, R) \qquad \text{and} \qquad \langle c, d\gamma\rangle = \{\gamma\}\,\langle dc, \gamma\rangle = -\{c\}\,\langle dc, \gamma\rangle \tag{1.3}$$

for $c \in C$ and $\gamma \in C^*$. (We set $\langle c, \gamma\rangle = 0$ if $|c| + |\gamma| \neq 0$.) This is again a complex because the grading is reversed. In doing so, we have no need to distinguish between chain and cochain complexes. This will be convenient later on when they appear together in the formulas. The price we have to pay for this is that homology and cohomology of a space cannot both live in positive degrees. Since chain complexes are the primary objects, we have chosen the homological setting. In order not to confuse the reader too much, we introduce the notation

$$C^n := (C^*)_{-n}.$$

This does not affect the grading! An element $\gamma \in C^n$ still has degree $-n$.

For any map $f\colon C \to B$ of complexes we define the **dual map** $f^*\colon B^* \to C^*$ by

$$\langle c, f^*(\beta)\rangle = \{c, f\}\,\langle f(c), \beta\rangle.$$

This illustrates the general principle to insert, whenever two objects are commuted, the factor -1 to the product of their degrees. This "sign rule" does *not* apply to the differential on the dual of a complex. As a consequence of the above definition, if $h\colon C \to B$ is a homotopy from f to f', then h^* is one from f^* to $(f')^*$.

The **tensor product** $B \otimes C$ of two complexes B, C is the tensor product (over R) of the underlying graded R-modules with total grading and differential

$$d(b \otimes c) = db \otimes c + \{b\}\, b \otimes dc.$$

The transposition of factors

$$T = T_{BC}\colon B \otimes C \to C \otimes B, \qquad b \otimes c \mapsto \{b, c\}\, c \otimes b, \tag{1.4}$$

the evaluation map

$$C \otimes C^* \to R, \qquad c \otimes \gamma \mapsto \langle c, \gamma\rangle \tag{1.5}$$

and the map

$$\iota = \iota_{BC}\colon C^* \otimes B^* \to (B \otimes C)^* \tag{1.6}$$

taking $\gamma \otimes \beta$ to the functional defined by

$$b \otimes c \mapsto \langle b, \beta\rangle\langle c, \gamma\rangle$$

are chain maps.

Given two maps of complexes $f\colon B \to B'$ and $g\colon C \to C'$, we define their **tensor product** $f \otimes g\colon B \otimes C \to B' \otimes C'$ by

$$(f \otimes g)(b \otimes c) = \{g, b\}\, f(b) \otimes g(c). \tag{1.7}$$

It is a map of complexes of degree $|f| + |g|$. The diagram

$$\begin{array}{ccc} (C')^* \otimes (B')^* & \xrightarrow{\iota_{B'C'}} & (B' \otimes C')^* \\ \Big\downarrow g^* \otimes f^* & & \Big\downarrow \{f,g\}\,(f \otimes g)^* \\ C^* \otimes B^* & \xrightarrow{\iota_{BC}} & (B \otimes C)^* \end{array}$$

commutes. Now suppose that f and g are chain maps. Then so is $f \otimes g$. If $h\colon B \to B'$ is a homotopy from f to another chain map f', then $h \otimes g$ is one from $f \otimes g$ to $f' \otimes g$. Analogously, $f \otimes k$ is a homotopy from $f \otimes g$ to $f \otimes g'$ if k is one from g to another chain map g'.

3. Algebras, coalgebras, and modules

An **algebra** is a complex A together with a chain map $A \otimes A \to A$ making A into an R-algebra with unit element. (This is what is usually called a "differential graded algebra." We have dropped the adjectives because most of our algebras will be of this kind, and any algebra without differential or grading can be considered as one with these structures, cf. the previous section.) A **map of algebras** is a chain map commuting with multiplication and taking 1 to 1. We call A **commutative** if commutativity holds in the graded sense, i. e., if one has

$$\forall a, b \in A \quad ab = \{a, b\}\, ba.$$

A **coalgebra** is a complex A together with chain maps $\Delta_A\colon A \to A \otimes A$ (called comultiplication or diagonal) and $\varepsilon_A\colon A \to R$ (counit or augmentation) such that

$$(\varepsilon_A \otimes 1)\Delta_A\colon A \to R \otimes A = A$$

is the identity, and likewise $(1 \otimes \varepsilon_A)\Delta_A = 1$. A **map of coalgebras** is a chain map commuting with the structure maps. A coalgebra A is called **coassociative** if the identity $(1 \otimes \Delta_A)\Delta_A = (\Delta_A \otimes 1)\Delta_A$ holds, and **cocommutative** if $T_{AA}\Delta_A = \Delta_A$, where T_{AA} is the transposition of factors (1.4).

The dual A^* of a coalgebra A is canonically an algebra with unit $\varepsilon_A^*(1)$ and product

$$\Delta_A^* \iota_{AA}\colon A^* \otimes A^* \to (A \otimes A)^* \to A^*.$$

It is associative or commutative if A is coassociative or cocommutative, respectively. Conversely, if an algebra A is a free R-module of finite rank in each degree, then A^* is canonically a coalgebra.

A **Hopf algebra** is a complex A with an algebra and a coalgebra structure such that the coalgebra structure maps ε_A and Δ_A are maps of algebras. A **map of Hopf algebras** is again a structure-preserving chain map.

Note that the tensor product of two algebras (coalgebras, Hopf algebras) is again an algebra (coalgebra, Hopf algebra). (Use the sign rule when permuting factors.)

A central role will be played in the sequel by symmetric and exterior algebras. Let V be a graded R-module with basis $(x_1, \dots, x_r)$ (of homogeneous elements). If all basis elements have even degree, then the symmetric algebra $S(V) = S(x_1, \dots, x_r)$ is defined as the associative and commutative algebra of all polynomials in x_1, … , x_r. If all basis elements have odd degree, one defines the exterior algebra $\bigwedge V = \bigwedge(x_1, \ldots, x_r)$, which is also associative and commutative. We write the (once a basis of V is chosen) canonical basis elements of $S(V)$ and $\bigwedge V$ as

$$x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r} \in S(V) \qquad \text{and} \qquad x_\mu = x_{i_1} \wedge \cdots \wedge x_{i_q} \in \bigwedge V$$

for $\alpha \in \mathbf{N}^r$ and $\mu = \{i_1 < \cdots < i_q\} \subset [r] := \{1, \ldots, r\}$. The multiplications are given in terms of these basis elements by

$$x^\alpha x^\beta = x^{\alpha+\beta} \qquad \text{and} \qquad x_\mu \wedge x_\nu = \begin{cases} \{(\mu, \nu)\}\, x_{\mu\cup\nu} & \text{if } \mu \cap \nu = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

(All this applies in particular to the case where the characteristic of R equals 2.) One has isomorphisms of algebras

$$S(x_1) \otimes \cdots \otimes S(x_r) \cong S(V), \qquad \bigwedge(x_1) \otimes \cdots \otimes \bigwedge(x_r) \cong \bigwedge V, \tag{1.8a}$$

$$s_1 \otimes \cdots \otimes s_r \mapsto s_1 \cdots s_r, \qquad a_1 \otimes \cdots \otimes a_r \mapsto a_1 \wedge \cdots \wedge a_r. \tag{1.8b}$$

Both $S(V)$ and $\bigwedge V$ can also be considered as coalgebras: The augmentations send all generators to zero, and the diagonals are for $r = 1$ given by

$$\Delta_{S(x)}(x^l) = \sum_{i+j=l} x^i \otimes x^j \qquad \text{and} \qquad \Delta_{\bigwedge(x)}(x) = x \otimes 1 + 1 \otimes x. \tag{1.9}$$

Interpreting equations (1.8) as the definition of the coalgebras on the left gives the general case.

With these definitions the exterior algebra $\bigwedge V$ becomes a (self-dual) Hopf algebra, but *not* the symmetric algebra. In this case both structures are dual to each other in the sense that $S(V^*) = S(V)^*$ with multiplication as above is the algebra dual to the coalgebra $S(V)$. One can give both $S(V)$ and $S(V^*)$ Hopf algebra structures, but we will not need this.

Let A be an associative algebra. A left ***A*-module** is a complex C equipped with a chain map $\mu_C \colon A \otimes C \to C$ written as a product and satisfying

$$\forall a_1, a_2 \in A, c \in C \quad (a_1 a_2)c = a_1(a_2 c) \qquad \text{and} \qquad 1 \cdot c = c.$$

Note that an R-module is just a complex.

Let $f \colon A \to A'$ be a map of associative algebras, and let C and C' be an A-module and an A'-module, respectively. A map $g \colon C \to C'$ is called a **(covariant) *f*-map** (or ***f*-covariant**) if it commutes in the graded sense with multiplication, i. e., if one has

$$g(ac) = \{g, a\}\, f(a) g(c)$$

for all $a \in A$ and $c \in C$. If f is the identity of A, we call g an A-map or ***A*-equivariant**. Suppose that we have a map of algebras $f' \colon A' \to A$ instead. Then g is a **contravariant *f′*-map** or ***f′*-contravariant** if we have

$$g(f'(a')c) = \{g, a'\}\, a' g(c)$$

for all $a' \in A'$ and $c \in C$. Note that these two flavours of equivariance do not coincide if the map of algebras in question is not invertible.

Right modules and maps between them are defined analogously; here covariance and contravariance mean

$$g(ca) = g(c) f(a) \qquad \text{and} \qquad g(cf'(a')) = g(c) a'$$

for all $a \in A$, $a' \in A'$ and $c \in C$, respectively. The notions of **chain map** and **homotopy** generalise from complexes to (left and right) modules.

If not specified further, a **module** will be a left module from now on. Moreover, if the type of equivariance of a map is not mentioned, it will always be covariant.

We denote by A-***Mod*** the category of (left) A-modules and A-equivariant homotopy classes of maps, and more generally by ***Mod*** the category with objects, (left) modules over associative algebras, and morphisms, pairs (f, h), where $f : A \to B$ is a map of associative algebras and h the f-covariant homotopy class of an f-chain map between an A-module and a B-module. The category ***Mod**** has the same objects as ***Mod***, but its morphisms are homotopy classes of *contravariant* maps.

If C is a right A-module, then its **dual** C^* is a left A-module via

$$\langle c, a\gamma\rangle = \langle ca, \gamma\rangle$$

for $a \in A$, $c \in C$ and $\gamma \in C^*$. Let f be a map of algebras. Then the dual of an f-equivariant map of right modules is again f-equivariant. If the first map is covariant then its dual is contravariant and vice versa.

We call a complex C an ***A*-*A'*-bimodule** if it is a left A-module and a right A'-module such that

$$(ac)a' = a(ca')$$

holds for all $a \in A$, $a' \in A'$ and $c \in C$.

The **tensor product** $B \otimes_A C$ of a right A-module B and a left A-module C is the quotient of the complex $B \otimes C$ by the subcomplex spanned by the elements

$$ba \otimes c - b \otimes ac, \qquad a \in A, b \in B, c \in C.$$

It is a complex. A pair of module maps induces as before a map of complexes. If B is in fact an A'-A-bimodule, then $B \otimes_A C$ is a left A'-module. Similarly, $B \otimes_A C$ is a right A'-module if C is an A-A'-bimodule. Note that taking multiple tensor products is associative (up to isomorphism), even over different algebras, and that the evaluation map of a right A-module C factors through $C \otimes_A C^*$.

If B is a, say, left A-module, and C a left A'-module, then their tensor product $B \otimes C$ over R is canonically a left $A \otimes A'$-module. (Use the sign rule.) If $A = A'$ is a Hopf algebra, then one can give $B \otimes C$ again an A-module structure via the diagonal Δ_A.

Let $S = S(V^*)$ be a symmetric algebra and A an arbitrary algebra which is also a (left or right) module over S. We call A an ***S*-algebra** if the multiplication of A is S-bilinear.

Let $\Lambda = \bigwedge V$ be an exterior algebra and B an algebra or coalgebra which is also a (left or right) module over Λ. We call B a (left or right) **Λ-(co)algebra** if the (co)multiplication and the (co)unit of B are Λ-equivariant. Here R is considered as trivial module via the augmentation $\Lambda \to R$. (The diligent reader will have noticed that this definition coincides with the previous one if Λ happens to be a symmetric algebra as well.) The dual of a Λ-coalgebra is a Λ-algebra.

LEMMA 1.3.1. *Let Λ be an exterior algebra, and B and C right and left Λ-coalgebras, respectively. Then $B \otimes_\Lambda C$ is a coalgebra. Moreover, any map of coalgebras $B \otimes C \to C'$ factoring through $B \otimes_\Lambda C$ as a map of complexes induces a map of coalgebras.*

PROOF. This simple exercise is done in Appendix 1. □

Later on, we will have to switch between left and right module structures. To do this in a systematic way, we introduce the notion of an opposition: An **opposition** of an associative algebra A is a chain map $A \to A$, $a \mapsto \bar{a}$, such that for all a, $b \in A$ one has

$$\bar{\bar{a}} = a \qquad \text{and} \qquad \overline{ab} = \{a,b\}\, \bar{b}\, \bar{a}. \tag{1.10}$$

Note that an opposition in A induces one in $H(A)$. If A is an exterior algebra (or, more generally, a commutative algebra with trivial differential), then it has a canonical opposition (apart from the scalings by ± 1), namely the assignment

$$a \mapsto \{a\}\, a. \tag{1.11}$$

With the help of an opposition, one can pass from left A-modules to right ones and back by setting

$$a \cdot c := \{a,c\}\, c \cdot \bar{a} \qquad \text{and} \qquad c \cdot a := \{c,a\}\, \bar{a} \cdot c.$$

4. Homology

As usual, the symbols $Z(C)$, $B(C)$, and $H(C)$ denote the cycles, boundaries, and the homology of a complex C, respectively. We write $[c] \in H(C)$ for the homology class represented by $c \in Z(C)$. Since $H(C)$ is a graded R-module (as are the other two), we may consider it as a complex.

We call complexes C and C' cohomologically equivalent (or **c-equivalent** for short), in symbols $C \sim C'$, if there is a sequence of chain maps (a "c-equivalence")

$$C = C_{(0)} \longrightarrow C_{(1)} \longleftarrow C_{(2)} \longrightarrow \cdots \longleftarrow C_{(k-1)} \longrightarrow C_{(k)} = C',$$

each inducing an isomorphism in homology. C-equivalence is an equivalence relation weaker than homotopy equivalence. This "poor man's version" of the derived category appears in [**GHV**, Sec. 0.10] and will be sufficient for our purposes.

Analogously, one may talk about c-equivalences of algebras, of modules over a fixed algebra, or of modules in general. For example, two objects in ***Mod*** are c-equivalent if there is a sequence in ***Mod*** as above which becomes a sequence of isomorphisms in ***Mod*** after passing to homology. This means that all maps $C_{(i)} \to C_{(i\pm 1)}$ are c-equivalences of complexes, equivariant with respect to c-equivalences $A_{(i)} \to A_{(i\pm 1)}$ of the corresponding algebras.

Furthermore, a complex (algebra, A-module) C is called **split** if it is c-equivalent to its own homology. If $A = R$ is a field for instance, then any complex is split. If C is split, we can and will assume that the sequence of maps above (a "splitting") induces the identity in homology.

Finally, we also call a natural transformation Φ between two functors to a category of modules a c-equivalence if for each object X in the source category the morphism Φ_X is a c-equivalence.

5. Koszul complexes

Let P be a free graded R-module with basis $(x_1, \ldots, x_r)$, and let $(\xi_1, \ldots, \xi_r)$ denote the dual basis. Throughout this chapter, we assume that all basis elements have odd degree. (Recall that this holds trivially if the characteristic of R is 2.) We denote by

$$\mathbf{\Lambda} = \bigwedge P$$

the exterior algebra over P with wedge product $\wedge$, and by

$$\mathbf{S} = S(P[1])$$

the symmetric (co)algebra over P, shifted in degree by 1. This means $x_i \in \mathbf{S}_{n+1}$ for a generator $x_i \in P_n \subset \mathbf{\Lambda}_n$. We consider $\mathbf{\Lambda}$ and its dual $\mathbf{\Lambda}^*$ as Hopf algebras, $\mathbf{S}$ as coalgebra and its dual $\mathbf{S}^*$ as algebra as described in Section 1.3. In addition to the canonical R-basis elements of $\mathbf{\Lambda}$ and $\mathbf{\Lambda}^*$,

$$x_\mu = x_{i_1} \wedge \cdots \wedge x_{i_q} \qquad \text{and} \qquad \xi_\mu = \xi_{i_q} \wedge \cdots \wedge \xi_{i_1}$$

for any subset $\mu = \{i_1 < \cdots < i_q\} \subset [r] = \{1, \ldots, r\}$, and those of $\mathbf{S}^*$ and $\mathbf{S}$,

$$\xi^\alpha = \xi_r^{\alpha_r} \cdots \xi_1^{\alpha_1} \qquad \text{and} \qquad x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$$

for any multi-index $\alpha \in \mathbf{N}^r$, we write ω for the "volume form" $\xi_{[r]}$, which generates $\mathbf{\Lambda}^*$ over $\mathbf{\Lambda}$, cf. formula (1.15a). Note that (x_μ) and (ξ_μ) are dual bases, as are (x^α) and (ξ^α). The signs of these basis elements are as follows:

$$\begin{aligned} &\{x_\mu\} = \{\xi_\mu\} = \{\mu\} &&\text{in } \mathbf{\Lambda} \text{ and } \mathbf{\Lambda}^*; \\ &\{x^\alpha\} = \{\xi^\alpha\} = 1 &&\text{in } \mathbf{S} \text{ and } \mathbf{S}^*. \end{aligned}$$

We introduce a right $\mathbf{S}^*$-module structure on $\mathbf{S}$ (the cap product) such that the dual operation is the usual multiplication, i. e.,

$$\langle s \cap \sigma, \tau \rangle = \langle s, \sigma\tau \rangle \tag{1.12}$$

for $s \in \mathbf{S}$ and σ, $\tau \in \mathbf{S}^*$. Hence, $x^\alpha \cap \xi_i$ reduces the exponent α_i by 1, or annihilates the monomial in case $\alpha_i = 0$.

The **homological Koszul complex** is the complex

$$K = K(P) = \mathbf{\Lambda} \mathbin{\tilde{\otimes}} \mathbf{S}$$

with differential

$$d(a \otimes s) = \sum_{i=1}^{r} x_i \wedge a \otimes s \cap \xi_i.$$

For now, the tilde over the tensor symbol is just a reminder that the differential is not the ordinary one (which would be zero in this case), but twisted. We will see in Section 1.7 that this fits into a more general context. Note the isomorphism of complexes

$$K = K(x_1) \otimes \cdots \otimes K(x_r), \tag{1.13}$$

which reduces to the one-dimensional case the proof that the map d is a differential, where it is readily verified (see also [**GS**, Sec. 3.1]). An easy corollary of Proposition 1.6.1 below will be that the differential is actually independent of the chosen basis of P, but this can also be checked directly without difficulty.

The homological Koszul complex being the tensor product of the Hopf algebra $\mathbf{\Lambda}$ and the coalgebra $\mathbf{S}$, we may consider it as left $\mathbf{\Lambda}$-coalgebra and right $\mathbf{S}^*$-module. The tensor product decomposition above is obviously compatible with these structures.

The canonical chain map $\mathbf{S}^* \otimes \mathbf{\Lambda}^* \to K^*$ from (1.6) is an isomorphism. The **cohomological Koszul complex**

$$\overline{K}^* = \mathbf{\Lambda}^* \otimes \mathbf{S}^*$$

is the dual complex K^* with *left* $\mathbf{\Lambda}$-module structure induced by the opposite, hence right structure on $\mathbf{\Lambda}$, which we write as $a \cdot \alpha$. This is up to sign the usual contraction of forms, i. e.,

$$\langle b, a \cdot \alpha \rangle = \langle b \cdot a, \alpha \rangle = \{\alpha\}\{a, b\} \langle a \wedge b, \alpha \rangle = \{a\} \langle b \wedge a, \alpha \rangle \tag{1.14}$$

for a, $b \in \mathbf{\Lambda}$ and $\alpha \in \mathbf{\Lambda}^*$. More explicitly, one has the identity

$$x_\nu \cdot \xi_\mu = \begin{cases} \{\nu\}\{(\mu \setminus \nu, \nu)\}\, \xi_{\mu\setminus\nu} & \text{if } \nu \subset \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{1.15a}$$

in particular

$$x_i \cdot \xi_i = -1. \tag{1.15b}$$

Note that $\overline{K}^*$ (like K^*) is canonically a left $\mathbf{S}^*$-module. Since $\mathbf{S}^*$ is commutative and concentrated in even degrees, the distinction between left and right module structures is not important. In what follows we will usually consider $\overline{K}^*$ as right $\mathbf{S}^*$-module.

As an illustration of the conventions stated at the beginning of this chapter, let us derive the differential on $\overline{K}^*$:

$$\begin{aligned} \langle a \otimes s, d(\sigma \otimes \alpha) \rangle &= -\{a\} \langle d(a \otimes s), \sigma \otimes \alpha \rangle = -\{a\} \sum_{i=1}^r \langle x_i \wedge a \otimes s \cap \xi_i, \sigma \otimes \alpha \rangle \\ &= \{x_i\}\{a, x_i\} \sum_{i=1}^r \langle x_i \wedge a, \alpha \rangle \langle s \cap \xi_i, \sigma \rangle = \sum_{i=1}^r \langle a \cdot x_i, \alpha \rangle \langle s \cap \xi_i, \sigma \rangle \end{aligned}$$

by (1.14),

$$= \sum_{i=1}^r \langle a, x_i \cdot \alpha \rangle \langle s, \xi_i \sigma \rangle = \sum_{i=1}^r \langle a \otimes s, x_i \cdot \alpha \otimes \xi_i \sigma \rangle.$$

Hence,

$$d(\alpha \otimes \sigma) = \sum_{i=1}^r x_i \cdot \alpha \otimes \xi_i \sigma.$$

The homological Koszul complex is actually the total complex of a free resolution of the trivial $\mathbf{\Lambda}$-module R, and $\overline{K}^*$ (like K^*) that of a free resolution of the constants over $\mathbf{S}^*$. This implies that the homology of both Koszul complexes is R. They are actually contractible, i. e., homotopy equivalent to the complex R. In the case of $\overline{K}^*$ this will be a corollary of the results of the following section.

We can now give a first definition of the **algebraic Koszul functors**: On objects they are defined by

$$\begin{aligned} \mathbf{t} = \mathbf{t}_P &: \mathbf{\Lambda}\text{-}\boldsymbol{Mod} \to \mathbf{S}^*\text{-}\boldsymbol{Mod}, & N &\mapsto L \otimes_{\mathbf{\Lambda}} N, \\ \mathbf{h} = \mathbf{h}_P &: \mathbf{S}^*\text{-}\boldsymbol{Mod} \to \mathbf{\Lambda}\text{-}\boldsymbol{Mod}, & M &\mapsto \overline{K}^* \otimes_{\mathbf{S}^*} M, \end{aligned}$$

where we have used the $\mathbf{S}^*$-$\mathbf{\Lambda}$-bimodule

$$L = \mathbf{S}^* \mathbin{\tilde{\otimes}} \mathbf{\Lambda}, \qquad d(\sigma \otimes a) = -\sum_{i=1}^r \xi_i \sigma \otimes x_i \wedge a. \tag{1.16}$$

More explicitly, we have

$$\mathbf{t}N = \mathbf{S}^* \mathbin{\tilde{\otimes}} N, \qquad d(\sigma \otimes n) = -\sum_{i=1}^{r} \xi_i \sigma \otimes x_i n + \sigma \otimes dn,$$

$$\mathbf{h}M = \mathbf{\Lambda}^* \mathbin{\tilde{\otimes}} M, \qquad d(\alpha \otimes m) = \sum_{i=1}^{r} x_i \cdot \alpha \otimes \xi_i m + \{\alpha\}\, \alpha \otimes dm.$$

An $\mathbf{\Lambda}$-equivariant map $f\colon N \to N'$ induces the $\mathbf{S}^*$-equivariant map

$$1 \otimes f\colon \mathbf{t}N \to \mathbf{t}N', \qquad \sigma \otimes n \mapsto \sigma \otimes f(n).$$

If f is a chain map or homotopy, then so is $1 \otimes f$. Similarly, an $\mathbf{S}^*$-equivariant map $g\colon M \to M'$ induces the $\mathbf{\Lambda}$-equivariant map

$$1 \otimes g\colon \mathbf{h}M \to \mathbf{h}M', \qquad \alpha \otimes m \mapsto \{\alpha, g\}\, \alpha \otimes g(m),$$

which is again a chain map or homotopy in case g is. (The sign is due to definition (1.7). It does not appear in the previous formula because $\mathbf{S}^*$ is evenly graded.) In particular, the Koszul functors are well-defined on morphisms. We will study their behaviour under more general maps in the next section. The proofs given there will in particular justify the above claims about maps.

The definition of $\mathbf{t}$ is motivated by the following observation, which will be used in Section 2.12:

LEMMA 1.5.1. *For all right $\mathbf{\Lambda}$-modules N the map*

$$\mathbf{t}N^* = \mathbf{S}^* \mathbin{\tilde{\otimes}} N^* \to (N \underset{\mathbf{\Lambda}}{\otimes} K)^* = (N \mathbin{\tilde{\otimes}} \mathbf{S})^*$$

sending $\sigma \otimes \nu$ to the functional

$$n \otimes s \mapsto \langle n, \nu \rangle \langle s, \sigma \rangle$$

is an isomorphism of $\mathbf{S}^*$-modules.

PROOF. This straightforward calculation can be found in Appendix 2. □

Note that the Koszul functors are compatible with direct sums $P = P' \oplus P''$ in the sense that

$$\mathbf{t}_{P' \oplus P''} N \cong \mathbf{t}_{P''} \mathbf{t}_{P'} N \qquad \text{and} \qquad \mathbf{h}_{P' \oplus P''} M \cong \mathbf{h}_{P''} \mathbf{h}_{P'} M.$$

Using the isomorphisms (1.8), one has in particular

$$\mathbf{t}_P N \cong \mathbf{t}_{(x_r)} \cdots \mathbf{t}_{(x_1)} N \qquad \text{and} \qquad \mathbf{h}_P N \cong \mathbf{h}_{(x_r)} \cdots \mathbf{h}_{(x_1)} M. \tag{1.17}$$

This often allows to reduce proofs to the case $r = 1$.

Let us remark how our construction of the Koszul functors relates to those of [**GKM**] and [**Fl**]: Our definition of $\mathbf{t}N$ agrees up to a minus sign in the differential with that of [**GKM**, Sec. 8.3], and the map

$$\mathbf{h}M \to \operatorname{Hom}_R(\mathbf{\Lambda}, M), \qquad \alpha \otimes m \mapsto \bigl(a \mapsto (-1)^{|\alpha|(|\alpha|+1)/2} \langle a, \alpha \rangle\, m\bigr)$$

is an isomorphism of $\mathbf{\Lambda}$-modules, again up to a minus sign in the differential.

If one takes the complex L from (1.16) as the complex T used in [**Fl**, Example 2.1.7], then the definitions of $\mathbf{t}N$ agree, and the map

$$\mathbf{h}M \to \operatorname{Hom}_{\mathbf{S}^*}(L, M), \quad \alpha \otimes m \mapsto \bigl(\sigma \otimes a \mapsto \{\alpha\} \langle a, \alpha \rangle\, \sigma m\bigr)$$

is an isomorphism of $\mathbf{\Lambda}$-modules if one generalises definition (1.3) to $f \in \operatorname{Hom}(C, B)$ by setting

$$\langle c, df\rangle = \{c\}\, d\langle c, f\rangle - \{c\}\langle dc, f\rangle.$$

6. First properties of the Koszul functors

Let P' be another free graded R-module of finite rank, giving rise to algebras $\mathbf{\Lambda}'$ and $(\mathbf{S}')^*$, and let $h\colon P' \to P$ be a linear map of degree 0. We call a map from a $\mathbf{\Lambda}$-module to a $\mathbf{\Lambda}'$-module ***h*-equivariant** if it is contravariant with respect to the induced map $h\colon \mathbf{\Lambda}' \to \mathbf{\Lambda}$. Similarly, we call a map from an $\mathbf{S}^*$-module to an $(\mathbf{S}')^*$-module h-equivariant if it is covariant with respect to the induced map $h^*\colon \mathbf{S}^* \to (\mathbf{S}')^*$.

Proposition 1.6.1. *The algebraic Koszul functors are natural with respect to maps $h\colon P' \to P$ as above. More precisely, an h-equivariant map $f\colon N \to N'$ induces the h-equivariant map*

$$\mathbf{t}f\colon \mathbf{t}N \to \mathbf{t}N', \qquad \sigma \otimes n \mapsto h^*(\sigma) \otimes f(n),$$

and an h-equivariant map $g\colon M \to M'$ the h-equivariant map

$$\mathbf{h}g\colon \mathbf{h}M \to \mathbf{h}M', \qquad \alpha \otimes m \mapsto \{\alpha, g\}\, h^*(\alpha) \otimes g(m).$$

These assignments preserve chain maps and homotopies.

Proof. It is fairly clear that the above maps are equivariant because the maps $h^*\colon \mathbf{\Lambda}^* \to (\mathbf{\Lambda}')^*$ and $h^*\colon \mathbf{S}^* \to (\mathbf{S}')^*$ are so. (Our slightly modified contraction (1.14) does not change this.) It is not difficult to check that chain maps and homotopies go over to the same kind of maps, see Appendix 3. ☐

The maps of complexes

$$\text{(1.18a)} \qquad \mathbf{ht}N \to N, \qquad \alpha \otimes \sigma \otimes n \mapsto \varepsilon(\alpha)\varepsilon(\sigma)\, n,$$

where ε denotes the canonical augmentations sending all generators of $\mathbf{\Lambda}^*$ and $\mathbf{S}^*$ to zero, and

$$\text{(1.18b)} \qquad M \to \mathbf{th}M, \qquad m \mapsto 1 \otimes 1 \otimes m,$$

can be understood as applications of Proposition 1.6.1: The identity mapping of any $\mathbf{\Lambda}$-module N is trivially contravariant with respect to the injection $h\colon 0 \to P$, whence a map

$$\text{(1.19)} \qquad \mathbf{t}N = \mathbf{t}_P N \to \mathbf{t}_0 N = N.$$

The induced map $h^*\colon \mathbf{S}^* \to R$ is the standard augmentation, which permits us to look at $\mathbf{t}_0 N = N$ as trivial $\mathbf{S}^*$-module. The map (1.19) then is $\mathbf{S}^*$-equivariant, we therefore obtain

$$\mathbf{ht}N = \mathbf{h}_P\mathbf{t}_P N \to \mathbf{h}_P\mathbf{t}_0 N \to \mathbf{h}_0\mathbf{t}_0 N = N,$$

which is the map (1.18a).

The projection $P \to 0$ induces the inclusion $h^*\colon R \to \mathbf{S}^*$, relative to which the identity of an $\mathbf{S}^*$-module M is equivariant. We thus have a map

$$M = \mathbf{h}_0 M \to \mathbf{h}_P M = \mathbf{h}M,$$

which is equivariant if we consider M as trivial $\mathbf{\Lambda}$-module via the augmentation $h\colon \mathbf{\Lambda} \to R$. The identity of $\mathbf{h}_0 M = M$ being h-equivariant, we end up with composition (1.18b),

$$M = \mathbf{t}_0\mathbf{h}_0 M \to \mathbf{t}_P\mathbf{h}_0 M \to \mathbf{t}_P\mathbf{h}_P M = \mathbf{th}M.$$

Let N be a $\mathbf{\Lambda}$-module and M an $\mathbf{S}^*$-module. Generalising the construction $\mathbf{S}^* \mathbin{\tilde\otimes} N$, we define for the sake of the next result the twisted tensor product $M \mathbin{\tilde\otimes} N$ to be the tensor product $M \otimes N$ with differential

$$d(m \otimes n) = dm \otimes n - \{m\} \sum_{i=1}^{r} \xi_i m \otimes x_i n + \{m\}\, m \otimes dn.$$

All our results about compositions of the algebraic Koszul functors will be consequences of the following technical lemma:

Lemma 1.6.2. *The maps*

$$a\colon M \mathbin{\tilde\otimes} \mathbf{ht}N \to M \mathbin{\tilde\otimes} N, \qquad m \otimes \alpha \otimes \sigma \otimes n \mapsto \varepsilon(\alpha)\, \sigma m \otimes n$$

and

$$b\colon M \mathbin{\tilde\otimes} N \to M \mathbin{\tilde\otimes} \mathbf{ht}N, \qquad m \otimes n \mapsto \sum_{\pi \subset [r]} \{\pi\}\, m \otimes \xi_\pi \otimes 1 \otimes x_\pi n$$

are homotopy equivalences over $\mathbf{S}^* \otimes \mathbf{\Lambda}$, inverse to each other. Here the actions of $\mathbf{S}^*$ and $\mathbf{\Lambda}$ on $M \mathbin{\tilde\otimes} \mathbf{ht}N$ and $M \mathbin{\tilde\otimes} N$ come from those on M and N, respectively.

For example, $\mathbf{S}^* \otimes \mathbf{\Lambda}$ acts on $M \mathbin{\tilde\otimes} N$ by

$$(\sigma \otimes a) \cdot (m \otimes n) = \{a, m\}\, \sigma m \otimes an.$$

Proof. Let $N_{(0)} = N$, and define for $i \in [r]$ inductively the $\mathbf{\Lambda}$-modules

$$N_{(i)} = \mathbf{h}_{(x_i)}\mathbf{t}_{(x_i)} N_{(i-1)} = \bigwedge(\xi_i) \mathbin{\tilde\otimes} S(\xi_i) \mathbin{\tilde\otimes} N_{(i-1)},$$

where x_i acts on the first factor of the tensor product on the right and all other generators of $\mathbf{\Lambda}$ on the last. Using the decompositions (1.17) and reordering the functors, we obtain

$$\begin{aligned} \mathbf{ht}N &\cong \mathbf{h}_{(x_r)} \cdots \mathbf{h}_{(x_1)} \mathbf{t}_{(x_r)} \cdots \mathbf{t}_{(x_1)} N \\ &\cong \mathbf{h}_{(x_r)}\mathbf{t}_{(x_r)} \cdots \mathbf{h}_{(x_1)}\mathbf{t}_{(x_1)} N = N_{(r)}, \end{aligned}$$

hence

$$M \mathbin{\tilde\otimes} N_{(0)} = M \mathbin{\tilde\otimes} N \qquad \text{and} \qquad M \mathbin{\tilde\otimes} N_{(r)} = M \mathbin{\tilde\otimes} \mathbf{ht}N$$

as complexes. Furthermore, define chain maps $a_{(i)}$ and $b_{(i)}$ similar to a and b above,

$$a_{(i)}\colon M \mathbin{\tilde\otimes} N_{(i)} \to M \mathbin{\tilde\otimes} N_{(i-1)}, \quad m \otimes \alpha \otimes \sigma \otimes n' \mapsto \begin{cases} \sigma m \otimes n' & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha = \xi_i, \end{cases}$$

$$b_{(i)}\colon M \mathbin{\tilde\otimes} N_{(i-1)} \to M \mathbin{\tilde\otimes} N_{(i)}, \quad m \otimes n' \mapsto m \otimes 1 \otimes 1 \otimes n' - m \otimes \xi_i \otimes 1 \otimes x_i n'.$$

Then

$$a = a_{(1)} \circ \cdots \circ a_{(r)} \qquad \text{and} \qquad b = b_{(r)} \circ \cdots \circ b_{(1)}.$$

This is obvious in the first case, the slightly more difficult verification for b is done in Appendix 4, as well as the proof that $a_{(i)}$ and $b_{(i)}$ are in fact chain maps. In order to show that a and b are homotopy equivalences of complexes and inverse to each other, it suffices therefore to consider a single pair $a_{(i)}$ and $b_{(i)}$.

We have $a_{(i)}b_{(i)} = 1$ and

$$b_{(i)}a_{(i)}(m \otimes \alpha \otimes \sigma \otimes n') = \begin{cases} \sigma m \otimes 1 \otimes 1 \otimes n' - \sigma m \otimes \xi_i \otimes 1 \otimes x_i n' & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha = \xi_i. \end{cases}$$

As proven in Appendix 4, a homotopy from the identity of $M \tilde{\otimes} N_{(i)}$ to $b_{(i)}a_{(i)}$ is given by

$$H_{(i)}(m \otimes \alpha \otimes \xi_i^l \otimes n') = \{m\} \sum_{p+q=l-1} \xi_i^p m \otimes \xi_i \wedge \alpha \otimes \xi_i^q \otimes n'. \tag{1.20}$$

Inspection finally shows that the actions of $\mathbf{S}^*$ and $\mathbf{\Lambda}$ on M and N, respectively, induce module structures on all $M \tilde{\otimes} N_{(i)}$ relative to which all maps $a_{(i)}$ and $b_{(i)}$ and all homotopies $H_{(i)}$ are equivariant. □

Theorem 1.6.3. *The compositions* **ht** *and* **th** *are c-equivalent to the respective identity functors of* $\mathbf{\Lambda}$-***Mod*** *and* $\mathbf{S}^*$-***Mod***. *They become isomorphic to them if composed with the forgetful functors to* R-***Mod***. *More precisely:*

1. *Let N be a $\mathbf{\Lambda}$-module. Then the map*

$$N \to \mathbf{ht}N = \mathbf{\Lambda}^* \tilde{\otimes} \mathbf{S}^* \tilde{\otimes} N, \qquad n \mapsto \sum_{\pi \subset [r]} \{\pi\}\, \xi_\pi \otimes 1 \otimes x_\pi n$$

is a $\mathbf{\Lambda}$-equivariant homotopy equivalence of complexes, natural in N. A strict left homotopy inverse is given by the canonical map (1.18a).

2. *Let M be an $\mathbf{S}^*$-module. Then the map*

$$\mathbf{th}M \to M, \qquad \sigma \otimes \alpha \otimes m \mapsto \varepsilon(\alpha)\, \sigma m$$

is an $\mathbf{S}^*$-equivariant homotopy equivalence of complexes, natural in M. A strict right homotopy inverse is given by the canonical map (1.18b).

Proof. Both statements are applications of the preceding lemma: In the first case simply choose $M = R$ to see that the map given in the statement is a homotopy equivalence of complexes with the claimed inverse.

In the second case one takes $N = R$ and reverts the order of the factors in the tensor product. Since we will make a different choice for N in the next proof, we nevertheless consider the case of general N. The isomorphism of graded R-modules

$$M \tilde{\otimes} \mathbf{ht}N = M \tilde{\otimes} \mathbf{\Lambda}^* \tilde{\otimes} \mathbf{S}^* \tilde{\otimes} N \to N \otimes \mathbf{S}^* \otimes \mathbf{\Lambda}^* \otimes M,$$
$$m \otimes \alpha \otimes \sigma \otimes n \mapsto \{m, \alpha \otimes \sigma \otimes n\}\{\alpha, \sigma \otimes n\}\{\sigma, n\}\, n \otimes \sigma \otimes \alpha \otimes m$$
$$= \{m, \alpha\}\{m, n\}\{\alpha, n\}\, n \otimes \sigma \otimes \alpha \otimes m$$

induces on the image the differential

$$\begin{aligned} d(n \otimes \sigma \otimes \alpha \otimes m) = {} & \{\alpha\}\{n\}\, n \otimes \sigma \otimes \alpha \otimes dm \\ & - \{n\} \sum_i n \otimes \sigma \otimes x_i \cdot \alpha \otimes \xi_i m \\ & + \{n\} \sum_i n \otimes \xi_i \sigma \otimes x_i \cdot \alpha \otimes m \\ & - \sum_i x_i n \otimes \xi_i \sigma \otimes \alpha \otimes m + dn \otimes \sigma \otimes \alpha \otimes m. \end{aligned} \tag{1.21}$$

Denote for a module C by $\overline{C}$ the same module, but with differential scaled by -1. Then formula (1.21) shows that the complex on the right hand side above is for $N = R$ equal to the $\mathbf{S}^*$-module

$$\overline{\mathbf{th}\overline{\overline{M}}}.$$

Similarly, the transposition of factors in $M \mathbin{\tilde{\otimes}} N$ gives the complex

$$N \mathbin{\tilde{\otimes}} M, \quad d(n \otimes m) = dn \otimes m - \sum_i x_i n \otimes \xi_i m + \{n\}\, n \otimes dm. \tag{1.22}$$

In particular, transposition of $M \mathbin{\tilde{\otimes}} R$ gives M. Therefore, the preceding lemma yields for any $M \in \mathbf{S}^*$-***Mod*** a homotopy equivalence

$$\overline{\mathbf{th}\overline{\overline{M}}} \to M.$$

Since the assignment $M \to \overline{M}$ preserves not only chain maps, but also homotopy equivalences, this implies that

$$\mathbf{th}M \to M$$

is such a map, too. It is readily verified that the explicit formula given in the statement is the correct one.

Finally, the equivariance of both maps is checked directly. □

We call a module over $\mathbf{\Lambda}$ or $\mathbf{S}^*$ **extended** if it is obtained by extension of scalars when forgetting the differential. I. e., as graded module it is of the form $N = \mathbf{\Lambda} \otimes C$ or $M = \mathbf{S}^* \otimes C$ for some graded R-module C, respectively. Note that in first case we may equally assume that $N = \mathbf{\Lambda}^* \otimes C'$ for some C'.

Corollary 1.6.4. *Any $\mathbf{\Lambda}$-module or $\mathbf{S}^*$-module is c-equivalent to an extended module.*

Proof. All modules in the image of the Koszul functors are extended. □

Corollary 1.6.5. *The algebraic Koszul functors form an adjoint pair $(\mathbf{t}, \mathbf{h})$.*

As remarked in [**Fl**], this is essentially a $\otimes$-Hom adjunction.

Proof. We have to show that the following compositions of the equivariant maps given in the theorem are the respective identities, cf. [**W**, Ex. A.6.2]:

$$\mathbf{t}N \longrightarrow \mathbf{t}(\mathbf{ht}N) = \mathbf{th}(\mathbf{t}N) \longrightarrow \mathbf{t}N,$$
$$\mathbf{h}M \longrightarrow \mathbf{ht}(\mathbf{h}M) = \mathbf{h}(\mathbf{th}M) \longrightarrow \mathbf{h}M.$$

This follows from the explicit description of the maps together with formula (1.15a). □

Note that in general there is no equivariant homotopy inverse to the maps given in the preceding theorem. (Take $N = R$ or $M = R$.) But one can even choose an equivariant homotopy if the module one starts with lies in the image of a Koszul functor:

Theorem 1.6.6. *The functors $\mathbf{t}$ and $\mathbf{tht}$ are isomorphic, as are $\mathbf{h}$ and $\mathbf{hth}$.*

Proof. Apply again Lemma 1.6.2: in the first case with $M = \mathbf{S}^*$, in the second with $N = \mathbf{\Lambda}^*$ and reversed factors. Equations (1.21) and (1.22) show that the reordering yields a homotopy equivalence between the complexes

$$\overline{\mathbf{hth}\overline{\overline{M}}} \qquad \text{and} \qquad \overline{\mathbf{h}\overline{\overline{M}}},$$

hence also between $\mathbf{hth}M$ and $\mathbf{h}M$. Note that we end up with the usual $\mathbf{\Lambda}$-actions on these modules. Therefore, the homotopy equivalence is one over $\mathbf{\Lambda}$. We thus obtain as in the first case a natural transformation between the functors which is for each object an isomorphism in the right category. □

Proposition 1.6.7. *For all $\mathbf{\Lambda}$-modules N the assignment*

$$\mathbf{ht}N = \mathbf{\Lambda}^* \mathbin{\tilde{\otimes}} \mathbf{S}^* \mathbin{\tilde{\otimes}} N \ni \omega \otimes \sigma \otimes n \mapsto (\omega \otimes \sigma) \otimes n \in \overline{K}^* \otimes N$$

extends uniquely to an isomorphism of $\mathbf{\Lambda}$-modules. The canonical map $N \to \mathbf{ht}N$ from Theorem 1.6.3 corresponds under this isomorphism to the canonical inclusion $N \hookrightarrow \overline{K}^* \otimes N$.

Since $\overline{K}^*$ is contractible, this result may be used to give a different proof of the homotopy equivalence of complexes $N \to \mathbf{ht}N$. Note that $\mathbf{\Lambda}$ acts on both factors of $\overline{K}^* \otimes N$.

Proof. Since any element of $\mathbf{ht}N$ is a sum of terms $x_\pi \cdot \omega \otimes c_\pi$ for uniquely determined $c_\pi \in \mathbf{t}N$, it is clear that we get a well-defined equivariant map f of $\mathbf{\Lambda}$-modules when forgetting the differentials. The action of $\mathbf{\Lambda}$ on $\overline{K}^* \otimes N$ is of the form

$$x_\pi \cdot \big((\omega \otimes \sigma) \otimes n\big) = (x_\pi \cdot \omega \otimes \sigma) \otimes n \\ + \text{terms of lower degree in the } \mathbf{\Lambda}^*\text{-component.}$$

(Recall that $\mathbf{\Lambda}^*$ is negatively graded.) This proves that f is bijective. It remains to show that it is compatible with the differentials. We have

$$\begin{aligned} &fd(\omega \otimes \sigma \otimes n) \\ &\quad = f\Big(\sum_i \big(x_i \cdot \omega \otimes \xi_i\sigma \otimes n - \{\omega\}\, \omega \otimes \xi_i\sigma \otimes x_i n\big) + \{\omega\}\, \omega \otimes \sigma \otimes dn\Big) \\ &\quad = \sum_i \Big(x_i \cdot f(\omega \otimes \xi_i\sigma \otimes n) - \{\omega\}\, f(\omega \otimes \xi_i\sigma \otimes x_i n)\Big) + \{\omega\}\, f(\omega \otimes \sigma \otimes dn) \end{aligned}$$

By definition of the $\mathbf{\Lambda}$-action on $\overline{K}^* \otimes N$ this simplifies to

$$= \sum_i (x_i \cdot \omega \otimes \xi_i\sigma) \otimes n + \{\omega\}\, (\omega \otimes \sigma) \otimes dn = df(\omega \otimes \sigma \otimes n),$$

as claimed.

It is readily verified that in the case $r = 1$ the element $1 \otimes n \in \overline{K}^* \otimes N$ corresponds to

$$1 \otimes 1 \otimes n - \xi_1 \otimes 1 \otimes x_1 n = \sum_{\pi \subset [r]} \{\pi\}\, \xi_\pi \otimes 1 \otimes x_\pi n \in \mathbf{ht}N,$$

which is the image of n under the canonical inclusion $N \hookrightarrow \mathbf{ht}N$. The general case follows by equations (1.17). □

7. Weak modules

The algebraic Koszul functors as defined in Section 1.5 are not yet suited for application to the topological situation we have in mind. We will see in the next chapter that the (co)chain complex of a space with an operation of a torus T can indeed be endowed with a module structure over $H(T)$, which is an exterior algebra. But unless the torus is a circle (or a point), the cochain complex of a space over BT in general is *not* a module over the symmetric algebra $H^*(BT)$. This is due to the fact that the cup product of cochains fails to be commutative, which reflects the lack of commutativity of the Alexander–Whitney map used to define it. This is unfortunate, but it gives rise to a beautiful construction to overcome the difficulty.

The key observation is that the Koszul functor $\mathbf{h}$ carries $\mathbf{S}^*$-modules to extended $\mathbf{\Lambda}$-modules, i. e., to modules obtained by extension of scalars when forgetting the differential, cf. Corollary 1.6.4. (The same applies of course to the functor $\mathbf{t}$.) Now one may ask: Given a graded R-module M, what is the most general $\mathbf{\Lambda}$-equivariant differential the graded R-module $\mathbf{\Lambda}^* \otimes M$ can carry? The differential being equivariant, it is of course determined by its values on elements from $\omega \otimes M$, where it must be of the form

$$
\begin{aligned}
d(\omega \otimes m) &= \sum_{\pi \subset [r]} \{\pi, \omega\}\{\pi\}\{\omega\}\, x_\pi \cdot \omega \otimes t_\pi(m) \\
&= \sum_{(\mu,\nu) \vdash [r]} \{\mu, \nu\}\{\omega\}\, x_\mu \cdot \omega \otimes t_\mu(m)
\end{aligned}
$$

for some maps of complexes $t_\pi \colon M \to M$ of degrees $-|x_\pi| - 1$. (The choice of signs will be convenient later on.) The requirement of d being a differential translates into the conditions

$$
\forall \pi \subset [r] \quad \sum_{(\mu,\nu) \vdash \pi} \{\nu\}\{(\mu, \nu)\}\, t_\nu t_\mu = 0, \tag{1.23}
$$

where the sum extends over all partitions $\mu \dot{\cup} \nu = \pi$. For small subsets π this means (omitting braces)

$$ t_\emptyset t_\emptyset = 0, \tag{1.24a} $$

$$ t_i t_\emptyset = t_\emptyset t_i, \tag{1.24b} $$

$$ t_{ij} t_\emptyset + t_\emptyset t_{ij} = t_j t_i - t_i t_j \qquad (i < j). \tag{1.24c} $$

We see that M is canonically a complex with differential $d = t_\emptyset$, that the t_i's commute with it strictly and with each other up to homotopy, these being supplied by the t_{ij}'s. The higher order terms can similarly be understood as "higher order homotopies" between the lower order ones.

Now suppose that M is a module over an associative algebra A. We can fulfil equations (1.24a) and (1.24b) by keeping the differential and defining $t_i(m) = -\xi'_i m$ for arbitrarily chosen cycles $\xi'_i \in A$ of (even) degree $|\xi'_i| = |\xi_i|$. If we want the higher order terms t_π also to be left multiplications by algebra elements, say

$$ t_\pi(m) = \{\pi\}\, \xi'_\pi m, $$

the equations (1.23) become

$$
\forall\, \emptyset \neq \pi \subset [r] \quad d\xi'_\pi = -\sum_{(\mu,\nu) \vDash \pi} \{\nu\}\{(\mu, \nu)\}\, \xi'_\nu \xi'_\mu, \tag{1.25}
$$

where the summation extends over all partitions of π with $\mu \neq \emptyset \neq \nu$. In a fancier language, the assignment

$$t\colon \mathbf{\Lambda}^* \ni \xi_\pi \mapsto t(\xi_\pi) = \{\pi\}\,\xi'_\pi \in A \quad \text{for } \mu \neq \emptyset, \quad t(1) = 0,$$

is a **twisting cochain** for $\mathbf{\Lambda}^*$ with values in A (see [**MC**, Def. 6.45] or [**M**, §30]). We take the twisting cochain t as a convenient tool to refer to the collection of the ξ'_π's. For example, note that the $\mathbf{\Lambda}$-equivariant differential $d = d_t$ on $\mathbf{\Lambda}^* \tilde{\otimes} M$ defined by t has the form

$$d_t = d_{\mathbf{\Lambda}^* \otimes M} + D_t = 1 \otimes d_M + D_t \tag{1.26a}$$

with

$$D_t = (1 \otimes \mu_M)(1 \otimes t \otimes 1)(\Delta \otimes 1)\colon \mathbf{\Lambda}^* \otimes M \to \mathbf{\Lambda}^* \otimes M, \tag{1.26b}$$

where Δ is the comultiplication of $\mathbf{\Lambda}^*$ and $\mu_M : A \otimes M \to M$ the structure map of the A-module M. Since d_t is $\mathbf{\Lambda}$-equivariant by construction, so is D_t, or, equivalently, the map of degree 1

$$(1 \otimes t)\Delta\colon \mathbf{\Lambda}^* \to \mathbf{\Lambda}^* \otimes M. \tag{1.27}$$

We record the explicit form of the differential for later use:

$$d(\omega \otimes m) = \sum_{\emptyset \neq \mu \subset [r]} \{\xi'_\mu, \omega\}\, x_\mu \cdot \omega \otimes \xi'_\mu m + \{\omega\}\, \omega \otimes dm, \tag{1.28a}$$

$$d(\xi_\pi \otimes m) = \{\pi\} \sum_{\substack{(\mu,\nu) \vdash \pi \\ \mu \neq \emptyset}} \{(\mu,\nu)\}\, \xi_\nu \otimes \xi'_\mu m + \{\pi\}\, \xi_\pi \otimes dm, \tag{1.28b}$$

where we have used equation (1.15a) to get the last line.

We call the triple (M, A, t) where M is a module over an associative algebra A and t a twisting cochain as above a **weak** (left) **S*-module**. (This concept is related to strongly homotopy multiplicative maps, cf. [**MC**, Sec. 8.1]. Though one could always take the algebra of all endomorphisms of M as A, it will be convenient to allow for other ones as well.) We again write $\mathbf{\Lambda}^* \tilde{\otimes} M$ for the resulting extended $\mathbf{\Lambda}$-module with twisted differential d_t. As a special case, any $\mathbf{S}^*$-module is also a weak $\mathbf{S}^*$-module: Simply set $\xi'_i = \xi_i \in \mathbf{S}^*$, and all higher order terms to zero.

A **map of weak S*-modules** $f\colon (M, A, t) \to (M', A', t')$ of degree d is a $\mathbf{\Lambda}$-equivariant map

$$f\colon \mathbf{\Lambda}^* \otimes M \to \mathbf{\Lambda}^* \otimes M', \qquad \omega \otimes m \mapsto \sum_\pi x_\pi \cdot \omega \otimes f_\pi(m),$$

where $f_\pi\colon M \to M'$ is a map of degree $d - |\pi|$. If the components f_π vanish for all $\pi \neq \emptyset$, we say that f (and, by abuse of language, also $f_\emptyset$) is **without higher order terms**. Chain maps and homotopies of weak $\mathbf{S}^*$-modules have the obvious meanings. This gives us the category $\mathbf{S}^*$-$\mathcal{M}\boldsymbol{od}$ of weak $\mathbf{S}^*$-modules and homotopy classes of maps. We will usually refer to a weak $\mathbf{S}^*$-module (M, A, t) by the module M.

The **homology** of (M, A, t) is by definition that of M. It carries a canonical (strict) $\mathbf{S}^*$-module structure, given on representative cycles by multiplication by the ξ'_i's. The equations (1.24) show that this is well-defined. If $f\colon M \to M'$

is a chain map of weak $\mathbf{S}^*$-modules, then this imposes certain conditions on its components f_π, for example:

$$f_\emptyset(dm) - df_\emptyset(m) = 0,$$
$$f_i(dm) + df_i(m) = \{\omega\}\big(\xi'_i f_\emptyset(m) - f_\emptyset(\xi'_i m)\big).$$

Therefore, $f_\emptyset\colon M \to M'$ is a chain map of complexes and induces an $\mathbf{S}^*$-equivariant map in homology. We define $H(f)\colon H(M) \to H(M')$ to be this map $H(f_\emptyset)$. If $h\colon M \to M'$ is a homotopy between two chain maps f and f', then one verifies similarly that $h_\emptyset$ is a homotopy between $f_\emptyset$ and $f'_\emptyset$. Hence $H(f) = H(f')$ in this case. Having defined the homology of weak $\mathbf{S}^*$-modules, we may talk about **c-equivalences** between them.

We extend the Koszul functor $\mathbf{h}$ to this new category $\mathbf{S}^*$-$\mathcal{M}\boldsymbol{od}$ in the obvious way: We assign to weak $\mathbf{S}^*$-modules and morphisms the corresponding $\mathbf{\Lambda}$-objects. This is somehow a “no-operation,” but the essential difference is the homology associated with M and $\mathbf{h}M$: It is $H(M)$ the former case and $H(\mathbf{\Lambda}^* \tilde{\otimes} M)$ in the latter.

The relevance of weak $\mathbf{S}^*$-modules for topological applications stems from the fact that the equations (1.25) are (non-trivially) soluble in the case of cochain algebras of topological spaces. As explained in the next chapter, any such algebra A is associative and comes with an additional product $*$ satisfying for all a, b, $c \in A$ the identities

(1.29a) $$d(a * b) = ab - \{a,b\}\, ba - da * b - \{a\}\, a * db,$$

(1.29b) $$ab * c = \{a\}\, a(b * c) + \{b,c\}\, (a * c)b.$$

(The “$*$” is supposed to bind weaker than the ordinary product and the differential.) We call any map of complexes $*\colon A \otimes A \to A$ of degree 1 satisfying these two equations a **Steenrod–Hirsch product** for A. The first line says that the Steenrod–Hirsch product is a homotopy between the product and the product with commuted factors. The second line is called the **Hirsch formula**. Note its asymmetry: There is no corresponding formula for $a * bc$. We remark in passing that the signs in the Hirsch formula are as predicted by the sign rule if we write the product as map $a * b = f(a \otimes b)$.

The following proposition justifies our interest in weak modules. It is of central importance to the present work in that it lays the ground on which almost all further developments are built upon.

PROPOSITION 1.7.1. *Assume that* $\mathbf{S}^*$ *is the homology of an associative algebra* A *with Steenrod–Hirsch product. Then any choice of representatives* $\xi'_i \in A$ *of the* ξ_i, $i \in [r]$, *canonically defines a twisting cochain* t *with* $t(\xi_i) = -\xi'_i$*; the higher* $\xi'_\pi \in A$ *are recursively defined by the formula*

(1.30) $$\xi'_\pi = -\xi'_{\pi'} * \xi'_{\pi^+},$$

where π^+ *is the maximum of* π *and* $\pi' \neq \emptyset$ *the other elements.*

Moreover, this defines a functor from the category A-$\boldsymbol{Mod}$ *to* $\mathbf{S}^*$-$\mathcal{M}\boldsymbol{od}$ *such that the* $\mathbf{S}^*$*-module structure on the homology of an* A*-module, considered as weak* $\mathbf{S}^*$*-module, is the original* $H(A)$*-module structure.*

The resulting differential d_t on $\mathbf{\Lambda}^* \tilde{\otimes} A$ appears already in [**GM**, Example 2.2], but it seems that its equivariance with respect to $\mathbf{\Lambda}$ has not been used before.

Proof. The proof that (1.30) defines a twisting cochain is rather straightforward. It is nevertheless given here due to the importance of the result.

We proceed by induction on $|\pi|$. For $\pi = \{i\}$ we need that ξ'_i be a cycle, which is true by hypothesis. Using properties (1.29) of the Steenrod–Hirsch product, we calculate for larger π

$$\begin{aligned}
-d\xi'_\pi &= d(\xi'_{\pi'} * \xi'_{\pi^+}) \\
&= \xi'_{\pi'}\xi'_{\pi^+} - \{\xi'_{\pi'}, \xi'_{\pi^+}\}\,\xi'_{\pi^+}\xi'_{\pi'} - d\xi'_{\pi'} * \xi'_{\pi^+} - \{\xi'_{\pi'}\}\,\xi'_{\pi'} * d\xi'_{\pi^+} \\
&= \xi'_{\pi'}\xi'_{\pi^+} - \xi'_{\pi^+}\xi'_{\pi'} + \sum_{(\mu,\nu)\vDash\pi} \{\nu\}\{(\mu,\nu)\}\,\xi'_\nu\xi'_\mu * \xi'_{\pi^+} \\
&= \xi'_{\pi'}\xi'_{\pi^+} - \xi'_{\pi^+}\xi'_{\pi'} \\
&\qquad + \sum_{(\mu,\nu)\vDash\pi'} \{\nu\}\{(\mu,\nu)\}\Big(\{\xi'_\nu\}\,\xi'_\nu(\xi'_\mu * \xi'_{\pi^+}) + \{\xi'_\mu, \xi'_{\pi^+}\}(\xi'_\nu * \xi'_{\pi^+})\xi'_\mu\Big)
\end{aligned}$$

and by substituting μ and ν for $\mu \cup \pi^+$ and $\nu \cup \pi^+$, respectively,

$$\begin{aligned}
&= \{\pi'\}\{(\pi^+, \pi')\}\,\xi'_{\pi'}\xi'_{\pi^+} + \sum_{\substack{(\mu,\nu)\vDash\pi \\ \{\pi^+\}\subsetneq\mu}} \{\nu\}\{(\mu,\nu)\}\,\xi'_\nu\xi'_\mu \\
&\qquad - \{(\pi', \pi^+)\}\,\xi'_{\pi^+}\xi'_{\pi'} + \sum_{\substack{(\mu,\nu)\vDash\pi \\ \{\pi^+\}\subsetneq\nu}} \{\nu\}\{(\mu,\nu)\}\,\xi'_\nu\xi'_\mu \\
&= \sum_{(\mu,\nu)\vDash\pi} \{\nu\}\{(\mu,\nu)\}\,\xi'_\nu\xi'_\mu,
\end{aligned}$$

as claimed.

The passage from A-modules to weak $\mathbf{S}^*$-modules is functorial: If $f\colon M \to M'$ is a chain map of A-modules, then

$$1 \otimes f\colon \mathbf{\Lambda}^* \mathbin{\tilde\otimes} M \to \mathbf{\Lambda}^* \mathbin{\tilde\otimes} M', \qquad \alpha \otimes m \mapsto \{f, \alpha\}\,\alpha \otimes f(m)$$

is a chain chain, and analogously for homotopies. Finally, the assertion about homology is clearly true, since this is how multiplication in $H(M)$ by elements of $H(A) = \mathbf{S}^*$ is defined. □

Let A' be another associative algebra with Steenrod–Hirsch product such that $H(A') = (\mathbf{S}')^*$ is a symmetric algebra corresponding to a graded R-module P' with basis $(y_1, \dots, y_s)$ and dual basis $(\chi_1, \dots, \chi_s)$ as in Section 1.6. Furthermore, let $\phi\colon A \to A'$ be a structure-preserving map such that $H(\phi) = h^*\colon \mathbf{S}^* \to (\mathbf{S}')^*$ is monotone in the following sense: Each ξ_i is mapped either to zero or to one of the χ_j's such that the relative order of the surviving basis elements is not changed. In the second case $\phi(\xi'_i)$ is a representative χ'_i of $\chi_i = h^*(\xi_i)$. Picking representatives of the other basis elements determines via Proposition 1.7.1 a twisting cochain t' for A'. By inspection of formula (1.30) one sees that under these assumptions any ϕ-equivariant chain map $M \to M'$ induces a chain map $\mathbf{h}_P M \to \mathbf{h}_{P'} M'$, contravariant with respect to the map $h\colon \mathbf{\Lambda}' \to \mathbf{\Lambda}$ corresponding to $H(\phi)$. The same applies to ϕ-equivariant homotopies. Taking $A = R$, i. e., $P' = 0$, we may in particular conclude that the assignment (1.18b),

$$M \hookrightarrow \mathbf{th}M, \qquad m \mapsto 1 \otimes 1 \otimes m \tag{1.31}$$

is still a chain map in this new context. We will take a closer look at this map in Section 1.9.

8. Multiplicativity

In this section we define algebra structures on $\mathbf{t}N$ and $\mathbf{h}M$, focusing on the cases relevant to our topological applications.

PROPOSITION 1.8.1.

1. *Let N be a $\mathbf{\Lambda}$-algebra. Then $\mathbf{t}N$ is canonically an $\mathbf{S}^*$-algebra with product*

$$(\sigma_1 \otimes n_1)(\sigma_2 \otimes n_2) = \sigma_1\sigma_2 \otimes n_1 n_2. \tag{1.32}$$

This product is associative if that of N is.

2. *If N is a right $\mathbf{\Lambda}$-coalgebra, then the map $\mathbf{t}N \to (N \otimes_{\mathbf{\Lambda}} K)^*$ given in Lemma 1.5.1 is an isomorphism of $\mathbf{S}^*$-algebras.*

PROOF. It is obvious that the product (1.32) is $\mathbf{S}^*$-bilinear. That it defines a chain map $\mathbf{t}N \otimes \mathbf{t}N \to \mathbf{t}N$ hinges on the fact that the generators $x_i \in \mathbf{\Lambda}$ act as antiderivations of the algebra N:

$$\begin{aligned}
d\big((\sigma_1 \otimes n_1)(\sigma_2 \otimes n_2)\big) &= d(\sigma_1\sigma_2 \otimes n_1 n_2)\\
&= -\sum_i \xi_i\sigma_1\sigma_2 \otimes x_i \cdot (n_1 n_2) + \sigma_1\sigma_2 \otimes d(n_1 n_2)\\
&= -\sum_i (\xi_i\sigma_1)\sigma_2 \otimes (x_i \cdot n_1)n_2 + \sigma_1\sigma_2 \otimes (dn_1)n_2\\
&\qquad - \{n_1\} \sum_i \sigma_1(\xi_i\sigma_2) \otimes n_1(x_i \cdot n_2) + \{n_1\}\, \sigma_1\sigma_2 \otimes n_1(dn_2)\\
&= d(\sigma_1 \otimes n_1)(\sigma_2 \otimes n_2) + \{\sigma_1 \otimes n_1\}(\sigma_1 \otimes n_1)d(\sigma_2 \otimes n_2).
\end{aligned}$$

The complex $N \otimes_{\mathbf{\Lambda}} K$ is a coalgebra by Lemma 1.3.1. Since the product on K^* is $\mathbf{S}^*$-bilinear, so are those of $(N \otimes K)^*$ and $(N \otimes_{\mathbf{\Lambda}} K)^*$. The multiplicativity of the isomorphism from Lemma 1.5.1 also follows directly from the definition of the comultiplication on $N \otimes_{\mathbf{\Lambda}} K$. □

One could give an analogous definition for an algebra M that is an $\mathbf{S}^*$-module via a map of algebras $\mathbf{S}^* \to M$ provided that the image of this map lies in the centre of M (cf. [**GHV**, Secs. 2.5 & 3.1]). This will not hold in our applications, so we proceed differently.

Let A be an associative algebra with Steenrod–Hirsch product. Assume $H(A) = \mathbf{S}^*$ and choose a twisting cochain t as in Proposition 1.7.1. Let $A \to M$ be a map of algebras, which we will suppress in our notation. To ensure that it gives M a left A-module structure, we assume the following partial associativity:

$$\forall a_1, a_2 \in A, m \in M \quad (a_1 a_2)m = a_1(a_2 m). \tag{1.33}$$

Assume further that A acts on M via another product $*\colon A \otimes M \to M$ compatible with the map $A \to M$ and satisfying the identities (1.29) of a Steenrod–Hirsch product (which both make sense in this new situation), and in addition

$$\forall a \in A \quad a * 1 = 0. \tag{1.34}$$

(We remark in passing that the identity $1 * m = 0$ follows from (1.29b).)

PROPOSITION 1.8.2. *Under the above assumptions* $\mathbf{h}M$ *is a* $\mathbf{\Lambda}$*-algebra with product*

$$(\wedge\otimes\mu_M)(1-1\otimes D'_t\otimes 1)(1\otimes T_{M\mathbf{\Lambda}^*}\otimes 1)\colon \mathbf{h}M\otimes\mathbf{h}M\to\mathbf{h}M, \tag{1.35}$$

where

$$D'_t=(1\otimes *)(1\otimes t\otimes 1)(\Delta\otimes 1)\colon \mathbf{h}M\to\mathbf{h}M,$$

and $\wedge$ *and* μ_M *denote the wedge product on* $\mathbf{\Lambda}^*$ *and the product of* M*, respectively. Explicitly,*

$$(\alpha\otimes m)(\alpha'\otimes m')= \{m,\alpha'\}\Big(\alpha\wedge\alpha'\otimes mm'-\sum_{\pi\neq\emptyset}\{\pi\}\{\pi,\alpha'\}\,\alpha\wedge(x_\pi\cdot\alpha')\otimes(\xi'_\pi * m)m'\Big). \tag{1.36}$$

Note that this product is not associative in general: For $r=1$ for instance, we have

$$(1\otimes m)\big((\xi\otimes 1)(\xi\otimes 1)\big)=0,$$

but

$$\begin{aligned}\big((1\otimes m)(\xi\otimes 1)\big)(\xi\otimes 1)&=\{m\}(\xi\otimes m+1\otimes\xi' * m)(\xi\otimes 1)\\&=-1\otimes\xi' *(\xi' * m)\end{aligned}$$

may not vanish.

PROOF. We first note that the product (1.35) is $\mathbf{\Lambda}$-equivariant because D'_t is so by (1.27) and by the fact that the map $1\otimes *$ does not affect the module structure. (But keep the sign convention in mind.)

The term $D_t(\alpha'\otimes m)$ vanishes for $\alpha'=1$, because $t(1)=0$, and for $m=1$ by (1.34). Hence

$$(\alpha\otimes m)(\alpha'\otimes m')=\{m,\alpha'\}\,\alpha\wedge\alpha'\otimes mm'=\alpha\wedge\alpha'\otimes mm' \tag{1.37}$$

in these cases, which shows that the new product extends the canonical $\mathbf{\Lambda}^*$-M-bimodule structure, where both structures come from left and right multiplication by elements of $\mathbf{\Lambda}^*$ and M, respectively.

The proof that formula (1.36) agrees with (1.35) can be found in Appendix 5, as well as the verification that the product is a chain map. □

I do not know how this product is related to the one defined in the (Russian) article [**Mi**] (for a more general topological application, cf. the remark following Proposition 2.11.3).

How do these new products fit together? The composition $\mathbf{t}\circ\mathbf{h}$ works fine, but in the other direction we need the additional ingredient $*\colon \mathbf{S}^*\otimes\mathbf{t}N\to\mathbf{t}N$ to repeat the construction. (The algebra $\mathbf{S}^*$ has of course the trivial Steenrod–Hirsch product.) Since $\mathbf{S}^*\otimes 1$ lies in the centre of $\mathbf{t}N$, we can simply set all Steenrod–Hirsch products to zero, as mentioned above. We then have:

PROPOSITION 1.8.3. *Notation being as above, the canonical maps*

$$\mathbf{ht}N\to N \qquad\text{and}\qquad M\to\mathbf{th}M$$

given by (1.18) *and* (1.31) *are multiplicative.*

9. Further properties of the Koszul functors

In this section we assume that all basis elements of P have *positive* degree, so that $\mathbf{\Lambda}^*$ and $\mathbf{S}^*$ are negatively graded. Furthermore, all $\mathbf{\Lambda}$-modules and weak $\mathbf{S}^*$-modules are supposed to be bounded from *above.* (Recall that this applies to cochain complexes of topological spaces by the convention stated at the beginning of this chapter.) Note that the image under a Koszul functor of a complex bounded above is again of this type (and with the same bound).

We begin with a partial generalisation of Theorem 1.6.3 (2).

LEMMA 1.9.1. *Let (M, A, t) be a weak $\mathbf{S}^*$-module. Then the map*

$$M \hookrightarrow \mathbf{th}M, \qquad m \mapsto 1 \otimes 1 \otimes m.$$

is a chain map inducing an isomorphism of $\mathbf{S}^*$-modules in homology.

PROOF. We have

$$\mathbf{th}M = \mathbf{S}^* \mathbin{\tilde\otimes} \mathbf{\Lambda}^* \mathbin{\tilde\otimes} M$$

with differential

$$d(\sigma \otimes \xi_\pi \otimes m) = \\ -\sum_{i=1}^{r} \xi_i \sigma \otimes x_i \cdot \xi_\pi \otimes m + \{\pi\} \sum_{\substack{(\mu,\nu)\vdash\pi \\ \mu\neq\emptyset}} \{(\mu,\nu)\}\, \sigma \otimes \xi_\nu \otimes \xi'_\mu m + \{\pi\}\, \sigma \otimes \xi_\pi \otimes dm.$$

This shows that $\iota\colon M \hookrightarrow \mathbf{th}M$ is a chain map. Now filter M by degree, i. e., $F_pM = M_{\le p}$, and the other complex by

$$F_p\mathbf{th}M = \mathbf{S}^* \otimes \mathbf{\Lambda}^* \otimes F_pM.$$

This leads to spectral sequences $E(M)$ and $E(\mathbf{th}M)$ with

$$E^0_{pq}(\mathbf{th}M) = (\mathbf{S}^* \mathbin{\tilde\otimes} \mathbf{\Lambda}^*)_q \otimes M_p,$$

hence

$$E^1(\mathbf{th}M) = R \otimes M$$

reduces to a single row because the bracketed factor in the preceding line is essentially the cohomological Koszul complex. The filtration-preserving map ι thus induces an isomorphism of spectral sequences on the E^1 level. This implies that $H(\iota)$ is an isomorphism of R-modules.

In order to prove that $H(\iota)$ is $\mathbf{S}^*$-equivariant, we have to show that the cycle

$$\xi_i \otimes 1 \otimes m - 1 \otimes 1 \otimes \xi'_i m$$

is a boundary in $\mathbf{th}M$ for any cycle $m \in M$. But this is indeed the case; the chain $1 \otimes \xi_i \otimes m$ does the job. □

Let N be a $\mathbf{\Lambda}$-module. We filter $\mathbf{t}N$ by

$$F_p\mathbf{t}N = \mathbf{S}^*_{\le p} \otimes N, \tag{1.38}$$

the first terms of the associated spectral sequence are

$$E^0_{pq}(\mathbf{t}N) = \mathbf{S}^*_p \otimes N_q,$$

$$E^1_{pq}(\mathbf{t}N) = E^2_{pq}(\mathbf{t}N) = \mathbf{S}^*_p \otimes H_q(N). \tag{1.39}$$

(The identity $E^1 = E^2$ follows from the assumption $|\xi_i| \le -2$ because this implies that the part of the differential coming from the Koszul differential decreases p-degree at least by 2.)

Similarly, given a weak $\mathbf{S}^*$-module M, we filter $\mathbf{h}M$ by

$$F_p\mathbf{h}M = \mathbf{\Lambda}^* \otimes M_{\le p}; \tag{1.40}$$

here we have

$$\begin{aligned} E^0_{pq}(\mathbf{h}M) = E^1_{pq}(\mathbf{h}M) &= \mathbf{\Lambda}^*_q \otimes M_p, \\ E^2_{pq}(\mathbf{h}M) &= \mathbf{\Lambda}^*_q \otimes H_p(M). \end{aligned} \tag{1.41}$$

If N is a $\mathbf{\Lambda}$-algebra or M an algebra with Steenrod–Hirsch product, then the products defined in the preceding section are compatible with the filtrations, so that we get spectral sequences of algebras. (Here we use that we assume the Steenrod–Hirsch product to be of degree 1, and again $|\xi_i| \le -2$.) Note that both spectral sequences lie essentially in the third quadrant, hence are bounded.

Proposition 1.9.2. *The algebraic Koszul functors preserve c-equivalence.*

Proof. We take the proof from [**GKM**, Sec. 9]: Any chain map of weak $\mathbf{S}^*$-modules $f\colon M \to M'$ induces a map $\mathbf{h}f\colon \mathbf{h}M \to \mathbf{h}M'$ compatible with the filtrations defined above and hence a morphism of spectral sequences. (Recall that $\mathbf{h}f$ "is" by definition the map f.) It becomes an isomorphism from the E^2 term on if $H(f) = H(f_\emptyset)$ is an isomorphism. The spectral sequence being bounded, $H(\mathbf{h}f)$ is an isomorphism as well.

The proof for the functor $\mathbf{t}$ is similar; here the isomorphism appears already on the E^1 level. □

Let C be a left $\mathbf{\Lambda}$-module or a weak $\mathbf{S}^*$-module. Inspired by [**GKM**, Sec. 9.2], we call C **split and trivial** if it is split and the module structure of $H(C)$ is trivial, i. e., if $\mathbf{\Lambda}$ resp. $\mathbf{S}^*$ act via the canonical augmentation ε. We call C **split and extended** if it is split and $H(C)$ is an extended $\mathbf{\Lambda}$-module or $\mathbf{S}^*$-module, respectively.

We continue to generalise results of [**GKM**, Sec. 9]:

Proposition 1.9.3. *The algebraic Koszul functors carry split and trivial modules to split and extended ones and vice versa.*

Proof. If the $\mathbf{\Lambda}$-module N is split and trivial, then by the preceding proposition $\mathbf{t}N \sim \mathbf{t}H(N) = \mathbf{S}^* \otimes H(N)$, which is split and extended, because its differential vanishes.

If N is split and extended, say $N \sim H(N) \cong \mathbf{\Lambda}^* \otimes C = \mathbf{h}C$ for some graded R-module C, considered as trivial $\mathbf{S}^*$-module, then $\mathbf{t}N \sim \mathbf{t}H(N) \cong \mathbf{th}C \sim C$ by Theorem 1.6.3 (1), and $H(\mathbf{t}N) \cong H(C) = C$ has trivial module structure.

The proof for weak $\mathbf{S}^*$-modules is identical. □

We assume for the rest of this section that R is a principal ideal domain.

Proposition 1.9.4. *Let N, N' be $\mathbf{\Lambda}$-modules, free as graded R-modules, and let $f\colon N \to N'$ be a c-equivalence. Then $\mathbf{t}f$ is an isomorphism in $\mathbf{S}^*$-**Mod**.*

The analogous statement holds for (strict) $\mathbf{S}^*$-modules.

Proof. We treat the case of $\mathbf{\Lambda}$-modules first, where we can imitate the proof of [$\mathbf{D}'$, Hilfssatz 3.9]: If $f\colon N \to N'$ is a map as above, then $\mathbf{t}f$ is a c-equivalence of free $\mathbf{S}^*$-modules by the preceding proposition. We have to show that it is a homotopy equivalence over $\mathbf{S}^*$. We know that its mapping cone $C = C(\mathbf{t}f)$ is free and acyclic because $H(\mathbf{t}f)$ is an isomorphism. Consider the exact sequence

$$0 \longrightarrow Z(C) \longrightarrow C \longrightarrow B(C) \longrightarrow 0$$

of $\mathbf{S}^*$-modules without differentials. It suffices by a standard argument [**D**, proof of Prop. II.3.6] to prove that $B(C) = Z(C)$ is projective as $\mathbf{S}^*$-module. (Note that the grading is not important because existence of a section to a graded map implies existence of a graded section.) By Hilbert's syzygy theorem [**W**, Thm. 4.3.7], $\mathbf{S}^*$ is of finite global dimension because so is R. Hence the last term in the exact sequence of $\mathbf{S}^*$-modules (no grading, no differentials)

$$0 \longleftarrow B(C) \longleftarrow C \longleftarrow \cdots \longleftarrow C \longleftarrow Z(P) \longleftarrow 0.$$

must be projective if we repeat C sufficiently many times. (This actually implies that $Z(P)$ is free, but we do not need this.)

Now let $f\colon M \to M'$ be a c-equivalence of $\mathbf{S}^*$-modules which are free as graded R-modules. Then $\mathbf{h}f$ is a c-equivalence of $\mathbf{\Lambda}$-modules, free as graded R-modules. By what we have just proven, $\mathbf{th}f$ is an isomorphism, hence $\mathbf{hth}f$ as well. Theorem 1.6.6 finally shows that $\mathbf{h}f$ is an isomorphism, too. □

Proposition 1.9.5. *Let N be a $\mathbf{\Lambda}$-module. If $H(N)$ is extended, then N is split. The analogous statement holds for (strict) $\mathbf{S}^*$-modules.*

Proof. Let N be a $\mathbf{\Lambda}$-module with $H(N) \cong \mathbf{\Lambda} \otimes C$ for an R-module C. Since R is a principal ideal domain, there exists a graded free bounded above resolution

$$0 \longleftarrow C \longleftarrow P^0 \longleftarrow P^1 \longleftarrow 0$$

of C. Tensoring it with $\mathbf{\Lambda}$ gives a graded free resolution of the $\mathbf{\Lambda}$-module $H(N)$ and therefore the (not uniquely determined) $\mathbf{\Lambda}$-equivariant vertical maps in the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathbf{\Lambda} \otimes C & \longleftarrow & \mathbf{\Lambda} \otimes P^0 & \longleftarrow & \mathbf{\Lambda} \otimes P^1 & \longleftarrow & 0 \\
 & & \cong\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longleftarrow & H(N) & \longleftarrow & Z(N) & \xleftarrow{\;d\;} & N. & &
\end{array}$$

This implies that the bounded above total complex $\mathbf{\Lambda} \otimes P$ is c-equivalent to both $H(N)$ and N.

The proof for $\mathbf{S}^*$-modules is identical. □

We remark that the corresponding statement for modules with trivial module structure in homology is false. We will see an example of this in Section 2.2.

Let N be a $\mathbf{\Lambda}$-module. Then the assignment

$$j_N\colon \mathbf{t}N \to N, \qquad \sigma \otimes n \to \varepsilon(\sigma)\, n, \tag{1.42}$$

is a map of complexes. It is even a map of algebras if N is a $\mathbf{\Lambda}$-algebra.

Proposition 1.9.6. *The following are equivalent for every* $\mathbf{\Lambda}$*-module* N*:*

1. N *is split and trivial.*
2. *There is a section of* R*-modules to the map* $H(j_N)\colon H(\mathbf{t}N) \to H(N)$*.*
3. *The spectral sequence* (1.39) *degenerates at the* E^2 *level and there is no composition problem, i. e.,*

$$H(\mathbf{t}N) \cong \mathbf{S}^* \otimes H(N)$$

as $\mathbf{S}^*$*-modules.*

If these conditions hold, then

$$H(N) \cong R \underset{\mathbf{S}^*}{\otimes} H(\mathbf{t}N) = H(\mathbf{t}N)/\mathbf{S}^{>0}H(\mathbf{t}N)$$

as complexes, and as algebras if N *is a* $\mathbf{\Lambda}$*-algebra.*

Proof. $1 \Rightarrow 2$ is trivial because in this case the map j_N induces the map $j_{H(N)}$ in homology. Since $\mathbf{S}^{>0}H(\mathbf{t}N)$ is the kernel of $H(j_N) = j_{H(N)}$, the conclusion follows.

$2 \Rightarrow 3$ is an algebraic version of the Leray–Hirsch theorem: If $s\colon H(N) \to H(\mathbf{t}N)$ is such a section, then the $\mathbf{S}^*$-equivariant map

$$\mathbf{S}^* \otimes H(N) \to H(\mathbf{t}N), \qquad \sigma \otimes [n] \mapsto \sigma s([n])$$

is an isomorphism, cf. [**L**, VI.8.2].

$3 \Rightarrow 1$ finally is a consequence of Propositions 1.9.5 and 1.9.3. □

CHAPTER 2

Simplicial Koszul duality

1. Simplicial sets

In this and the following sections we review some elements of the theory of simplicial sets. We refer to [**M**], [**L**] and [**ML**, Ch. VIII] for comprehensive expositions.

A **simplicial set** X is a family of sets X_n, indexed by the natural numbers, together with **face maps** $\partial_i\colon X_n \to X_{n-1}$, $0 \le i \le n$, for all positive n, and **degeneracy maps** $s_i\colon X_n \to X_{n+1}$, $0 \le i \le n$, for all n, satisfying the relations

$$
\begin{aligned}
\partial_i\partial_j &= \partial_{j-1}\partial_i && (i<j),\\
s_is_j &= s_{j+1}s_i && (i\le j),\\
\partial_is_j &= s_{j-1}\partial_i && (i<j),\\
\partial_is_i &= \partial_{i+1}s_i = \mathrm{id}, &&\\
\partial_is_j &= s_j\partial_{i-1} && (i>j+1).
\end{aligned}
\tag{2.1}
$$

The elements of X_n are called **simplices** of degree n or n-simplices, those of X_0 also **vertices**. A **simplicial map** $f\colon X \to Y$ is a collection of maps $f_n\colon X_n \to Y_n$ commuting with the face and degeneracy maps. We postpone the introduction of simplicial homotopies until we have the notion of simplicial products at our disposal.

Examples of simplicial sets are the simplicial models of standard simplices: The k-simplices of the simplicial n-model $\Delta^{(n)}$ are the weakly increasing functions $\{0,\dots,k\} \to \{0,\dots,n\}$. The i-th face operator drops the i-th value, and the i-th degeneracy operator duplicates it. We write $*$ for the (up to isomorphism unique) "one-point space" $\Delta^{(0)}$ having exactly one simplex in each degree.

A simplicial set X is called **connected** if for any two vertices x, $x' \in X_0$ there is a sequence $x^0, \dots, x^k$ of 1-simplices such that

$$\partial_0x^0 = x, \qquad \partial_1x^i = \partial_0x^{i+1} \quad (0 \le i \le k-1), \qquad \text{and} \qquad \partial_1x^k = x'.$$

To simplify notation later on, we finally introduce the abbreviations

$$\partial_i^j = \partial_i \circ \partial_{i+1} \circ \dots \circ \partial_j, \quad \partial_i^{i-1} = \mathrm{id}, \qquad \text{and} \qquad s_\mu = s_{i_q} \circ \dots \circ s_{i_1}, \quad s_\emptyset = \mathrm{id}$$

for $i \le j$ and any set $\mu = \{i_1 < i_2 < \dots < i_q\} \subset \mathbf{N}$.

The **chain functor** C from simplicial sets to complexes is defined as follows: For a simplicial set X the n-th degree $C_n(X)$ of $C(X)$ is the free R-module with basis X_n, and the differential of an n-simplex x is

$$dx = \sum_{i=0}^{n}(-1)^i\partial_ix.$$

With a simplicial map $f\colon X \to Y$ one associates its extensions $f_* = C(f)$ to $C_n(X) \to C_n(Y)$ for all $n \in \mathbf{N}$. The **cochain functor** C^* is the composition of C

with the functor taking a complex to its dual. Here we write f^* for $C^*(f)$. The elements of $C(X)$ are called **chains**, those of $C^*(X)$ **cochains**. We abbreviate the homology $H(C(X))$ and the cohomology $H(C^*(X))$ of X by $H(X)$ and $H^*(X)$, respectively. A **c-equivalence** of simplicial sets is a sequence

$$X = X_{(0)} \longrightarrow X_{(1)} \longleftarrow X_{(2)} \longrightarrow \cdots \longleftarrow X_{(k-1)} \longrightarrow X_{(k)} = Y,$$

of simplicial maps, each inducing a c-equivalence of cochain complexes, i. e., an isomorphism in cohomology.

Starting from a topological space X, one obtains a simplicial set $\mathcal{S}X$ by letting $\mathcal{S}_nX$ be the set of all singular n-simplices in that space. For a singular n-simplex $x\colon \Delta_n \to X$ we define $\partial_i x = x \circ s_i^*$, where $s_i^*\colon \Delta_{n-1} \to \Delta_n$ is the affine map sending the vertices of Δ_{n-1} in ascending order to those of Δ_n, omitting the i-th one. Similarly, we set $s_i x = x \circ \partial_i^*$, where $\partial_i^*\colon \Delta_{n+1} \to \Delta_n$ is the affine projection mapping vertices to vertices, maintaining their order and identifying the i-th one with its successor. Associating in the obvious way a simplicial map with any continuous map of topological spaces, one arrives thus at a functor $\mathcal{S}$ from the category of topological spaces to that of simplicial sets. Composed with the functor C described above, this yields nothing but the chain complex of a topological space with coefficients in R.

A simplex x in a simplicial set X is called **degenerate** if it is of the form $s_i x'$ for some simplex x' and some i. More precisely, it is called q-fold degenerate if it is of the form $s_\mu x'$ with $|\mu| = q$. It follows from the commutation relations (2.1) that the degenerate simplices span a subcomplex $C_D(X)$ of $C(X)$. The quotient $C_N(X) = C(X)/C_D(X)$ is called the **normalised chain complex** of X, its elements normalised chains. Note that $C_N(X)$ is again a free R-module in each degree. Intuitively, degenerate simplices should not contribute to the homology of X because they factor through lower-dimensional ones. This is indeed the case: The projection $C(X) \to C_N(X)$ is a homotopy equivalence commuting with maps of complexes induced by simplicial maps [**ML**, Sec. VIII.6]. Note that the normalised chain complex of a one-point space is canonically isomorphic to R.

Since we will not leave the simplicial setting any more (apart from Section 2.9) and hardly use non-normalised chains, we streamline our notation and also our terminology:

*From now up to the end, the term **space** means simplicial set,*
*and a **map of spaces** a simplicial one.*
Moreover, the letter C denotes the normalised chain functor,
and C^* the normalised cochain functor. Similarly, all chains and
cochains will be normalised ones, unless stated otherwise.

2. Products

The **product** $X \times Y$ of two spaces X and Y is the space with $X_n \times Y_n$ as the set of n-simplices, and componentwise face and degeneracy maps, i. e.,

$$\partial_i(x, y) = (\partial_i x, \partial_i y) \quad \text{and} \quad s_i(x, y) = (s_i x, s_i y).$$

This again mimics the topological situation, so that the simplicial set associated with a product of topological spaces is the product of the simplicial sets corresponding to the factors.

The following so-called Eilenberg–Zilber maps relate the chain complex of a product of spaces to the tensor product of the chain complexes of the factors:

The Mac Lane or **shuffle map** $\nabla = \nabla_{XY} \colon C(X) \otimes C(Y) \to C(X \times Y)$ carries the non-normalised chain $x \otimes y$, $x \in X_m$, $y \in Y_n$, to

$$\nabla(x \otimes y) = \sum_{(\mu,\nu) \vdash (m,n)} \{(\mu,\nu)\}(s_\nu x, s_\mu y),$$

where the sum is meant to extend over all (m,n)-shuffles of the set $\{0,\dots,m{+}n{-}1\}$, i. e., all partitions $\mu \dot{\cup} \nu$ of this set with $|\mu| = m$ and $|\nu| = n$. (Note that $s_\nu x$ and $s_\mu y$ are always well-defined.)

The **Alexander–Whitney map** $AW = AW_{XY} \colon C(X \times Y) \to C(X) \otimes C(Y)$ is defined on the non-normalised complexes by

$$AW(x,y) = \sum_{i=0}^{n} \partial_{i+1}^{n} x \otimes \partial_0^{i-1} y$$

for $(x,y) \in (X \times Y)_n$.

The shuffle map as well as the Alexander–Whitney map are chain maps and pass to the normalised complexes, where they are homotopy inverse to each other according to the Eilenberg–Zilber theorem. More precisely, there is a chain homotopy $H = H_{XY} \colon C(X \times Y) \to C(X \times Y)$ such that

$$AW \nabla = 1, \tag{2.2a}$$

$$\nabla AW - 1 = d \circ H + H \circ d, \tag{2.2b}$$

$$AW H = 0, \tag{2.2c}$$

$$H \nabla = 0, \tag{2.2d}$$

$$HH = 0, \tag{2.2e}$$

see [**EML**, Thm. 2.1a]. An explicit non-recursive formula for H has been given by Rubio and Morace, cf. [**GDR**, Sec. 2]. All three maps are natural with respect to pairs of simplicial maps, simply because they are defined in terms of face and degeneracy maps.

We can now return to simplicial homotopies: It is obvious from the definition that the "simplicial interval" $\Delta^{(1)}$ has exactly two vertices (0) and (1) and exactly one non-degenerate 1-simplex (01), which satisfies $d(01) = (1) - (0)$. Now a **simplicial homotopy** from one simplicial map $f \colon X \to Y$ to another map f' is a simplicial map $h \colon \Delta^{(1)} \times X \to Y$ that restricts on the subspace $(0) \times X \approx X$ to f and on $(1) \times X \approx X$ to f'.

To see the relation with topology, note that $\Delta^{(1)}$ is isomorphic to a subspace of the space (i. e., simplicial set) $\mathcal{S}\Delta_1$ associated with the topological 1-simplex, namely the one generated by the identity mapping of Δ_1. By 'subspace' we mean the smallest subset of $\mathcal{S}\Delta_1$ containing this simplex and closed under face and degeneracy maps. Given a homotopy $h \colon \Delta_1 \times X \to Y$ from a map f of topological spaces to another map f', one restricts the induced simplicial map from $\mathcal{S}(\Delta_1 \times X) = \mathcal{S}\Delta_1 \times \mathcal{S}X$ to $\mathcal{S}Y$ to the subspace $\Delta^{(1)} \times \mathcal{S}X$ and arrives thus at a simplicial homotopy.

As in topology, a simplicial homotopy $h \colon \Delta^{(1)} \times X \to Y$ induces the chain homotopy $h_\diamond \colon C(X) \to C(Y)$, $c \mapsto h_\diamond(c) = h_* \nabla\big((01) \otimes c\big)$.

The shuffle and the Alexander–Whitney map are associative, i. e., the diagrams

$$\begin{array}{ccc} C(X)\otimes C(Y)\otimes C(Z) & \xrightarrow{\nabla_{XY}\otimes 1} & C(X\times Y)\otimes C(Z) \\ {\scriptstyle 1\otimes\nabla_{YZ}}\big\downarrow & & \big\downarrow{\scriptstyle \nabla_{X\times Y,Z}} \\ C(X)\otimes C(Y\times Z) & \xrightarrow[\nabla_{X,Y\times Z}]{} & C(X\times Y\times Z) \end{array}$$

and

$$\begin{array}{ccc} C(X)\otimes C(Y)\otimes C(Z) & \xleftarrow{AW_{XY}\otimes 1} & C(X\times Y)\otimes C(Z) \\ {\scriptstyle 1\otimes AW_{YZ}}\big\uparrow & & \big\uparrow{\scriptstyle AW_{X\times Y,Z}} \\ C(X)\otimes C(Y\times Z) & \xleftarrow[AW_{X,Y\times Z}]{} & C(X\times Y\times Z) \end{array}$$

commute for all spaces X, Y, and Z; the shuffle map is also commutative in the sense that

$$\begin{array}{ccc} C(X)\otimes C(Y) & \xrightarrow{\nabla_{XY}} & C(X\times Y) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \tau_*} \\ C(Y)\otimes C(X) & \xrightarrow{\nabla_{YX}} & C(Y\times X) \end{array}$$

commutes, where T denotes the transposition of factors (1.4) and

$$\tau(x,y) = \tau_{XY}(x,y) = (y,x).$$

The Alexander–Whitney map is *not* commutative in general. This will be the subject of the following section.

For each space X there are canonical maps $X \to X \times X$ (called the **diagonal** of X) and $X \to *$. Applying to them the chain functor, we get a comultiplication $AW\Delta_*\colon C(X) \to C(X)\otimes C(X)$ and an augmentation $C(X) \to C(*) = R$ turning the chain complex of a space into a coassociative coalgebra.

Proposition 2.2.1. *For all spaces X, Y, Z, and W the following diagram commutes:*

$$\begin{array}{ccc} C(X\times Y)\otimes C(Z\times W) & \xrightarrow{\nabla_{X\times Y,Z\times W}} & C(X\times Y\times Z\times W) \\ {\scriptstyle AW_{X\times Y}\otimes AW_{Z\times W}}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id},\tau,\mathrm{id})_*} \\ C(X)\otimes C(Y)\otimes C(Z)\otimes C(W) & & C(X\times Z\times Y\times W) \\ {\scriptstyle 1\otimes T\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle AW_{X\times Z,Y\times W}} \\ C(X)\otimes C(Z)\otimes C(Y)\otimes C(W) & \xrightarrow{\nabla_{XZ}\otimes\nabla_{YW}} & C(X\times Z)\otimes C(Y\times W). \end{array}$$

PROOF. This is done by direct computation, as shown in Appendix 7. □

COROLLARY 2.2.2. *The shuffle map is a map of coalgebras.*

PROOF. Precede the diagram from Proposition 2.2.1 by

$$\begin{array}{ccc} C(X)\otimes C(Y) & \xrightarrow{\nabla} & C(X\times Y) \\ {\scriptstyle \Delta_*\otimes\Delta_*}\downarrow & & \downarrow{\scriptstyle (\Delta,\Delta)_*} \\ C(X\times X)\otimes C(Y\times Y) & \xrightarrow{\nabla} & C(X\times X\times Y\times Y). \end{array}$$ □

Alternatively, one can derive Proposition 2.2.1 from this result: Set $X = X \times Y$ and $Y = Z \times W$ and map everything to $C(X \times Z) \otimes C(Y \times W)$. The original proof of Corollary 2.2.2 in [**EM**, §17] is essentially identical to the one we have given for Proposition 2.2.1. A very elegant one appears in [**P′**].

COROLLARY 2.2.3. *For all spaces X, Y, and Z the diagrams*

$$\begin{array}{ccc} C(X\times Y)\otimes C(Z) & \xrightarrow{\nabla_{X\times Y,Z}} & C(X\times Y\times Z) \\ {\scriptstyle AW_{XY}\otimes 1}\downarrow & & \downarrow{\scriptstyle AW_{X,Y\times Z}} \\ C(X)\otimes C(Y)\otimes C(Z) & \xrightarrow[1\otimes\nabla_{YZ}]{} & C(X)\otimes C(Y\times Z) \end{array}$$

and

$$\begin{array}{ccc} C(X)\otimes C(Y\times Z) & \xrightarrow{\nabla_{X,Y\times Z}} & C(X\times Y\times Z) \\ {\scriptstyle 1\otimes AW_{YZ}}\downarrow & & \downarrow{\scriptstyle AW_{X\times Y,Z}} \\ C(X)\otimes C(Y)\otimes C(Z) & \xrightarrow[\nabla_{XY}\otimes 1]{} & C(X\times Y)\otimes C(Z) \end{array}$$

commute.

The first diagram can be found in [**Sh**, § II.4].

PROOF. Set $Z = *$ and $Y = *$ in Proposition 2.2.1, respectively. □

Note that by choosing $Y = *$ in Corollary 2.2.3 we get equation (2.2a). Of course, this is not the simplest proof of that identity. A very short one is due to [**P**].

With the help of the Eilenberg–Zilber maps one can introduce a whole lot of products, cf. [**D**, Ch. VII]: The **homological cross product**

$$\times\colon C(X)\otimes C(Y)\to C(X\times Y)$$

is just the shuffle map. The **cohomological cross product**

$$\times\colon C^*(Y)\otimes C^*(X)\to C^*(X\times Y)$$

is the composition of the canonical map $C^*(Y)\otimes C^*(X) \to (C(X)\otimes C(Y))^*$ defined by equation (1.6) with the dual of the Alexander–Whitney map. The **cup product**

$$\cup\colon C^*(X)\otimes C^*(X) \to C^*(X)$$

is the cross product, followed by the chain map $\Delta_*\colon C^*(X\times X)\to C^*(X)$. This endows $C^*(X)$ with the canonical algebra structure dual to the coalgebra structure on $C(X)$. Since the latter is coassociative, $C^*(X)$ is associative. The **cap product**

$$\cap\colon C(X)\otimes C^*(X) \to C(X)$$

is the composition of the chain map

$$AW\Delta_*\otimes 1\colon C(X)\otimes C^*(X) \to C(X)\otimes C(X)\otimes C^*(X)$$

and the partial evaluation

$$C(X)\otimes C(X)\otimes C^*(X) \to C(X), \qquad b\otimes c\otimes\gamma \mapsto \langle c,\gamma\rangle b.$$

The various products are related by the formulas

$$c\cap(\gamma_1\cup\gamma_2) = (c\cap\gamma_1)\cap\gamma_2, \tag{2.3a}$$

$$\langle c,\gamma_1\cup\gamma_2\rangle = \langle c\cap\gamma_1,\gamma_2\rangle, \tag{2.3b}$$

$$(a\times b)\cap(\beta\times\alpha) = \{b\cap\beta,\alpha\}\,(a\cap\alpha)\times(b\cap\beta), \tag{2.3c}$$

$$\langle a\times b,\beta\times\alpha\rangle = \langle a,\alpha\rangle\langle b,\beta\rangle. \tag{2.3d}$$

The first two identities exhibit $C(X)$ as right $C^*(X)$-module with dual operation the cup product.

3. More products

We now come back to the lack of commutativity of the Alexander–Whitney map and study what structure remains. To do so, we have to introduce yet another pair of products, which are defined with the help of what I call the **Steenrod map** $ST = ST_{XY}\colon C(X\times Y)\to C(X)\otimes C(Y)$. It is not as fundamental as the previous maps, but defined as the composition of the "commuted Alexander–Whitney map"

$$\widetilde{AW}_{XY} = T_{C(Y),C(X)}AW_{YX}\tau_{XY*}\colon C(X\times Y)\to C(X)\otimes C(Y)$$

and the chain homotopy H from the Eilenberg–Zilber theorem:

$$ST_{XY} = \widetilde{AW}_{XY}H_{XY}.$$

Using the explicit description for H alluded to above, one can verify that the Steenrod map carries the (non-normalised) simplex $(x,y)\in(X\times Y)_n$ to

$$ST(x,y) = -\sum_{0\le i<j\le n}(-1)^{(i+1)(j+1)}\,\partial_0^{i-1}\partial_{j+1}^{n}x\otimes\partial_{i+1}^{j-1}y \tag{2.4}$$

of degree $n+1$, see [**GDR**, Thm. 3.1]. (And this formula is all we need.) The Steenrod map is of course natural with respect to pairs of maps of spaces.

Lemma 2.3.1. *Let X, Y, and Z be spaces.*

1. *The Steenrod map is a chain homotopy from the commuted to the ordinary Alexander–Whitney map, i. e.,*

$$d\circ ST_{XY} + ST_{XY}\circ d = AW_{XY} - \widetilde{AW}_{XY}.$$

2. *The composition $ST_{XY}\nabla_{XY}$ vanishes.*

3. *The following identity holds:*

$$(1 \otimes AW_{YZ}) ST_{X,Y\times Z} = (ST_{XY} \otimes 1) AW_{X\times Y,Z} + (T_{C(Y),C(X)} \otimes 1)(1 \otimes ST_{XZ}) AW_{Y,X\times Z}(\tau_{XY}, \mathrm{id}_Z)_*.$$

4. *The following diagram commutes:*

$$\begin{array}{ccc} C(X) \otimes C(Y \times Z) & \xrightarrow{\nabla_{X,Y\times Z}} & C(X \times Y \times Z) \\ \downarrow{\scriptstyle 1 \otimes ST_{YZ}} & & \downarrow{\scriptstyle ST_{X\times Y,Z}} \\ C(X) \otimes C(Y) \otimes C(Z) & \xrightarrow[\nabla_{XY} \otimes 1]{} & C(X \times Y) \otimes C(Z). \end{array}$$

5. *For all $c \in C(X \times Y)$ and $z \in C(Z)$ with $|z| \le 1$ one has*

$$ST_{X,Y\times Z} \nabla_{X\times Y,Z}(c \otimes z) = (1 \otimes \nabla_{YZ})(ST_{XY} \otimes 1)(c \otimes z).$$

Parts 4 and 5 are analogues of Corollary 2.2.3 for the Steenrod map. But this correspondence is only partial because part 5 above is *false* for general z.

PROOF. We will use both descriptions of the Steenrod map, although we have not proven their equivalence. Alternatively, one could stick to the explicit formula (2.4), but this would increase further the number of long calculations in the appendix.

1. Multiplying equation (2.2b) from the left by $\widetilde{AW}$ and using equation (2.2a) and the shuffle map's commutativity, one gets

$$\begin{aligned} d \circ ST_{XY} + ST_{XY} \circ d &= TAW\tau_* \nabla AW - \widetilde{AW} = TAW \nabla TAW - \widetilde{AW} \\ &= TTAW - \widetilde{AW} = AW - \widetilde{AW}. \end{aligned}$$

2. This follows from equation (2.2d).

The remaining claims are verified by direct calculations, see Appendix 8. □

The **cross$_1$ product**

$$\times_1 \colon C^*(Y) \otimes C^*(X) \to C^*(X \times Y)$$

and **cup$_1$ product**

$$\cup_1 \colon C^*(X) \otimes C^*(X) \to C^*(X)$$

are derived from the Steenrod map like cross and cup product from the Alexander–Whitney map, i. e.,

$$\alpha \times_1 \beta = ST^* \iota(\alpha \otimes \beta) \qquad \text{and} \qquad \alpha \cup_1 \beta = \Delta^*(\alpha \times_1 \beta).$$

LEMMA 2.3.2.

1. *A cross$_1$ product or cup$_1$ product vanishes if one factor is of degree zero.*
2. *The cup$_1$ product is a Steenrod–Hirsch product in the sense of Section 1.7.*
3. *The cross$_1$ product vanishes on the image of the shuffle map for all pairs of spaces.*

The cup$_1$ product has been introduced by Steenrod, as well as further products of higher degree. The important formula (1.29b) for the cup$_1$ product is due to G. Hirsch.

PROOF. The first assertion follows directly from the explicit formula (2.4), and the others from the previous lemma by dualisation (when checking this, one has to be careful about signs). □

4. Groups and group actions

A simplicial group G is a group object in the category of simplicial sets, i. e., a space G together with maps $* = 1 \to G$, $G \times G \to G$ and $G \to G$ satisfying the usual identities between unit, multiplication and inversion. This means that all sets G_n are groups and all face and degeneracy maps group homomorphisms. When we are careful about notation, we write 1_n for the unit element of G_n. For simplicity we only consider connected simplicial groups, to which we refer by the term "**group**" from now on. A **map of groups** is a simplicial map between two groups that is a group homomorphism in each degree.

A left (resp. right) G-action on a space X is a simplicial map $\alpha\colon G \times X \to X$ (resp. $X \times G \to X$) enjoying the usual properties. We call X a (left or right) ***G*-space** and will often use it to refer to the group operation. If $f\colon G \to H$ is a map of groups and $X \to X'$ a map from the G-space X to an H-space X' commuting with the operations, we call the latter map f-equivariant, or simply G-equivariant if f is the identity.

The **quotient** X/G of X by the action of G is the space with X_n/G_n as set of n-simplices and the induced face and degeneracy operators.

For any group G we denote the category of right G-actions and equivariant homotopy classes of maps by ***Op***-G. The category with objects, all right group actions, and morphisms, pairs (h, f) with h the f-equivariant homotopy class of a map between spaces with right actions, is denoted by ***Op***.

Note that all these definitions are compatible with the functor $\mathcal{S}$ from topological spaces to simplicial sets in that it carries the analogous topological objects to the simplicial ones.

The **Pontryagin product** $C(G) \otimes C(G) \to C(G)$ of a group G is the composition of shuffle map and the chain map $C(G \times G) \to C(G)$ induced by the group multiplication. It turns $C(G)$ into an associative algebra. The associativity of $C(G)$ follows directly from those of G and of the shuffle map and the shuffle map's naturality. By Corollary 2.2.2 and naturality the Pontryagin product is a map of coalgebras. This means that $C(G)$ is in fact an associative and coassociative Hopf algebra. By commutativity of the shuffle map $C(G)$ is commutative if G is.

The chain map $\iota_*\colon C(G) \to C(G)$ induced by the group inversion $\iota = \iota_G$ is an opposition in the sense of Section 1.3. The defining properties (1.10) follow from the identities

$$(g^{-1})^{-1} = g \qquad \text{and} \qquad (gh)^{-1} = h^{-1}g^{-1}$$

for all $g, h \in G$, and commutativity of the shuffle map. Note that for $G = T$ a torus the map $H(\iota_T)$ is the canonical opposition (1.11) of the exterior algebra $H(T)$.

If X is a, say, right G-space, then $C(X)$ is canonically a right $C(G)$-module, again by naturality and associativity. If the action of G on X is trivial, then so is that of $C(G)$ on $C(X)$, because it factors through the augmentation $C(G) \to C(1) = R$. (This is another advantage of normalised complexes.)

Note that we now have two ways of considering $C(X)$ as left $C(G)$-module: We can first give X the canonical left G-space structure,

$$g \cdot x = x \cdot g^{-1},$$

and then pass to algebra, or we can give the right $C(G)$-module $C(X)$ a left module structure by using the opposition ι_*, as explained in Section 1.3. A look at the definitions shows that both structures coincide.

LEMMA 2.4.1. *Let G be a group and X and Y both left or both right G-spaces.*

1. *The Alexander–Whitney map AW_{XY} is $C(G)$-equivariant.*
2. *The shuffle map ∇_{XY} is $C(G)$-equivariant if G operates trivially on either space.*
3. *The Steenrod map ST_{XY} is $C(G)$-equivariant if G operates trivially on Y.*
4. *It is also equivariant with respect to an $a \in C(G)$ if $|a| \leq 1$ and G operates trivially on X.*
5. *The diagonal and the augmentation of the coalgebra $C(X)$ are $C(G)$-equivariant.*

(Recall that $C(X) \otimes C(Y)$ has a canonical $C(G)$-module structure because $C(G)$ is a Hopf algebra.)

PROOF. Notice first that it is not important whether G acts from the left or from the right because we may always pass from one to the other by redefining the action and then back to the original one on the chain level with the help of the canonical opposition.

The non-trivial part of the last assertion follows from the first, which is once again a consequence of Proposition 2.2.1. By the same token Lemma 2.3.1 (4) & (5) prove the third and fourth claim. The second assertion is a consequence of associativity, commutativity, and naturality of the shuffle map. □

PROPOSITION 2.4.2. *The cochain functor is a well-defined contravariant functor*

$$C^*\colon \boldsymbol{Op} \to \boldsymbol{Mod}^*$$

which restricts for any group G to a functor

$$C^*\colon \boldsymbol{Op}\text{-}G \to C(G)\text{-}\boldsymbol{Mod}.$$

Recall that $\boldsymbol{Mod}^*$ is the category of modules with *contravariant* homotopy classes of chain maps.

PROOF. We actually prove that the chain functor is well-defined as functor $\boldsymbol{Op} \to \boldsymbol{Mod}$ and $\boldsymbol{Op}\text{-}G \to C(G)\text{-}\boldsymbol{Mod}$. The above claims then follow by applying the dualising functor.

We have already explained how the chain complex of a right G-space acquires a canonical right $C(G)$-module structure. Using again the naturality of the shuffle map, one easily shows that f-equivariant maps are carried to f_*-equivariant chain maps for any map $f\colon G \to H$ of groups.

By associativity and naturality of the shuffle map the diagram

$$\begin{array}{ccccc}
C(\Delta^{(1)}) \otimes C(X) \otimes C(G) & \xrightarrow{1 \otimes \nabla} & C(\Delta^{(1)}) \otimes C(X \times G) & \xrightarrow{1 \otimes \alpha_*} & C(\Delta^{(1)}) \otimes C(X) \\
\downarrow{\scriptstyle \nabla \otimes 1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} \\
C(\Delta^{(1)} \times X) \otimes C(G) & \xrightarrow{\nabla} & C(\Delta^{(1)} \times X \times G) & \xrightarrow{(\mathrm{id}, \alpha)_*} & C(\Delta^{(1)} \times X) \\
\downarrow{\scriptstyle h_* \otimes f_*} & & \downarrow{\scriptstyle (h,f)_*} & & \downarrow{\scriptstyle h_*} \\
C(X') \otimes C(H) & \xrightarrow{\nabla} & C(X' \times H) & \xrightarrow{\alpha'_*} & C(X'),
\end{array}$$

commutes for any f-equivariant homotopy $h\colon \Delta^{(1)} \times X \to X'$, where α and α' denote the respective actions. Hence the chain homotopy induced by h is f_*-equivariant. □

A **c-equivalence** in ***Op*** or ***Op***-G is a map or morphism inducing via the cochain functor a c-equivalence in ***Mod**** or $C(G)$-***Mod***, respectively.

5. Spaces over a base space

Let B be a space. A **space over** $\boldsymbol{B}$ is simply a map of spaces $p\colon Y \to B$. We will usually refer to p by Y, the map p being understood. A **map** $\boldsymbol{f\colon p \to p'}$ **of spaces over** $\boldsymbol{B}$ is a map $f\colon Y \to Y'$ such that $p' \circ f = p$. Similarly, **homotopies between spaces over** $\boldsymbol{B}$ are homotopies $\Delta^{(1)} \times Y \to Y'$ commuting with the projections to B. This generalises to maps over maps $B \to B'$. In this case we always assume that homotopies are compatible with a map $B \to B'$, not with a homotopy $\Delta^{(1)} \times B \to B'$.

We denote the category of spaces over B and homotopy classes over B of maps by ***Map***-B, and by ***Map*** the category of all spaces over base spaces (i. e., of all simplicial maps) and homotopy classes of maps over maps.

For any space $p\colon Y \to B$ over B the algebra $C^*(Y)$ acts on $C(Y)$ from the right by the cap product, hence also $C^*(B)$ via the map of algebras $p^*\colon C^*(B) \to C^*(Y)$. We will occasionally need the canonical map of spaces $Y \to Y \times B$, which we denote by $\tilde{\Delta} = \tilde{\Delta}_Y$.

Proposition 2.5.1. *The cochain functor is a well-defined contravariant functor*

$$C^*\colon \boldsymbol{Map} \to \boldsymbol{Mod}$$

which restricts for any space B to a functor

$$C^*\colon \boldsymbol{Map}\text{-}B \to C^*(B)\text{-}\boldsymbol{Mod}.$$

PROOF. The naturality of the Alexander–Whitney map implies for any space Y over B the commutativity of the diagram

$$\begin{array}{ccccc}
C(Y) & \xrightarrow{\Delta_*} & C(Y\times Y) & \xrightarrow{AW} & C(Y)\otimes C(Y) \\
 & \searrow^{\tilde\Delta_*} & \Big\downarrow{\scriptstyle (\mathrm{id},p_Y)_*} & & \Big\downarrow{\scriptstyle 1\otimes p_{Y*}} \\
 & & C(Y\times B) & \xrightarrow[AW]{} & C(Y)\otimes C(B);
\end{array}$$

together with what we already know about cup and cap products this proves all claims about spaces over base spaces except for the one regarding homotopies. Now for any homotopy $h\colon \Delta^{(1)}\times Y\to Y'$ over $f\colon B\to B'$ the diagram

$$\begin{array}{ccccc}
C(\Delta^{(1)})\otimes C(Y) & \xrightarrow{1\otimes\tilde\Delta_*} & C(\Delta^{(1)})\otimes C(Y\times B) & \xrightarrow{1\otimes AW} & C(\Delta^{(1)})\otimes C(Y)\otimes C(B) \\
\Big\downarrow{\scriptstyle\nabla} & & \Big\downarrow{\scriptstyle\nabla} & & \Big\downarrow{\scriptstyle\nabla\otimes 1} \\
C(\Delta^{(1)}\times Y) & \xrightarrow{(\mathrm{id},\tilde\Delta)_*} & C(\Delta^{(1)}\times Y\times B) & \xrightarrow{AW} & C(\Delta^{(1)}\times Y)\otimes C(B) \\
\Big\downarrow{\scriptstyle h_*} & & \Big\downarrow{\scriptstyle (h,f)_*} & & \Big\downarrow{\scriptstyle h_*\otimes f_*} \\
C(Y') & \xrightarrow{\tilde\Delta'_*} & C(Y'\times B') & \xrightarrow{AW} & C(Y')\otimes C(B')
\end{array}$$

commutes by naturality and Corollary 2.2.3. (Note that this would not be the case without normalisation!) For the chain homotopy $h_\diamond$ induced by h we therefore find

$$h_\diamond(c)\cdot\beta' = h_*((01)\times c)\cap p_{Y'}^*\beta' = h_*\big((01)\times(c\cap p_Y^*(f^*\beta'))\big) = h_\diamond(c\cdot f^*\beta')$$

for $c\in C(Y)$ and $\beta'\in C^*(B')$, as was to be shown. □

A **c-equivalence** in ***Map*** or ***Map*-B** is a map or morphism inducing via the cochain functor a c-equivalence in ***Mod*** or $C(B)$-***Mod***, respectively.

6. Fibre bundles

A simplicial fibre bundle is a generalisation of the notion of a product. We will only consider principal and associated bundles. Given a space B and a map of graded sets $\tau\colon B_{>0}\to G$ of degree -1 to a group G, one defines for every right G-space F the space $F\times_\tau B$ almost like the ordinary product $F\times B$, only the zeroth face map is twisted by τ, so that one has

$$\begin{aligned}
\partial_0(f,b) &= \big((\partial_0 f)\tau(b),\partial_0 b\big), \\
\partial_i(f,b) &= (\partial_i f,\partial_i b) \qquad (i>0), \\
s_i(f,b) &= (s_i f, s_i b) \qquad (i\geq 0).
\end{aligned}$$

That this be a space imposes some conditions on τ, cf. [**M**, §18], namely

$$\partial_0\tau(b) = \tau(\partial_1 b)\big(\tau(\partial_0 b)\big)^{-1}, \tag{2.5a}$$

$$\partial_i\tau(b) = \tau(\partial_{i+1} b) \qquad (i > 0), \tag{2.5b}$$

$$s_i\tau(b) = \tau(s_{i+1} b) \qquad (i \geq 0), \tag{2.5c}$$

$$1_n = \tau(s_0 b) \tag{2.5d}$$

for all $b \in B_n$. (Note that in [**M**] the fibre F is assumed to be a left G-space.) If these identities hold, one calls τ a **twisting function** and the map $F \times_\tau B \to B$ a **fibre bundle** with **base** B, **structure group** G, **fibre** F, and **total space** $F \times_\tau B$. It is called **principal** if $F = G$. Since there is no difference to the usual product in the second factor of a fibre bundle, the projection $F \times_\tau B \to B$ is a map of spaces, i. e., $F \times_\tau B$ is canonically a space over B. By abuse of language, we often do not distinguish between a fibre bundle and its total space, the projection onto the base being understood.

The key ingredient (in a way, the only one) to prove the main theorems of this chapter is the simplicial version of the Leray–Serre spectral sequence of a fibre bundle. We will describe it in the context of Brown's (or "twisted Eilenberg–Zilber") theorem, because this fits nicely into the general theme of the present work.

The classical Eilenberg–Zilber theorem compares the complex $C(F \times B)$ to the tensor product $C(F) \otimes C(B)$. This is done via the shuffle map and the Alexander–Whitney map, as described in Section 2.2. Since a fibre bundle $E = F \times_\tau B$ is a somehow "twisted" Cartesian product, the idea is to "twist" the differential on $C(F) \otimes C(B)$ as well, such that both complexes are still homotopy equivalent one to another. Brown's theorem states that this is possible.

We follow the treatment in [**Sh**, Ch. 2], but see also [**MC**, p. 223], [**M**, Ch. IV], and [**Sz**]: Starting from the twisting function τ one can define a twisting cochain $C(B) \to C(G)$ giving rise to a differential on $C(F) \otimes C(B)$ very much like in definition (1.26b). There is one difference once we apply the algebraic apparatus developed in the previous chapter to topology: The complex M from Section 1.7 will then be the cochain complex of the base space of a fibre bundle and $\mathbf{\Lambda}^*$ the cohomology of the fibre. So while in equation (1.26b) the twisting goes from the fibre to the base, it now goes the other way round because we deal with *chain* complexes. With this twisting cochain at hand, one introduces twisted versions of the Eilenberg–Zilber maps (shuffle map, Alexander–Whitney map, and the homotopy H) such that the identities (2.2) hold for these new maps. In particular, the complexes $C(F \times_\tau B)$ and $C(F) \tilde{\otimes} C(B)$ are homotopy equivalent.

We now define filtrations on both complexes such that the twisted Eilenberg–Zilber maps are filtration-preserving. We filter $A = C(F) \tilde{\otimes} C(B)$ by the degree of the base component,

$$F_p(A) = C(F) \otimes C_{\leq p}(B) \tag{2.6}$$

(which is to be compared with equations (1.38) and (1.40)). The complex $C(E)$ is filtered by the p-skeletons of the base B. More precisely, a (non-degenerate) simplex $(f, b) \in E_n$ belongs to $F_pC(E)$ for $p \in \mathbf{Z}$ if b is $(n-p)$-fold degenerate. (Hence $F_pC(E) = 0$ for $p < 0$.) The term E^0_{pq} of the resulting spectral sequence is just the free R-module generated by the non-degenerate $(p+q)$-simplices (f, b) with exactly q-fold degenerate b. The E^0 term of the filtration (2.6) looks like the original complex, but with differential reduced to the one on $C(F)$. Consequently, we have an isomorphism of bigraded R-modules

$$E^1(A) = H(F) \otimes C(B).$$

On the E^0 level, the maps induced by the twisted Eilenberg–Zilber maps still enjoy properties (2.2), hence induce isomorphisms on the E^1 level. (This is somewhat implicit in [**Sh**, § II.2], but stated more clearly in [**Sz**, §5]. The maps on the E^0 level are actually equal to those induced by the non-twisted counterparts.)

One knows that the differential d^1 on $E^1(A)$ is of the form

$$d^1([f] \otimes b) = [f] \otimes db \pm [f \cdot (\tau(b')^{-1} - 1)] \otimes b''$$

for $b \in B$ and $[f] \in H(F)$, where $b' \otimes b''$ is the component of $AW(b)$ with first factor of degree 1, see [**M**, eq. (T.1) on p. 143] or [**Sz**, Thm. 2.2]. Since we assume G to be connected, $\tau(b')^{-1} - 1$ is always homologous to zero, hence $E^1(A)$ is isomorphic to the chain complex of B with coefficients in $H(F)$. Therefore,

$$E^2(E) := E^2(C(E)) \cong E^2(A) \cong H(B, H(F)). \tag{2.7a}$$

The filtrations on $C(E)$ and A canonically induce filtrations on the dual complexes, and one finds similarly

$$E_2(E) := E^2(C^*(E)) \cong E^2(A^*) \cong H^*(B, H^*(F)). \tag{2.7b}$$

We will repeatedly use the isomorphism

$$H^*(B, H^*(F)) = H^*(F) \otimes H^*(B),$$

which holds if the cohomology of either space is free of finite rank in each degree.

The identities (2.7a) and (2.7b) are referred to as the Leray–Serre theorem, because they are simplicial analogues of results due to Leray and Serre. (See [**MC**, Chs. 5 & 6] for a comprehensive exposition of the Leray–Serre theorem in singular homology.) We remark that one can extend the Leray–Serre theorem to non-connected groups by introducing local coefficient systems.

The question of multiplicativity in the cohomological Leray–Serre spectral sequence is a delicate one. It is not difficult to see that the Alexander–Whitney map on $C(E)$ is filtration-preserving, so that the associated cohomological spectral sequence is one of algebras. But one has to work hard to relate this to a product structure on the spectral sequence $E(A)$. Luckily, we will only need the module structure of $C(E)$ over $C^*(B)$, where things are much easier – at least conceptually: The coalgebra structure on $C(B)$ induces a chain map

$$C(F) \tilde{\otimes} C(B) \xrightarrow{1 \otimes \Delta_*} C(F) \tilde{\otimes} C(B \times B) \xrightarrow{1 \otimes AW} C(F) \tilde{\otimes} C(B) \otimes C(B)$$

compatible with the twisted shuffle map. Similarly, if $F = G$, then the differential on A and the twisted shuffle map are compatible with group multiplication. (See [**Sh**, § II.3, Prop. 1, and § II.4, Props. 1 & 2] for proofs.) Hence the isomorphisms (2.7a) and (2.7b) are $H^*(B)$-equivariant and, for $F = G$, also $H(G)$-equivariant.

Moreover, they are natural with respect to (appropriately defined) bundle maps [**M**, Def. 20.1]. In particular, the following is a commutative diagram of

$H^*(B)$-modules for any inclusion $i\colon * \hookrightarrow B$:

$$\begin{array}{ccccc} E^2(F,*) & \longleftarrow & E^2(F,B) & \longleftarrow & E^2(*,B) \\ \big\downarrow = & & \big\downarrow \cong & & \big\downarrow = \\ H^*(F) & \xleftarrow{H^*(i)} & H^*(B,H^*(F)) & \longleftarrow & H^*(B). \end{array}$$

The outer vertical maps are the respective identity mappings because these filtrations are trivial.

7. Universal bundles and classifying spaces

In order to define Koszul functors in the simplicial setting, we need a simplicial construction of universal bundles and classifying spaces [**M**, §21]. We begin with the latter.

For any group G the **classifying space** BG is the space with set of n-simplices

$$BG_n = G_{n-1} \times \cdots \times G_0$$

for $n \in \mathbf{N}$. We write the simplices of BG in the form

$$[g_{n-1}, \ldots, g_0] \in BG_n, \qquad \text{also} \qquad b_0 := [] \in BG_0$$

for the unique vertex of BG. The face and degeneracy maps are given by

$$\partial_i[g_{n-1}, \ldots, g_0] = [\partial_{i-1}g_{n-1}, \ldots, \partial_1 g_{n-i+1}, (\partial_0 g_{n-i})g_{n-i-1}, g_{n-i-2}, \ldots, g_0],$$
$$s_i[g_{n-1}, \ldots, g_0] = [s_{i-1}g_{n-1}, \ldots, s_1 g_{n-i+1}, s_0 g_{n-i}, 1_{n-i}, g_{n-i-1}, \ldots, g_0].$$

(Undefined components, such as $\partial_{-1}g_{n-1}$, are supposed to be omitted when applying these formulas for given values of i and n.) The map of graded sets

$$\tau_G\colon BG_{>0} \to G, \qquad [g_{n-1}, \ldots, g_0] \mapsto g_{n-1}$$

is a twisting function for BG.

We call the principal bundle $EG = G \times_{\tau_G} BG \to BG$ the **universal *G*-bundle**. G acts freely on its total space EG, which has the canonical base point $e_0 = (1_0, [])$. Both BG and EG are connected: This is trivial for BG; for EG it suffices to observe the identities

$$\partial_0(s_0 g, [g^{-1}h]) = (h, []) \qquad \text{and} \qquad \partial_1(s_0 g, [g^{-1}h]) = (g, [])$$

for $g, h \in G_0 \approx EG_0$.

Note that the construction of classifying spaces and universal bundles is functorial. Moreover, it is compatible with products, i. e.,

$$B(G \times H) = BG \times BH \qquad \text{and} \qquad E(G \times H) = EG \times EH \tag{2.8}$$

for any pair G, H of groups.

An important role is played by the following map of degree 1

$$S = S_G\colon EG \to EG, \qquad (g_n, [g_{n-1}, \ldots, g_0]) \mapsto (1_{n+1}, [g_n, g_{n-1}, \ldots, g_0]), \tag{2.9}$$

which satisfies for all $e \in EG_n$ and $0 \le i \le n$ the identities

$$\partial_0 Se = e, \qquad \partial_{i+1} Se = \begin{cases} S\partial_i e & \text{if } n > 0, \\ e_0 & \text{if } n = 0, \end{cases} \tag{2.10a}$$

$$s_0 Se = SSe, \qquad s_{i+1} Se = S s_i e. \tag{2.10b}$$

In particular, S passes to a map $C(EG) \to C(EG)$ on the normalised complex, which we continue to denote by the same letter. There the composition $S \circ S$ vanishes, and $Se_0 = s_0 e_0 = 0$. Note that S is compatible with products, i. e., $S_{G\times H} = S_G \times S_H$.

A more detailed study of this map than done in [**M**] yields the following result:

PROPOSITION 2.7.1. *The space EG is contractible. The map S is the chain homotopy induced from some contraction to e_0, i. e.,*

$$Sde + dSe = \begin{cases} e & \text{if } |e| > 0, \\ e - e_0 & \text{if } |e| = 0 \end{cases} \tag{2.11}$$

for all (non-degenerate) $e \in EG$. Moreover, the following identities hold for all groups G and H:

$$AW_{EG,EH} S_{G\times H} = e_0 \otimes S_H p_2 + (S_G \otimes 1) AW_{EG,EH}, \tag{2.12a}$$

where $e_0 \otimes S_H p_2$ is the map $(e, e') \mapsto e_0 \otimes S_H e'$,

$$ST_{EG,EH} S_{G\times H} = -(S_G \otimes S_H) AW_{EG,EH} - (1 \otimes S_H) ST_{EG,EH}, \tag{2.12b}$$

$$\nabla_{EG,EH}(S_G \otimes S_H) = S_{G\times H} \nabla_{EG,EH}(1 \otimes S_H - S_G \otimes 1). \tag{2.12c}$$

PROOF. Define a map $h\colon \Delta^{(1)} \times EG \to EG$ by

$$h(x, e) = S^k (\partial_0)^k e.$$

where k is the number of zeros in the sequence $x \in \Delta^{(1)}$, cf. the definition of $\Delta^{(1)}$ in Section 2.1. Note that $\partial_0 e$ is not defined for $|e| = 0$, but we agree to interpret $S\partial_0 e$ as e_0 in this case. Then $h(x, e) = e$ if x if x contains only ones, i. e., if $(x, e) \in (1) \times EG$, and $h(x, e) = (s_0)^{|e|} e_0$ if x contains only zeros, i. e., if $(x, e) \in (0) \times EG$. The relations (2.10) imply that h is simplicial, hence a homotopy from the identity to the constant map $EG \to e_0$. We do not check all details here because a similar, but more general calculation will appear in the proof of Theorem 2.8.2.

The induced chain homotopy $H\colon C(EG) \to C(EG)$ maps $e \in EG_n$ to

$$H(e) = \sum_{k=0}^{n} (-1)^k h\big((0^{k+1} 1^{n-k}), s_k e\big),$$

where $(0^{k+1} 1^{n-k}) \in \Delta^{(1)}$ is the sequence with $k+1$ zeros followed by $n-k$ ones. But the relation $S \circ S = 0$ implies that

$$h\big((0^{k+1} 1^{n-k}), s_k e\big) = S^{k+1} (\partial_0)^{k+1} s_k e$$

is degenerate for $k > 0$, whence $H = S$. This gives (2.11).

The proofs of the identities given for the shuffle, the Alexander–Whitney, and the Steenrod map are simple calculations, see Appendix 9. □

8. Simplicial Koszul functors

We are now in the position to introduce the **simplicial Koszul functors** between the category ***Op*** and the subcategory ***MapCl*** of ***Map*** with objects, spaces over classifying spaces, and morphisms, homotopy classes of maps over maps Bf induced by maps $f\colon G \to H$ of groups.

Let G be a group and X a right G-space. The **Borel construction**

$$p_{X_G}\colon X_G = X \times_{\tau_G} BG \to BG$$

is a space over BG. The **equivariant cohomology** $H^*_G(X)$ of X is by definition the cohomology of X_G. It is a module over $H^*(BG) = H^*_G(*)$ via $H^*(p_{X_G})$.

The map of spaces

$$q_X\colon X \times EG \to X_G, \qquad \bigl(x, (g, b)\bigr) \mapsto (xg, b),$$

is the quotient of $X \times EG$ by the G-action $(x, e)g = (xg, g^{-1}e)$. This justifies the notation $X_G = X \times_G EG$. We record the following observation, which will be used in Section 2.12:

LEMMA 2.8.1. *The composition*

$$q_{X*} \circ \nabla\colon C(X) \otimes C(EG) \to C(X \times EG) \to C(X \underset{G}{\times} EG)$$

is a chain map of right $C^*(BG)$-modules.

Here $C^*(B)$ operates trivially on $C(X)$ and by taking cap products on all other chain complexes.

PROOF. This follows from the commutative diagram

$$\begin{array}{ccccc}
C(X) \otimes C(EG) & \xrightarrow{1 \otimes \tilde{\Delta}_*} & C(X) \otimes C(EG \times BG) & \xrightarrow{1 \otimes AW} & C(X) \otimes C(EG) \otimes C(BG) \\
\downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla \otimes 1} \\
C(X \times EG) & \xrightarrow{(\mathrm{id}, \tilde{\Delta})_*} & C(X \times EG \times BG) & \xrightarrow{AW} & C(X \times EG) \otimes C(BG) \\
\downarrow{\scriptstyle q_{X*}} & & \downarrow{\scriptstyle (q_X, \mathrm{id})_*} & & \downarrow{\scriptstyle q_{X*} \otimes 1} \\
C(X \underset{G}{\times} EG) & \xrightarrow{\tilde{\Delta}_*} & C(X \underset{G}{\times} EG \times BG) & \xrightarrow{AW} & C(X \underset{G}{\times} EG) \otimes C(BG),
\end{array}$$

which is essentially a special case of the large diagram appearing in the proof of Proposition 2.5.1. □

The Koszul functor $\boldsymbol{t}\colon \boldsymbol{Op} \to \boldsymbol{MapCl}$ assigns to each right G-space X the space X_G (more precisely, the map p_{X_G}), and to each morphism $X \to X'$ the induced morphism $X_G \to X'_H$.

For any space Y over BG, i. e., any map $p_Y\colon Y \to BG$, we can form the fibre product

$$Y \overset{BG}{\times} EG = \{\, (y, e) : p_Y(y) = p_{EG}(e) \,\},$$

which is readily seen to be a principal G-bundle with base Y and twisting function $\tau_G \circ p_Y$. The group G operates on it from the left.

The Koszul functor $\boldsymbol{h}\colon \boldsymbol{MapCl} \to \boldsymbol{Op}$ assigns to each space Y over BG the space $Y \times^{BG} EG$ with the *opposite*, hence right G-action, and to each morphism $Y \to Y'$ the induced morphism $Y \times^{BG} EG \to Y' \times^{BH} EH$. (Note that Bf determines $f\colon G \to H$ uniquely.)

If we want to indicate the group, we write the functors as $\boldsymbol{t}_G$ and $\boldsymbol{h}_G$, respectively.

The naturality of the Koszul functors with respect to the group gives us in analogy with Section 1.6 canonical maps of spaces

$$X = \boldsymbol{h}_1\boldsymbol{t}_1X \to \boldsymbol{h}_G\boldsymbol{t}_1X \to \boldsymbol{h}_G\boldsymbol{t}_GX = \boldsymbol{ht}X, \tag{2.13a}$$

$$\boldsymbol{th}Y = \boldsymbol{t}_G\boldsymbol{h}_GY \to \boldsymbol{t}_G\boldsymbol{h}_1Y \to \boldsymbol{t}_1\boldsymbol{h}_1Y = Y. \tag{2.13b}$$

The following result is a (partial) analogue of [**AP′**, Remark 1.7] in the simplicial setting. As remarked there, it parallels the duality between the algebraic Koszul functors. The reader may want to compare the present results with those obtained in Section 1.6.

THEOREM 2.8.2. *The compositions* $\boldsymbol{ht}$ *and* $\boldsymbol{th}$ *are c-equivalent to the identity functors of* $\boldsymbol{Op}$ *and* $\boldsymbol{MapCl}$. *They become isomorphic to them if composed with the forgetful functors to the homotopy category of spaces. More precisely:*

1. *Let* $X \in \boldsymbol{Op}$-G. *Then* $\boldsymbol{ht}X$ *is (up to isomorphism) a bundle with base* EG *and fibre* X *and admits a* G*-equivariant trivialising map*

$$\boldsymbol{ht}X \to X \times EG.$$

The canonical map (2.13a) *corresponds under this isomorphism to the inclusion of* X *over* $e_0 \in EG$. *In particular, it is a homotopy equivalence with strict left* G*-equivariant inverse, and both maps are natural in* G *and* X.

2. *Let* $(p_Y : Y \to BG) \in \boldsymbol{Map}$-$BG$. *Then the map* (2.13b) *is (up to isomorphism) a bundle with fibre* EG. *It is also a homotopy equivalence possessing as strict right inverse a map over* BG. *Both maps are natural in* BG *and* Y.

PROOF. The space

$$\boldsymbol{ht}X = (X \underset{G}{\times} EG) \overset{BG}{\times} EG$$

is by definition a bundle with fibre G and base, a bundle over BG with fibre X. It is therefore isomorphic as graded set to the Cartesian product $G \times X \times BG$. The only difference lies in the face map ∂_0, which is now

$$\partial_0(g, x, b) = \big((\partial_0 g)\tau_G(b), (\partial_0 x)\tau_G(b), \partial_0 b\big). \tag{2.14}$$

This exhibits $\boldsymbol{ht}X$ as a bundle with base EG, fibre X and twisting function $\tau_G \circ p_{EG}$. Inspection of the definitions shows that the composition (2.13a) is just the inclusion of the fibre X over $e_0 \in EG$. The maps

$$\begin{aligned} X \times EG &\leftrightarrow \boldsymbol{ht}X \\ (x, g, b) &\mapsto (g, xg, b) \\ (xg^{-1}, g, b) &\leftarrow\!\shortmid (g, x, b) \end{aligned} \tag{2.15}$$

are isomorphisms of right G-spaces, inverse to each other. Here the G-action on $X \times EG$ is the diagonal of the action on X and the opposite action on EG. (See Appendix 10 for a proof that they are both maps of spaces.) Under this isomorphism the map (2.13a) still corresponds to the inclusion of X over e_0 in $X \times EG$. This is by Proposition 2.7.1 a homotopy equivalence with the canonical G-equivariant projection onto X as inverse.

We now consider the map (2.13b). The space

$$\boldsymbol{th}Y = (Y \overset{BG}{\times} EG) \underset{G}{\times} EG$$

is isomorphic to $G \times Y \times B$ as graded set, only the zeroth face map is different,

$$\partial_0(g, y, b) = \big(\tau_G(b)^{-1}(\partial_0 g)\tau_Y(y), \partial_0 y, \partial_0 b\big)$$

with $\tau_Y = \tau_G \circ p_Y$. It will be convenient to apply the group inversion to G and reorder the factors in the Cartesian product, so that one has

$$\partial_0(g, b, y) = \big(\tau_Y(y)^{-1}(\partial_0 g)\tau_G(b), \partial_0 b, \partial_0 y\big).$$

This shows that the canonical map $\boldsymbol{th}Y \to Y$ is (essentially) a bundle with fibre EG, base Y and twisting function τ_Y. Note that for $Y = *$ the total space is just EG. We will therefore keep the promise made in the proof of Proposition 2.7.1 and show now in detail that EG is contractible.

We claim that the projection to the base Y is a homotopy equivalence with inverse

$$q_Y \colon Y \to \boldsymbol{th}Y, \qquad y \mapsto (1, p_Y(y), y),$$

which is a map over BG. (Recall that $\boldsymbol{th}Y$ is a space over BG via projection onto the second coordinate in our representation.)

The fact that q_Y is simplicial is readily verified. The composition $p_Y \circ q_Y$ is the identity of Y, and

$$q_Y(p_Y(g, b, y)) = (1, p_Y(y), y).$$

We define a homotopy $h \colon \Delta^{(1)} \times \boldsymbol{th}Y \to \boldsymbol{th}Y$ from $q_Y \circ p_Y$ to the identity recursively by

$$h(x, g, b, y) = \begin{cases} (1, b_0, y) & \text{if } x = (0), \\ \big(S(\tau_Y(y)g', b'), y\big) & \text{if } x_0 = 0, \text{ but } x \neq (0), \\ (g, b, y) & \text{if } x_0 = 1, \end{cases}$$

where S is the map introduced in (2.9) and g' and b' are determined by

$$(g', b', \partial_0 y) = h(\partial_0(x, g, b, y)).$$

Moreover, x_0 denotes the leading element of the sequence $x \in \Delta^{(1)}$. (Recall that the simplices in $\Delta^{(1)}$ are the weakly increasing sequences composed of zeros and ones.) See Appendix 10 for the verification that h is a homotopy as claimed. □

COROLLARY 2.8.3. *Any G-space is c-equivalent to a free one, and any space over BG to a fibre bundle with base BG.*

COROLLARY 2.8.4. *The simplicial Koszul functors form an adjoint pair $(\boldsymbol{h}, \boldsymbol{t})$.*

PROOF. Using the explicit formulas for the canonical equivariant maps given in the proof of the theorem, it is easily verified that the following compositions are the respective identities:

$$\begin{aligned} \boldsymbol{h}Y &\longrightarrow \boldsymbol{h}(\boldsymbol{th}Y) = \boldsymbol{ht}(\boldsymbol{h}Y) \longrightarrow \boldsymbol{h}Y, \\ \boldsymbol{t}X &\longrightarrow \boldsymbol{th}(\boldsymbol{t}X) = \boldsymbol{t}(\boldsymbol{ht}X) \longrightarrow \boldsymbol{t}X. \end{aligned}$$

□

Proposition 2.8.5. *The simplicial Koszul functors preserve c-equivalence.*

Proof. We have to show the following for any map $f\colon G \to H$ of (connected) groups:

1. Let $g\colon X \to X'$ be f-equivariant. If f and g are c-equivalences, then so are $\boldsymbol{t}g\colon X_G \to X'_H$ and Bf.
2. Let $g\colon Y \to Y'$ be a map over Bf. If Bf and g are c-equivalences, then so are $\boldsymbol{h}g\colon Y \times_{BG} EG \to Y' \times_{BH} EH$ and f.

To begin with, f is a c-equivalence if and only if Bf is. This follows from Zeeman's comparison theorem applied to the f-equivariant map $Ef\colon EG \to EH$ over Bf: The E_2 term of the associated Leray–Serre spectral sequence for G is

$$E_2^{pq} = H^q(G) \otimes H^p(BG),$$

and similar for H. The total spaces EG and EH being contractible, the map Ef is trivially a c-equivalence. Now Zeeman's comparison theorem states that $E_2^{0*} = H^*(f)$ is an isomorphism if and only if $E_2^{*0}(f) = H^*(Bf)$ is.

The above claims follow immediately from a further application of the Leray–Serre theorem and the usual comparison of spectral sequences. □

9. Relation to topology

The simplicial Koszul functors $\boldsymbol{t}$ and $\boldsymbol{h}$ have well-known topological analogues, which we call $\mathfrak{t}$ and $\mathfrak{h}$ in this section. As remarked in the previous section, they enjoy properties analogous to those stated in Theorem 2.8.2. We now choose a definite topological model for EG, namely the Milnor construction [**tD**, Abschnitt IX.4], i. e., the infinite join $G * G * \cdots$. The (topological) principal bundle $EG \to BG = EG/G$ has a natural countable trivialising cover $\mathcal{U}$. For any map $p\colon Y \to BG$ we denote by $\mathcal{S}'Y$ the space of all singular simplices in Y compatible with $\mathcal{U}$, i. e., each contained in $p^{-1}(U)$ for some $U \in \mathcal{U}$. We will repeatedly use the fact that the inclusion $C(\mathcal{S}'Y) \hookrightarrow C(\mathcal{S}Y)$ is a chain homotopy equivalence, cf. [**BT**, p. 186].

Lemma 2.9.1. *Let G be a connected topological group. Then there is a unique $\mathcal{S}G$-equivariant homotopy class $\mathcal{S}'EG \to E\mathcal{S}G$. Given a map of connected topological groups $G \to H$, one can choose representatives of these homotopy classes such that the diagram*

$$\begin{array}{ccc} \mathcal{S}'EG & \longrightarrow & E\mathcal{S}G \\ \Big\downarrow{\scriptstyle \mathcal{S}'Ef} & & \Big\downarrow{\scriptstyle E\mathcal{S}f} \\ \mathcal{S}'EH & \longrightarrow & E\mathcal{S}H \end{array}$$

commutes. Moreover, the induced morphism $\mathcal{S}'BG \to B\mathcal{S}G$ is a c-equivalence.

Proof. This draws very much upon the results of [**M**], to which all citations in this proof refer.

The projection $q\colon \mathcal{S}'EG \to \mathcal{S}'BG$ is a principal G-fibration in the sense of Definition 18.1. By Proposition 18.7 and Theorem 21.13 q is (up to isomorphism) induced from a map $F_G\colon \mathcal{S}'BG \to B\mathcal{S}G$, unique up to homotopy. Any such F_G equips $\mathcal{S}'EG \to \mathcal{S}'BG$ with the structure of a principal $\mathcal{S}G$-bundle with twisting function $\tau = \tau_G \circ F$, hence gives rise to a pseudo-cross section $\sigma_G\colon \mathcal{S}'BG \to \mathcal{S}'EG$.

Conversely, any pseudo-cross section determines such a map F_G by the construction of Theorem 21.7.

To see that F_G is a c-equivalence, consider the map of principal bundles $\mathcal{S}'EG \to E\mathcal{S}G$: It trivially induces an isomorphism in cohomology because both spaces are contractible. Since $\mathcal{S}'BG$ and $B\mathcal{S}G$ are connected, the induced map between the E_2 terms of the associated Leray–Serre spectral sequences is an isomorphism on the vertical axis. This implies by Zeeman's comparison theorem that we also have an isomorphism on the horizontal axis. But this map is just $H^*(F_G)$.

It remains to show the claimed naturality. Let $f\colon G \to H$ be a map of topological groups and denote kernel and image of $\mathcal{S}f\colon \mathcal{S}G \to \mathcal{S}H$ by K and H', respectively. Then $\mathcal{S}f\colon \mathcal{S}G \to H'$ is a principal K-fibration.

The map f induces an f-equivariant map $Ef\colon EG \to EH$, hence maps of spaces $\mathcal{S}'Ef\colon \mathcal{S}'EG \to \mathcal{S}'EH$ and $\mathcal{S}'Bf\colon \mathcal{S}'BG \to \mathcal{S}'BH$. (Here we use the naturality of the cover $\mathcal{U}$.) The above diagram commutes with F_G and F_H as horizontal maps if (and only if)

$$\begin{array}{ccc} \mathcal{S}'BG & \xrightarrow{\sigma_G} & \mathcal{S}'EG \\ {\scriptstyle \mathcal{S}'Bf}\downarrow & & \downarrow{\scriptstyle \mathcal{S}'Ef} \\ \mathcal{S}'BH & \xrightarrow[\sigma_H]{} & \mathcal{S}'EH \end{array} \tag{2.16}$$

does, where σ_G and σ_H are the pseudo-cross sections corresponding to F_G and F_H, respectively.

Let E' and B' be the images of $\mathcal{S}'Ef$ and $\mathcal{S}'Bf$, respectively. It is readily seen that $E' \to B'$ is a principal H'-fibration. Choose a pseudo-cross section σ' for this fibration and extend it to $\mathcal{S}'EH \to \mathcal{S}'BH$ by Lemma 18.6. This gives σ_H.

Let $E'' \to \mathcal{S}'BG$ be the pull back of $E' \to B'$ along $\mathcal{S}'Bf$ with induced pseudo-cross section σ''. Then $\mathcal{S}'EG \to E''$ is a principal K-fibration. The composition of any pseudo-cross section for it with σ'' is a pseudo-cross section σ_G making (2.16) commutative. □

As a consequence, if $Y \to BG$ is a continuous map then we may consider $\mathcal{S}'Y$ as a space over $B\mathcal{S}G$. The resulting module structure of $H^*(\mathcal{S}Y)$ over $H^*(B\mathcal{S}G)$ is that over $H^*(\mathcal{S}BG)$. Note that, by the preceding lemma, $H^*(B\mathcal{S}G)$ and $H^*(\mathcal{S}BG)$ are naturally isomorphic. We write $H^*(BG)$ for both of them. Furthermore, the pull back $\boldsymbol{h}_{\mathcal{S}G}\mathcal{S}Y$ does (up to isomorphism) not depend on the chosen map $\mathcal{S}'Y \to B\mathcal{S}G$.

PROPOSITION 2.9.2. *Let G be a connected topological group.*

1. *Let X be a (topological) G-space. Then there is an isomorphism*
$$H^*(\mathcal{S}\mathfrak{t}_G X) = H^*(\boldsymbol{t}_{\mathcal{S}G}\mathcal{S}X)$$
of $H^*(BG)$-modules, natural in G and X.
2. *Let Y be a (topological) space over BG. Then there is an isomorphism*
$$H^*(\mathcal{S}\mathfrak{h}_G Y) = H^*(\boldsymbol{h}_{\mathcal{S}G}\mathcal{S}Y)$$
of $H(G)$-modules, natural in G and Y.

Proof. Let X be a G-space. By the preceding lemma there is a commutative diagram

$$\begin{array}{ccc} \mathcal{S}'(X \underset{G}{\times} BG) & \longrightarrow & \mathcal{S}X \underset{\mathcal{S}G}{\times} E\mathcal{S}G \\ \downarrow & & \downarrow \\ \mathcal{S}'BG & \longrightarrow & B\mathcal{S}G. \end{array}$$

The top row induces an isomorphism in cohomology by the Leray–Serre theorem, because the bottom row does and the fibres of both bundles are identical. The naturality of this isomorphism follows from that of the map $\mathcal{S}'EG \to E\mathcal{S}G$ established in the lemma.

Since $\mathcal{S}'EG \to \mathcal{S}'BG$ is induced from $\mathcal{S}'BG \to B\mathcal{S}G$ and we use this c-equivalence to consider a continuous map $Y \to BG$ as a space over $B\mathcal{S}G$, the second assertion is clear. □

10. Tori

We focus on tori for the rest of this work. Let us begin with the definition of a "simplicial circle," which may replace a topological one, very much like the "simplicial interval" $\Delta^{(1)}$, which we often use instead of the topological one $\Delta_1 \approx [0,1]$. In contrast to $\Delta^{(1)}$, we do not give a combinatorial description, but start from the topological circle S^1. A **simplicial circle** is a group isomorphic to the subgroup (i. e., subsimplical group) of the group $\mathcal{S}S^1$ generated by a simplex $x' \in (S^1)_1$ representing a generator x of the abelian group $H_1(S^1;\mathbf{Z})$. (This simplicial circle is essentially the simplicial construction of the Eilenberg–Mac Lane space $K(\mathbf{Z},1)$, cf. [**M**, §23].) We define a **circle** as a group containing a simplicial circle as a subgroup such that the inclusion is a c-equivalence. A **torus** of **rank** r is a group isomorphic to an r-fold product of circles. Examples of tori are the compact tori $\cong (S_1)^r$, the algebraic tori $\cong (\mathbf{C}^*)^r$, and the **simplicial tori** isomorphic to products of simplicial circles. There exists by definition a c-equivalence from a simplicial torus to any other torus of the same rank. By Proposition 2.8.5 this restriction does not affect cohomology.

We call a torus of rank r together with a fixed decomposition into circles a **standard torus** and denote it by T^r, and we write x_i if we consider the homology class x as an element of the i-th factor of $H(S^1) \otimes \cdots \otimes H(S^1)$ and also for the corresponding element in $H(T)$. The same applies to x'. Moreover, we now take $H_1(T)$ as the free R-module P (concentrated in degree 1) that was the starting point for the constructions of the previous chapter. Note that, according to our choice of P, we have a canonical isomorphism of Hopf algebras $H(T) = \mathbf{\Lambda}$. One can even improve on this result:

Lemma 2.10.1. *The assignment*

$$\mathbf{\Lambda} \to C(T), \quad x_{i_1} \wedge \cdots \wedge x_{i_q} \mapsto x'_{i_1} \cdots x'_{i_q}$$

is a c-equivalence of Hopf algebras.

Proof. We may assume $R = \mathbf{Z}$. Since $C(T)$ is free over $\mathbf{Z}$, the above map is well-defined if for all i, $j \in [r]$ we have

$$x'_j \cdot x'_i = -x'_i \cdot x'_j.$$

This equation follows from the commutativity of the shuffle map and that of the group multiplication μ (expressed by the identity $\mu \circ \tau_{TT} = \mu$):

$$x'_j \cdot x'_i = \mu_* \nabla(x'_j \otimes x'_i) = -\mu_* \tau_* \nabla(x'_i \otimes x'_j) = -\mu_* \nabla(x'_i \otimes x'_j) = -x'_i \cdot x'_j.$$

The shuffle map being a map of coalgebras by Corollary 2.2.2, it suffices to verify this property for the above map in the case $r = 1$. Since x' is a simplex, we find

$$AW \Delta_* x' = x' \otimes 1 + 1 \otimes x',$$

which matches (1.9). The augmentations are compatible, too, because x' is of positive degree. It is clear that the induced map in homology is an isomorphism because we map generators to generators. □

A map of groups $T^r \to T^{r'}$ is called **componentwise** if it deletes some of the components and permutes the others, with the possible insertion of $1's$. It is called **monotone** if it is componentwise and keeps the order of the remaining components. For example, the map

$$T^4 \to T^3, \qquad (g_1, g_2, g_3, g_4) \mapsto (1, g_4, g_1)$$

is componentwise, but not monotone. Moreover, a map over a map $Bf\colon BT^r \to BT^{r'}$ is **monotone** if $f\colon T^r \to T^{r'}$ is.

11. Three important maps

In this section we are going to introduce certain maps

$$f\colon K \to C(ET) \qquad \text{and} \qquad \phi\colon \mathbf{\Lambda}^* \tilde{\otimes} C^*(BT) \to C^*(ET)$$

which will enable us to compare the algebraic and simplicial Koszul functors in the next section. To be definite, we fix once and for all a decomposition $T \cong (S^1)^r$, which determines a basis $(x_1, \ldots, x_r)$ of $P = H_1(T)$ with representatives $x'_1, \ldots, x'_r$. We also need suitable representatives of the classes generating the algebra $H^*(BT)$, which we will show to be isomorphic to $\mathbf{S}^*$. By the Eilenberg–Zilber and Künneth theorems it suffices to treat the case of a circle, which we denote by S^1. (Recall our definition of a circle from the preceding section.)

Consider the Leray–Serre spectral sequence for the universal principal S^1-bundle $p\colon ES^1 \to BS^1$. Its E_2 term is

$$E_2 = E_2(ES^1) = \mathbf{\Lambda}^* \otimes H^*(BS^1),$$

where $\mathbf{\Lambda}^q$ is isomorphic to R for $q = 0$ or 1 and zero otherwise. This implies that the spectral sequences collapses on the E_3 level. Since ES^1 is contractible, the $E_3 = E_\infty$ term vanishes outside the origin. The differential

$$d_2\colon E_2^{p,1} \to E_2^{p+2,0}$$

must therefore be an isomorphism for all $p \in \mathbf{N}$.

By definition the loop x' represents a homology class x generating $H_1(S^1)$. Now also choose a cocycle $\chi \in C^1(ES^1)$ pulling back under the canonical inclusion $i_{S^1} : S^1 \to ES^1$ to a representative of the class in $H^1(S^1)$ dual to $-x$. We then have in particular

$$\langle x', i_{S^1}^* \chi \rangle = -1. \tag{2.17}$$

This $[\chi]$ is a generator of $E^2_{0,-1}(ES^1)$, and

$$\xi = d^2[\chi]$$

generates $E_2^{20} = H^2(BS^1)$. By changing χ by an element from $F_{-1}C^1(ES^1)$, which lies in the kernel of $i^*_{S^1}$, we may therefore assume

$$d\chi = p^*\xi' \tag{2.18}$$

for a representative ξ' of ξ. In short, $i^*_{S^1}\chi$ corresponds to ξ' under transgression.

As mentioned in Section 2.6, the differentials in the Leray–Serre spectral sequence commute with cup products with cochains pulled back from the base. Multiplying repeatedly by ξ, it follows inductively that $H^p(BS^1) = E_2^{p,0} \cong R$ with generator ξ^p. This gives a (after choosing ξ) canonical isomorphism of algebras

$$H^*(BT) = \mathbf{S}^*$$

for $T = S^1$, consequently for all tori T by the Künneth theorem.

We again write ξ_i if we consider ξ as an element of the i-th factor of the tensor product $H^*(BS^1) \otimes \cdots \otimes H^*(BS^1)$ or as the corresponding element in $H^*(BT)$, and similar for ξ' and χ.

Our choice of the x'_i determines by Lemma 2.10.1 functors from the categories of left and right $C(G)$-modules to those over $\mathbf{\Lambda}$. These functor are natural with respect to componentwise maps $T^r \to T^{r'}$ by the shuffle map's commutativity. In [**GKM**] the resulting $\mathbf{\Lambda}$-module structure on chain and cochain complexes of spaces is called the "sweep action." Analogously, Proposition 1.7.1 gives us a functor from left $C^*(BT)$-modules to weak $\mathbf{S}^*$-modules by our choice of $\xi'_1, \ldots, \xi'_r$. Here naturality does only hold for monotone maps over classifying spaces of standard tori. In order to avoid a too clumsy notation, we incorporate these functors into the cochain functor. Hence, $C^*(X) \in \mathbf{\Lambda}$-***Mod*** for $X \in$ ***Op***-T and $C^*(Y) \in \mathbf{S}^*$-***Mod*** for $Y \in$ ***Map***-BT. Note that $C^*(X)$ is in fact a $\mathbf{\Lambda}$-algebra by Lemma 2.4.1 (5), and $C^*(Y)$ fulfils all assumptions made in Section 1.8 to define a product on $\mathbf{h}C^*(Y)$.

We now construct the map $f\colon K \to C(ET)$, where $K = K(P)$ is the homological Koszul complex defined in Section 1.5. For $r = 1$ we recursively set

$$\begin{aligned} f(1 \otimes 1) &= e_0, \\ f(x \otimes x^l) &= x f(1 \otimes x^l) = x' f(1 \otimes x^l), \\ f(1 \otimes x^{l+1}) &= S f(x \otimes x^l), \end{aligned}$$

where $l \in \mathbf{N}$ and x' is the chosen representative simplex of x. For arbitrary r we compose this construction with the cross product,

$$\begin{aligned} f\colon K = K(x_1) \otimes \cdots \otimes K(x_r) & \\ &\to C(ES^1) \otimes \cdots \otimes C(ES^1) \\ &\to C(ES^1 \times \cdots \times ES^1) = C(ET), \end{aligned}$$

using (2.8). In particular, f is the identity of R for $r = 0$.

Proposition 2.11.1. *The map f is a map of $\mathbf{\Lambda}$-coalgebras, and its dual is a chain map of weak $\mathbf{S}^*$-modules without higher order terms.*

Recall that the second assertions means that

$$1 \otimes f^*\colon \mathbf{\Lambda}^* \tilde{\otimes} C^*(ET) \to \mathbf{\Lambda}^* \tilde{\otimes} K^*, \qquad \alpha \otimes \gamma \mapsto \alpha \otimes f^*\gamma,$$

is a $\mathbf{\Lambda}$-equivariant chain map. Here $\mathbf{\Lambda}$ acts of course only on the first factor of both tensor products.

PROOF. We may assume $r = 1$ for the first claim because the shuffle map is a map of coalgebras and in addition equivariant if all but one space of a product have trivial group action, see Lemma 2.4.1 (2).

We start by verifying that f is a chain map. By induction and Proposition 2.7.1 we have for all $l \in \mathbf{N}$ (with the convention $x \otimes x^{-1} := 0$)

$$\begin{aligned} df(1 \otimes x^l) &= dSf(x \otimes x^{l-1}) = f(x \otimes x^{l-1}) - Sdf(x \otimes x^{l-1}) \\ &= f(x \otimes x^{l-1}) - Sf(d(x \otimes x^{l-1})) = f(x \otimes x^{l-1}), \\ df(x \otimes x^l) &= d(xf(1 \otimes x^l)) = -xdf(1 \otimes x^l) = -xf(d(1 \otimes x^l)) \\ &= -xf(x \otimes x^{l-1}) = -(x \wedge x)f(1 \otimes x^{l-1}) = 0, \end{aligned}$$

which proves the claim. The $\mathbf{\Lambda}$-equivariance of f follows directly from the definition. (Note that $C(ET)$ is a $\mathbf{\Lambda}$-coalgebra by Lemma 2.4.1 (5).) To show that f commutes with comultiplication, we have to establish the identities

$$AW\Delta_* f(1 \otimes x^l) = \sum_{m+n=l} f(1 \otimes x^m) \otimes f(1 \otimes x^n), \tag{2.19a}$$

$$\begin{aligned} AW\Delta_* f(x \otimes x^l) &= \sum_{m+n=l} f(x \otimes x^m) \otimes f(1 \otimes x^n) \\ &\quad + \sum_{m+n=l} f(1 \otimes x^m) \otimes f(x \otimes x^n). \end{aligned} \tag{2.19b}$$

They follow again by induction, Proposition 2.7.1 and the fact that the Alexander–Whitney map is $\mathbf{\Lambda}$-equivariant. Hence f is a map of left $\mathbf{\Lambda}$-coalgebras.

That f^* be a map of weak $\mathbf{S}^*$-modules without higher order terms translates into the conditions

$$f(k) \cap p^*\xi'_\mu = \begin{cases} f(k \cap \xi_i) & \text{if } \mu = \{i\}, \\ 0 & \text{otherwise} \end{cases}$$

for all $k \in K$ and $\emptyset \neq \mu \subset [r]$. Since we already know f to be a map of coalgebras, the first alternative simplifies for $r = 1$ to

$$\langle f(1 \otimes x), p^*\xi' \rangle = 1.$$

This holds by our choice of ξ' and χ, because

$$\begin{aligned} \langle f(1 \otimes x), p^*\xi' \rangle &= \langle S(x'e_0), d\chi \rangle = -\langle dS(x'e_0), \chi \rangle \\ &= \langle Sd(x'e_0) - x'e_0, \chi \rangle = -\langle x'e_0, \chi \rangle = 1. \end{aligned}$$

The general case now follows by equation (2.3c).

As for the second alternative, note that the cup_1 products determining in equation (1.30) the higher order elements ξ'_μ, $|\mu| > 1$, are here cross_1 products of cochains coming from different factors of the induced decomposition of the classifying space BT. Hence Lemma 2.3.2 (3) gives the desired result. □

The composition $p \circ f \colon K \to C(BT)$ factors through $\mathbf{S} = R \otimes_{\mathbf{\Lambda}} K$ because the action of T on BT, hence also that of $\mathbf{\Lambda}$, is trivial. Call this new map

$$\bar{f} \colon \mathbf{S} \to C(BT).$$

PROPOSITION 2.11.2. *The map $\bar{f}^* \colon C^*(BT) \to \mathbf{S}^*$ is a multiplicative c-equivalence of weak $\mathbf{S}^*$-modules without higher order terms. Moreover, it annihilates all cup_1 products and is natural with respect to componentwise maps.*

The existence of such a map is due to Gugenheim and May [**GM**, Thm. 4.1]. Though somewhat technical in nature, this result is of great importance, for instance to the study of the cohomology of homogeneous spaces of Lie groups, cf. *op. cit.* or [**MC**, §8.1]. Our derivation of the cohomology of toric varieties in Section 3.3 also depends crucially on it. The present construction of such a map is considerably simpler than the original one given in the appendix to [**GM**], but see also [**M′**, Remarks 13.7]. Before trying to prove this result, I have checked some examples with the help of the "Kenzo" program [**DS**].

Proof. Since f is a map of coalgebras by the preceding proposition, so is $\bar{f}$. Hence $\bar{f}^*$ is a map of algebras. Using again the previous result and the commutativity of the diagram

$$\begin{array}{ccc} C^*(ET) & \xrightarrow{\;f^*\;} & K^* \\ {\scriptstyle p^*}\big\uparrow & & \big\uparrow \\ C^*(BT) & \xrightarrow[\;\bar{f}^*\;]{} & \mathbf{S}^*, \end{array}$$

we see that $\bar{f}^*$ is also a map of weak $\mathbf{S}^*$-modules without higher order terms because $\mathbf{S}^*$ injects into K^*. Having chosen the ξ_i' as representatives of the $\xi_i \in \mathbf{S}^* = H^*(BT)$, we conclude that $H^*(f)$ is an isomorphism. The naturality of $\bar{f}^*$ with respect to componentwise maps of standard tori is clear.

That $\bar{f}^*$ annihilates all cup_1 products is equivalent to the vanishing of

$$ST_{BT,BT}\Delta_{BT*}\bar{f}\colon \mathbf{S} \to C(BT) \otimes C(BT)$$

and to that of

$$(p_* \otimes p_*)ST_{ET,ET}\Delta_{ET*}f\colon K \to C(BT) \otimes C(BT).$$

We actually prove the stronger statement that

$$ST_{BT,ET}\hat{\Delta}_* f = (p_* \otimes 1)ST_{ET,ET}\Delta_{ET*}f\colon K \to C(BT) \otimes C(ET)$$

vanishes, where $\hat{\Delta}$ is the canonical map $ET \to BT \times ET$. We proceed by double induction on the rank r of T and the degree of $c = x_\pi \otimes x^\alpha \in K$, the case $r = 0$ being trivial. If $r > 0$ and π non-empty, then $x_\pi = x_i \cdot a$ for some $a \in \mathbf{\Lambda}$ and some i. By Lemma 2.4.1 (4), the Steenrod map $ST_{BT,ET}$ is equivariant with respect to multiplication by x_i because the latter is of degree 1. Hence

$$\begin{aligned} ST\hat{\Delta}_* f(c) &= ST\hat{\Delta}_*\big(x_i \cdot f(a \otimes x^\alpha)\big) = ST\big(x_i \cdot \hat{\Delta}_* f(a \otimes x^\alpha)\big) \\ &= -x_i \cdot ST\hat{\Delta}_* f(a \otimes x^\alpha) = 0 \end{aligned}$$

by induction.

It remains the case $\pi = \emptyset$, i.e., $c = 1 \otimes x^\alpha$. We may assume all $\alpha_i > 0$. (Otherwise use the result for smaller r). Formula (2.12c) shows that the cross product of two chains lying in the image of the respective cone operators does so itself. This generalises readily to several factors. Since $SS = 0$ and $f(1 \otimes x_i^{\alpha_i}) = Sf(x_i \otimes x_i^{\alpha_i - 1})$ by definition, we conclude that

$$f(c) = Sdf(c) + dSf(c) = Sf(dc)$$

by Proposition 2.7.1. Applying equation (2.12b) yields

$$ST\Delta_* f(c) = ST\Delta_* Sf(dc) = -(S\otimes S)AW\Delta_* f(dc) - (1\otimes S)ST\Delta_* f(dc).$$

The first summand vanishes because f is map of coalgebras and $SS = 0$, cf. the explicit form (2.19b) of $AW\Delta_* f$. We project the remaining term to $C(BT)\otimes C(ET)$:

$$\begin{aligned} ST\hat{\Delta}_* f(c) &= -(p_*\otimes 1)ST\Delta_* f(c) \\ &= -(1\otimes S)(p_*\otimes 1)ST\Delta_* f(dc) \\ &= -(1\otimes S)ST\hat{\Delta}_* f(dc) = 0, \end{aligned}$$

again by induction. This finishes the proof. □

We now construct the map $\phi\colon \mathbf{\Lambda}^* \tilde{\otimes} C^*(BT) \to C^*(ET)$. This is considerably more involved. For a subset $\pi\subset[r]$ with at least two elements, we set

$$\chi_\pi = -p^*\xi'_{\pi'} \cup_1 \chi_{\pi^+}, \tag{2.20}$$

where π^+ again denotes the maximum of π and π' its complement in π, and finally

$$\zeta_\emptyset = 1, \qquad \zeta_\pi = -\sum_{\substack{(\mu,\nu)\vdash\pi\\ \pi^+\in\mu}} \{(\mu,\nu)\}\,\zeta_\nu\cup\chi_\mu \tag{2.21}$$

for non-empty π. Note that we have $\{\chi_\pi\} = \{\zeta_\pi\} = \{\pi\}$.

PROPOSITION 2.11.3. *The assignment*

$$\begin{aligned} \phi\colon \mathbf{\Lambda}^* \tilde{\otimes} C^*(BT) &\to C^*(ET) \\ \xi_\pi\otimes\beta &\mapsto \zeta_\pi\cup p^*\beta \end{aligned}$$

is a chain map of $\mathbf{\Lambda}$-$C^*(BT)$*-bimodules.*

Recall that the left $\mathbf{\Lambda}$-module structure of $C^*(ET)$ is defined via the canonical opposition of $C(T)$ on the chain complex of the left T-space ET. Moreover, $\mathbf{\Lambda}^*\tilde{\otimes} C^*(BT)$ is a $\mathbf{\Lambda}$-$C^*(BT)$-bimodule, because the right $C^*(BT)$-module structure of $C^*(BT)$ carries over, cf. formulas (1.26a) and (1.28).

PROOF. The map ϕ is a chain map if (and only if) the relations

$$\{\pi\}\,d\zeta_\pi = \sum_{\substack{(\mu,\nu)\vdash\pi\\ \mu\neq\emptyset}} \{(\mu,\nu)\}\,\zeta_\nu\cup p^*\xi'_\mu \tag{2.22}$$

hold for all $\pi\subset[r]$. These can be verified inductively using the formula

$$\{\pi\}\,d\chi_\pi = -p^*\xi'_\pi - \sum_{\substack{(\mu,\nu)\vDash\pi\\ \pi^+\in\mu}} \{\mu\}\{(\mu,\nu)\}\,p^*\xi'_\nu\cup\chi_\mu + \sum_{\substack{(\mu,\nu)\vDash\pi\\ \pi^+\in\nu}} \{(\mu,\nu)\}\,\chi_\nu\cup p^*\xi'_\mu \tag{2.23}$$

for non-empty π, which is a consequence of equation (1.25) and the definition of χ_π. Details for this and the following calculations appear in Appendix 11.

It is obvious that ϕ is a map of right $C^*(BT)$-modules. By formula (1.15a), the $\mathbf{\Lambda}$-equivariance of ϕ is equivalent to the identities

$$x_i\cdot\zeta_\pi = \begin{cases} -\{(\pi\setminus i, i)\}\,\zeta_{\pi\setminus i} & \text{if } i\in\pi,\\ 0 & \text{otherwise}\end{cases} \tag{2.24}$$

for all $i \in [r]$ and all $\pi \subset [r]$. They follow by induction from

$$x_i \cdot \chi_\pi = \begin{cases} 1 & \text{if } \pi = \{i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.25}$$

Let us prove the preceding line: We clearly have $x_i \cdot \chi_j = 0$ for $i \neq j$ because in this case the χ_j comes from a factor of ET on which x_i acts trivially. Similarly, $x_i \cdot \chi_\pi$ vanishes for $|\pi| \geq 2$: The cup_1 product in (2.20) is in fact a cross_1 product with the first factor coming from a trivial T-space, namely a factor of BT. Hence

$$x_i \cdot \chi_\pi = -\{\pi\}\, p^*\xi'_{\pi'} \cup_1 x_i \cdot \chi_{\pi^+} = 0$$

by Lemmas 2.4.1 (3) and 2.3.2 (1). It remains to show $x_i \cdot \chi_i = x \cdot \chi = 1$. Note that $x \cdot \chi$ is a cocycle because

$$d(x \cdot \chi) = -x \cdot d\chi = -x \cdot p^*\xi' = -p^*(x \cdot \xi') = 0.$$

Since ET is connected it suffices therefore to evaluate $x \cdot \chi$ on a single vertex. By our choice of χ and the fact that the "inverted simplex" $\iota_* x'$ is homologous to $-x'$, we obtain

$$\langle e_0, x \cdot \chi \rangle = \langle e_0 \cdot x, \chi \rangle = \langle 1 \cdot x, i^*_{S^1}\chi \rangle = \langle \iota_* x', i^*_{S^1}\chi \rangle = -\langle x', i^*_{S^1}\chi \rangle = 1, \tag{2.26}$$

which finally proves (2.25). □

We remark that ϕ (as well as the maps Φ_Y to be defined below) fall into a class of maps $U\colon H^*(F) \otimes C^*(B) \to C^*(F \times_\tau B)$ considered in [**Hi**] for (almost) arbitrary fibres. This is also the setting for the article [**Mi**] referred to in the previous chapter.

12. Comparing the Koszul functors

With the help of the maps f and ϕ introduced in the last section we can now compare the simplicial Koszul functors with their algebraic counterparts.

We define the natural transformation

$$\Psi\colon C^* \circ \boldsymbol{t} \to \mathbf{t} \circ C^*$$

by letting Ψ_X be the dual of the bottom row of the following commutative diagram:

$$\begin{array}{ccccc} C(X) \otimes K & \xrightarrow{1 \otimes f} & C(X) \otimes C(ET) & \xrightarrow{\nabla} & C(X \times ET) \\ \downarrow & & \downarrow & & \downarrow q_{X*} \\ C(X) \underset{\mathbf{\Lambda}}{\otimes} K & \longrightarrow & C(X) \underset{C(T)}{\otimes} C(ET) & \longrightarrow & C(X \underset{T}{\times} ET). \end{array} \tag{2.27}$$

(Recall from Lemma 1.5.1 that $\mathbf{t}C^*(X)$ is the dual of $C(X) \otimes_{\mathbf{\Lambda}} K$.) Note that Ψ_X is the map f^* for $X = T$ and $\bar{f}^*$ for $X = *$. We conclude from Lemma 2.8.1 and Proposition 2.11.1 that Ψ_X is a morphism of weak $\mathbf{S}^*$-modules without higher order terms.

Lemma 2.12.1. *The map Ψ_X is a map of algebras for all right T-spaces X.*

Proof. The top row of diagram (2.27) is a map of coalgebras by Proposition 2.11.1 and Corollary 2.2.2. The projection to the coalgebra $C(X \times_T ET)$ factors through $C(X) \otimes_{\mathbf{\Lambda}} K$ by Lemma 1.3.1. Proposition 1.8.1 finally asserts that the algebra structure on $\mathbf{t}C^*(X)$ is dual to the coalgebra structure on $C(X) \otimes_{\mathbf{\Lambda}} K$. □

For the natural transformation

$$\Phi\colon \mathbf{h}\circ C^* \to C^* \circ \boldsymbol{h},$$

the $\mathbf{\Lambda}$-equivariant morphism Φ_Y is defined as the composition along the (by Proposition 2.11.3 well-defined) bottom row of the commutative diagram

$$\begin{array}{ccccc}
 & \mathbf{\Lambda}^* \tilde{\otimes} C^*(BT) \otimes C^*(Y) & \xrightarrow{\phi\otimes 1} & C^*(ET)\otimes C^*(Y) & \xrightarrow{\times} & C^*(Y\times ET) \\
 & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbf{h}C^*(Y) = & \mathbf{\Lambda}^* \tilde{\otimes} C^*(BT) \underset{C^*(BT)}{\otimes} C^*(Y) & \longrightarrow & C^*(ET) \underset{C^*(BT)}{\otimes} C^*(Y) & \longrightarrow & C^*(Y \overset{BT}{\times} ET).
\end{array}$$

Note that Ψ is natural with respect to componentwise maps and Φ natural with respect to monotone maps.

We can now state the main results of the present work. Recall that we have defined a c-equivalence of functors as a natural transformation all of whose morphisms are c-equivalences.

THEOREM 2.12.2. *The natural transformations Ψ and Φ are c-equivalences.*

PROOF. Let X be a T-space. We want to use the Leray–Serre theorem to show that Ψ_X is a c-equivalence of $\mathbf{S}^*$-modules. This map respects the filtrations because it is the dual of the composition

$$F\colon C(X)\tilde{\otimes}\,\mathbf{S} \to C(X)\underset{C(T)}{\otimes} C(ET) \to C(X\underset{T}{\times} ET),$$

which is filtration-preserving: For $c \in C_q(X)$ and $l \in \mathbf{N}$ the "base component" of

$$F(c\otimes x^l) = q_{X*}\nabla\bigl(c\otimes f(1\otimes x^l)\bigr)$$

is q-fold degenerate by the definition of the shuffle map. Hence Ψ_X induces a map of spectral sequences. We know from equation (1.39) and the Leray–Serre theorem that both E^2 terms are (abstractly) isomorphic, but we need that the map

$$E^2(\Psi_X)\colon \mathbf{S}^*\tilde{\otimes} H^*(X) \cong E^2(X,B) \to E^2(\mathbf{t}C^*(X)) = \mathbf{S}^*\tilde{\otimes} H^*(X) \tag{2.28}$$

is such an isomorphism. The componentwise map $1\to T$ induces the canonical inclusion of the fibre $X \hookrightarrow X_T$, and the corresponding inclusion $0 \hookrightarrow H_1(T) = P$ induces the augmentation $\mathbf{S}^* \to R$. Since all the constructions involved are natural with respect to these maps, we may conclude from the commutative diagram

$$\begin{array}{ccc}
H^*(X) & \longleftarrow & E^2(X,B) \\
= \big\downarrow & & \big\downarrow E^2(\Psi_X) \\
H^*(X) & \longleftarrow & E^2(\mathbf{t}C^*(X))
\end{array}$$

that Ψ_X is an isomorphism on the column $p=0$. (Here we use $\mathbf{S}^0 = R$.) By the preceding lemma and the Leray–Serre theorem the map (2.28) is $\mathbf{S}^*$-equivariant. The $p=0$ columns generating both E^2 terms over $\mathbf{S}^*$, the connecting map must be an isomorphism for all p and q. Hence $H(\Psi_X)$ is an isomorphism, too.

Though one could give a similar proof for Φ, we prefer to deduce the dual result in Corollary 2.12.4 (2) from our analysis of how both natural transformations relate. □

THEOREM 2.12.3.
1. *For all right T-spaces X the left diagram below is a commutative diagram of chain maps of complexes which induce isomorphisms of $\mathbf{\Lambda}$-algebras in homology.*
2. *For all spaces Y over BT the right diagram below is a commutative diagram of chain maps of complexes which induce isomorphisms of $\mathbf{S}^*$-algebras in homology.*

$$
\begin{array}{ccc}
C^*(\boldsymbol{ht}X) & & \\
\Phi_{\boldsymbol{t}X}\uparrow & \searrow & \\
\mathbf{h}C^*(\boldsymbol{t}X) & & C^*(X) \\
\mathbf{h}\Psi_X\downarrow & \nearrow & \\
\mathbf{ht}C^*(X) & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & C^*(\boldsymbol{th}Y) \\
 & \nearrow & \downarrow\Psi_{\boldsymbol{h}Y} \\
C^*(Y) & & \mathbf{t}C^*(\boldsymbol{h}Y) \\
 & \searrow & \uparrow\mathbf{t}\Phi_Y \\
 & & \mathbf{th}C^*(Y)
\end{array}
$$

PROOF. We write $\mathbf{t}_T$ for $\mathbf{t}_{H_1(T)}$ and similarly for $\mathbf{h}$. The left diagram above is a condensed version of the diagram of chain maps

$$
\begin{array}{ccccc}
C^*(\boldsymbol{h}_T\boldsymbol{t}_TX) & \longrightarrow & C^*(\boldsymbol{h}_1\boldsymbol{t}_TX) & \longrightarrow & C^*(\boldsymbol{h}_1\boldsymbol{t}_1X) \\
\Phi_{T,\boldsymbol{t}_TX}\uparrow & & \Phi_{1,\boldsymbol{t}_TX}\uparrow & & \Phi_{1,\boldsymbol{t}_1X}\uparrow \\
\mathbf{h}_TC^*(\boldsymbol{t}_TX) & \longrightarrow & \mathbf{h}_1C^*(\boldsymbol{t}_TX) & \longrightarrow & \mathbf{h}_1C^*(\boldsymbol{t}_1X) \\
\mathbf{h}_T\Psi_{T,X}\downarrow & & \mathbf{h}_1\Psi_{T,X}\downarrow & & \mathbf{h}_1\Psi_{1,X}\downarrow \\
\mathbf{h}_T\mathbf{t}_TC^*(X) & \longrightarrow & \mathbf{h}_1\mathbf{t}_TC^*(X) & \longrightarrow & \mathbf{h}_1\mathbf{t}_1C^*(X),
\end{array}
$$

which commutes by naturality of the Koszul functors with respect to the monotone homomorphism $1 \to T$. (More precisely, the homomorphism $1 \to T^r$ to our chosen (in fact, to any) decomposition of T is monotone.) Note that on the right hand side all complexes are equal to $C^*(X)$ and all maps to the identity. The compositions along the top and bottom row are just the canonical maps

$$C^*(\boldsymbol{ht}X) \to C^*(X) \qquad \text{and} \qquad \mathbf{ht}C^*(X) \to C^*(X),$$

from (2.13a) and (1.18a), which induce isomorphisms of $\mathbf{\Lambda}$-algebras in homology by Theorem 2.8.2 (1), Theorem 1.6.3 (1), and Proposition 1.8.3.

The middle row is multiplicative as well: Denoting the canonical inclusion $X \hookrightarrow X_T$ by i_X, one finds for the former map the formula

$$\mathbf{\Lambda}^* \mathbin{\tilde{\otimes}} C^*(\boldsymbol{t}X) \to C^*(X), \quad \alpha \otimes \gamma \mapsto \varepsilon(\alpha)\, i_X^*\gamma.$$

Thus, the multiplicativity follows from the explicit formula (1.36) and the fact that $i_X^* \xi'_\pi = 0$ for all $\pi \neq \emptyset$.

Taking homology therefore transforms the big outer square into the commutative diagram

$$\begin{array}{ccc}
H^*(\boldsymbol{ht}X) & & \\
{\scriptstyle H(\Phi_{\boldsymbol{t}X})}\big\uparrow & \searrow & \\
H(\mathbf{h}C^*(\boldsymbol{t}X)) & \longrightarrow & H^*(X). \\
{\scriptstyle H(\mathbf{h}\Psi_X)}\big\downarrow & \nearrow & \\
H(\mathbf{ht}C^*(X)) & &
\end{array}$$

Now Ψ_X is a map of weak $\mathbf{S}^*$-modules (without higher order terms), hence $\mathbf{h}\Psi_X$ is $\boldsymbol{\Lambda}$-equivariant. By Theorem 2.12.2 and Proposition 1.9.2 the map $H(\mathbf{h}\Psi_X)$ is an isomorphism of $\boldsymbol{\Lambda}$-modules, hence so are $H(\Phi_{\mathbf{t}X})$ and the middle row.

The horizontal and diagonal maps being multiplicative, the vertical maps on the left must finally be maps of algebras, too.

The second part follows analogously from naturality with respect to the projection $T \to 1$. The proof uses Lemma 1.9.1 and the yet unproven part of Theorem 2.12.2, but is itself not needed for the following corollary. $\square$

Corollary 2.12.4.

1. *The map $H(\Psi_X)$ is an isomorphism of $\mathbf{S}^*$-algebras for all right T-spaces X.*
2. *The map $H(\Phi_Y)$ is an isomorphism of $\boldsymbol{\Lambda}$-algebras for all spaces Y over BT.*

Proof. The first assertion is a weakening of Lemma 2.12.1 and Theorem 2.12.2. By Theorem 2.12.3 the second claim is true for all Y in the image of the functor $\boldsymbol{t}$. For arbitrary Y there is a c-equivalence $Y \to \boldsymbol{th}Y$ over BT, hence a commutative diagram

$$\begin{array}{ccc}
\mathbf{h}C^*(Y) & \longleftarrow & \mathbf{h}C^*(\boldsymbol{th}Y) \\
{\scriptstyle \Phi_Y}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{\boldsymbol{th}Y}} \\
C^*(\boldsymbol{h}Y) & \longleftarrow & C^*(\boldsymbol{hth}Y),
\end{array}$$

the rows of which are multiplicative c-equivalences by Propositions 1.9.2 and 2.8.5, whence the assertion. $\square$

One cannot expect the map Φ_Y itself to be multiplicative, as the example $Y = *$ shows: Here $\Phi_Y \colon \boldsymbol{\Lambda}^* \to C^*(T)$ maps from a commutative to a non-commutative algebra.

CHAPTER 3

Applications

1. The Cartan model

In the introduction we have mentioned the Cartan model expressing the equivariant cohomology of a G-manifold X in terms of the de Rham complex. Now we want to show how to recover this result from our theory in the case of torus actions.

Recall that the exterior derivative of differential forms, the contraction of forms by vector fields, and the Lie derivative are related by the formulas

$$\mathbf{i}_Y\, \mathbf{i}_Z\, \gamma + \mathbf{i}_Z\, \mathbf{i}_Y\, \gamma = 0, \tag{3.1a}$$

$$L_Y\, \mathbf{i}_Z\, \gamma - \mathbf{i}_Z\, L_Y \gamma = \mathbf{i}_{L_Y Z}\, \gamma, \tag{3.1b}$$

$$d\, \mathbf{i}_Y\, \gamma + \mathbf{i}_Y\, d\gamma = L_Y \gamma, \tag{3.1c}$$

$$dL_Y \gamma - L_Y d\gamma = 0. \tag{3.1d}$$

(See for example [**AMR**, §6.4], or [**GS**, Sec. 2.2] for a nice interpretation in terms of superalgebras.)

Let X be a manifold with a smooth left action of a compact torus T of rank r. Then T acts by diffeomorphisms on X, hence on vector fields and differential forms. In Section 2.11 we have chosen representative loops $x_i' \in C_1(T)$ of a basis of $H_1(T)$. We now assume that they are actually one-parameter subgroups, i. e., of the form $t \mapsto \lambda_i(t) = e^{y_i t}$ for some y_i in the Lie algebra of T. A differential form γ on X is T-invariant if and only if its Lie derivative $L_{Y_i}\gamma$ vanishes for all generating vector fields

$$Y_i(x) = \left.\frac{d}{dt} x e^{y_i t}\right|_{t=0} \tag{3.2}$$

on X. It follows from equation (3.1d) that the T-invariant differential forms are a subcomplex $\Omega^T(X)$ of the de Rham complex $\Omega(X)$. One has $L_{Y_i} Y_j = 0$ by [**AMR**, Prop. 4.2.27] because T is commutative. Hence equation (3.1b) implies that $\Omega^T(X)$ is stable under contraction by the vector fields Y_i. Unfortunately, the exterior differential turns out not to be compatible with our definition (1.3) of the differential on $C^*(X)$. We therefore introduce the new differential

$$D\gamma = \{\gamma\}\, d\gamma$$

on $\Omega(X)$ and $\Omega^T(X)$. Equations (3.1a) and (3.1c) show that $\Omega^T(X)$ (with new differential) becomes a $\mathbf{\Lambda}$-module if we define

$$x_i \cdot \gamma = \{\gamma\}\, \mathbf{i}_{Y_i}\, \gamma.$$

PROPOSITION 3.1.1. *The $\mathbf{\Lambda}$-module $\Omega^T(X)$ is c-equivalent to the singular cochain complex of X with real coefficients. Here $C^*(X)$ is a $\mathbf{\Lambda}$-module by the sweep action induced from the opposite, hence right T-action on X.*

Corollary 3.1.2. *The Cartan model*

$$\mathbf{S}^* \tilde{\otimes} \Omega^T(X), \qquad d(\sigma \otimes \gamma) = -\sum_{i=1}^{r} \xi_i \sigma \otimes \mathbf{i}_{Y_i} \gamma + \sigma \otimes d\gamma,$$

computes the real equivariant cohomology of the T-manifold X as $\mathbf{S}^*$-module.

Note that this is the complex (0.1) since T operates trivially on $\mathbf{S}^* = H^*(BT)$.

Proof (of the corollary). The above complex is just $\mathbf{t}\Omega^T(X)$ with differential scaled by $\{\gamma\} = \{\sigma \otimes \gamma\}$. Hence the assertion follows immediately from Theorem 2.12.2 and the fact that the Koszul functors preserve c-equivalence, see Proposition 1.9.2. □

The multiplicative structure is the right one, too, but the comparison of the wedge product of differential forms and the cup product of cochains is more involved.

Proof (of the proposition). The restriction

$$C^*(X) \to C^*_\infty(X) \tag{3.3}$$

of singular cochains to smooth simplices is a c-equivalence of complexes, as are the inclusion

$$\Omega^T(X) \hookrightarrow \Omega(X)$$

and the integration map

$$\Omega(X) \to C^*_\infty(X), \qquad \gamma \mapsto \Big(c \mapsto \int_c \gamma\Big),$$

see for example [**Bre**, Secs. V.9 & V.12]. (The last map would not be a chain map if we had not scaled the exterior differential appropriately, cf. [**Bre**, Sec. V.5].) Note that we use a negative grading on the de Rham complex.

The subcomplex of smooth simplices in X is $\mathbf{\Lambda}$-stable because T acts smoothly. Hence $C^*_\infty(X)$ is a $\mathbf{\Lambda}$-module, and the map (3.3) equivariant. It remains to show the $\mathbf{\Lambda}$-equivariance of the composition

$$I\colon \Omega^T(X) \hookrightarrow \Omega(X) \to C^*_\infty(X).$$

(This is a special case of [**GKM**, Prop. 18.4].) Let γ be a T-invariant differential form on X and $c\colon \Delta_n \to X$ a smooth simplex of degree $n = -|\gamma| - 1$, and denote by $\alpha\colon X \times T \to X$ the T-action on X. The left and right $\mathbf{\Lambda}$-action on simplices are related by the opposition ι_* induced from the group inversion, cf. Section 2.4. Hence

$$\langle c, x_i \cdot I(\gamma)\rangle = \langle c \cdot x_i, I(\gamma)\rangle = \{c\} \left\langle \iota_* x_i' \cdot c, I(\gamma)\right\rangle$$

where $\iota_* x_i'$ is the "inverted" representative $t \mapsto \lambda_i^{-1}(t) = e^{-2\pi i t}$,

$$= \{c\} \int_{\iota_* x_i' \cdot c} \gamma = \{c\} \int_{\lambda^{-1} \times c} \alpha^* \gamma = \{c\} \int_{\Delta_1 \times \Delta_n} (\lambda^{-1}, c)^* \alpha^* \gamma.$$

More precisely, one has to integrate over the simplicial subdivision of $\Delta_1 \times \Delta_n$ determined by the shuffle map. Since the sign of the permutation in the formula for the shuffle map compensates for the different orientations of the simplices, this is the same as integrating over the product $\Delta_1 \times \Delta_n$. Now $\beta = (\lambda^{-1}, c)^* \alpha^* \gamma$ is constant with respect to the first coordinate because α is T-invariant. This means $\beta = dy \wedge \beta'$ for a form β' on Δ_n and the canonical volume form dy on the interval Δ_1. Let Y be the unit vector field on $\Delta_1 \times \Delta_n$ along the y coordinate. Then $\mathbf{i}_Y\,\beta = (\mathbf{i}_Y\,dy) \wedge \beta' = \beta'$ and $(\lambda_* Y)(1) = y_i$. We may therefore continue

$$\begin{aligned}\langle c, x_i \cdot I(\gamma)\rangle &= \{c\} \int\limits_{\Delta_1 \times \Delta_n} dy \wedge \beta' = \{c\} \int\limits_{\Delta_1} dy \int\limits_{\Delta_n} \beta' = \{c\} \int\limits_{\Delta_n} \mathbf{i}_Y\,\beta \\ &= \{c\} \int\limits_{\Delta_n} \mathbf{i}_Y (\lambda^{-1}, c)^* \alpha^* \gamma = -\{c\} \int\limits_{\Delta_n} c^*(\mathbf{i}_{Y_i}\,\gamma)\end{aligned}$$

by the definition of the generating vector field Y_i,

$$= \{\gamma\} \int\limits_{c} \mathbf{i}_{Y_i}\,\gamma = \langle c, I(x_i \cdot \gamma)\rangle,$$

which was to be shown. □

2. Equivariantly formal spaces

We have seen in the previous chapter that the equivariant cohomology of a space X with an action of a torus T can be computed from the ordinary cochain complex $C^*(X)$ together with the action of $H(T) = \mathbf{\Lambda}$ on it. We now turn to a class of spaces where one cohomology already determines the other, at least at modules. We assume as in the second part of Section 1.9 that R is a principal ideal domain.

A T-space X is called **equivariantly formal** if its cochain complex is a split and trivial $\mathbf{\Lambda}$-module, i. e., if $C^*(X) \sim H^*(X)$ and the $\mathbf{\Lambda}$-action on the latter is trivial.

Recall that there is a canonical inclusion $i_X \colon X \hookrightarrow X_T$ over the unique vertex $b_0 \in BT$ inducing a map of algebras $H^*(i_X)\colon H^*_T(X) \to H^*(X)$.

PROPOSITION 3.2.1. *The following are equivalent for every T-space X:*

1. *X is equivariantly formal.*
2. *$H^*_T(X)$ is an extended $\mathbf{S}^*$-module.*
3. *There exists to $H^*(i_X)$ a section of graded R-modules.*
4. *The Leray–Serre spectral sequence for X_T degenerates on the E_2 level, and there is no composition problem, i. e., $H^*_T(X) \cong \mathbf{S}^* \otimes H^*(X)$ as $\mathbf{S}^*$-modules.*

Under these conditions there is an isomorphism of algebras

$$H^*(X) \cong R \underset{\mathbf{S}^*}{\otimes} H^*_T(X) = H^*_T(X)/\mathbf{S}^{>0} H^*_T(X). \tag{3.4}$$

PROOF. These are essentially reformulations of Propositions 1.9.5 and 1.9.6 because $\Psi_X \colon C^*(X_T) \to \mathbf{t}C^*(X)$ is a c-equivalence of $\mathbf{\Lambda}$-modules by Theorem 2.12.2.

To see that condition (3) is equivalent to Proposition 1.9.6 (2) note that the inclusion $i_{X*}\colon C(X) \hookrightarrow C(X_T)$ factors through $C(X) \otimes_{\mathbf{\Lambda}} K$, hence $i_X^* = j_{C^*(X)} \Psi_X$. This gives in particular (3.4).

Since we know from the proof of Theorem 2.12.2 that the Leray–Serre spectral sequence for X_T is isomorphic to the spectral sequence (1.39) from the E^2-term on, the last condition above is the same as Proposition 1.9.6 (3). □

In the literature there seems to be no universal definition of equivariant formality of a space with respect to a coefficient ring that is not a field. Some authors only assume that the Leray-Serre spectral sequence degenerates on the E^2 level.

The list of T-spaces known to be equivariantly formal over the reals includes:

- spaces with vanishing cohomology in odd degrees,
- smooth complete complex algebraic varieties with algebraic torus actions,
- compact symplectic manifolds with Hamiltonian torus actions,

cf. [**GKM**, Thm. 14.1] and [**We**].

Suppose that the T-space X satisfies the assumptions of the localisation theorem for singular cohomology [**AP**, Sec. 3.1]. For instance, X might be a finite-dimensional T-CW-complex. Suppose further that R is a field and that the Betti sum $\dim H^*(X)$ is finite. Then the Betti sum of the fixed point set X^T of X satisfies

$$\dim H^*(X^T) \leq \dim H^*(X);$$

equality holds if and only if X is equivariantly formal [**AP**, Cor. 3.1.15]. Since the Euler characteristics of X and X^T coincide [**AP**, Cor. 3.1.13], one may replace H^* by H^{even} or H^{odd} in this assertion. These conditions for equivariant formality carry over to $R = \mathbf{Z}$ if in addition $H^*(X)$ is free. (Here the Betti sum of X^T is the dimension of $H^*(X^T) \otimes \mathbf{Q}$.)

For rational (or real) coefficients, the localisation theorem implies a nice description of the equivariant cohomology for equivariantly formal spaces. To wit, by a result of Chang and Skjelbred (cf. [**Hs**, §VI.2], or [**GKM**, Sec. 6.3]) the sequence

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X^T) \longrightarrow H_T^*(X_1, X^T)$$

is exact, where X_1 denotes the equivariant 1-skeleton of X, i. e., the union of all orbits of dimension not greater than 1. The last map above is the boundary operator of the long exact cohomology sequence of the pair (X_1, X^T).

We finally give an example, due to [**GKM**, Secs. 1.5 & 11.3], that a T-space X may not be equivariantly formal, although $H^*(X)$ is trivial as $\mathbf{\Lambda}$-module. This shows in particular that a $\mathbf{\Lambda}$-module N is not necessarily split if the module structure on $H(N)$ is trivial: For $T = S^1$ the $\mathbf{\Lambda}$-module structure on $H^*(S^{2k+1})$, $k \geq 2$, is trivial for all actions, in particular for the free "Hopf action" with quotient $\mathbf{C}P^k$. But in this case $H_{S^1}^*(S^{2k+1}) = H^*(\mathbf{C}P^k)$ is not a free $\mathbf{S}^*$-module (unless $R = 0$).

3. Toric varieties

In this section we apply our general theory to toric varieties, on which we assume a basic knowledge, see for example [**Fu**], [**Ew**], or [**K**]. We will need the Mayer–Vietoris double complex for the cover of a space by finitely many subspaces [**BT**, §15]. This is a generalisation of the well-known Mayer–Vietoris exact sequence.

Let $\mathcal{U} = (U_i)_{i \in I}$ be a **cover** of the space (i. e., simplicial set) X by finitely many subspaces. This means that each simplex of X lies in some U_i. We assume the index set I to be totally ordered. Let $U_{i_0,\dots,i_p}$ denote the intersection $U_{i_1} \cap \dots \cap U_{i_p}$, and

$$\phi_i \colon U_i \hookrightarrow X \qquad \text{and} \qquad \phi_{i_0,\dots,i_p;i_k} \colon U_{i_0,\dots,i_p} \hookrightarrow U_{i_0,\dots,\widehat{i_k},\dots,i_p}$$

the canonical inclusions for $i_1 < \dots < i_p$, $p > 0$. Define complexes

$$E^p(\mathcal{U}) = \bigoplus_{i_0 < \dots < i_p} C(U_{i_0,\dots,i_p})$$

with differential D, the direct sum of the differentials of the respective chain complexes, and maps between them,

$$\delta\colon E^{p+1}(\mathcal{U}) \to E^p(\mathcal{U}), \quad C(U_{i_0,\dots,i_{p+1}}) \ni c \mapsto \bigoplus_{k=0}^{p+1}(-1)^k(\phi_{i_0,\dots,i_p;i_k})_*(c),$$

$$\varepsilon = \bigoplus_i (\phi_i)_*\colon E^0 \to C(X).$$

Then the sequence of complexes

$$0 \longleftarrow C(X) \xleftarrow{\ \varepsilon\ } E^0(\mathcal{U}) \xleftarrow{\ \delta\ } E^1(\mathcal{U}) \xleftarrow{\ \delta\ } \cdots$$

is exact. In particular, $\varepsilon\colon E(\mathcal{U}) \to C(X)$ is a c-equivalence, where $E(\mathcal{U})$ is the total complex associated to the sequence $(E^p(\mathcal{U}))$ above. This is a double complex with horizontal differential δ, vertical differential D and total differential

$$d_{pq} = D + (-1)^q\delta.$$

Consequently, $\varepsilon^*\colon C^*(X) \to E^*(\mathcal{U})$ is a c-equivalence as well. (Since we will only need $E^*(\mathcal{U})$ in the sequel, we will be somewhat sloppy as far as the sign convention (1.3) is concerned.) We call $E^*(\mathcal{U})$ the (cohomological) Mayer–Vietoris complex associated to the cover $\mathcal{U}$.

The complex $E^*(\mathcal{U})$ is actually a bigraded algebra such that ε^* is a map of algebras: The product of $\alpha \in C^q(U_{i_0,\dots,i_p})$ and $\beta \in C^{q'}(U_{j_0,\dots,j_{p'}})$ is defined to be 0 if $i_p \neq j_0$, and $(-1)^{pq'}\alpha \cup \beta$ otherwise. Here the product is taken after restriction of both cochains to $U_{i_0,\dots,i_p=j_0,j_1,\dots,j_{p'}}$. (Note that this is the Alexander–Whitney map in a different guise.)

Filtering $E^*(\mathcal{U})$ by p-degree, we get the (cohomological) Mayer–Vietoris spectral sequence with E_1 term

$$E_1^{pq}(\mathcal{U}) = \bigoplus_{i_0<\cdots<i_p} H^q(U_{i_0,\dots,i_p})$$

and differential

$$d_1^{pq}([\alpha]) = \sum_{k=0}^{p}(-1)^k H^*(\phi_{i_0,\dots,i_p;i_k})([\alpha]).$$

The inclusion ε^* induces a map of algebras

$$E_1(\varepsilon)\colon H^*(X) \to E_1(\mathcal{U}). \tag{3.5}$$

This establishes a link between the cohomology of a space and that of its subspaces. Hence if the latter is known, one might deduce results about the former.

If $\mathcal{U} = (U_i)$ is a finite cover of a topological space, then the space $\mathcal{S}X$ is not the union of the subspaces $\mathcal{S}U_i$ in general, because a singular simplex in X may not lie entirely in one U_i. But if all U_i are open in X then the inclusion

$$\mathcal{S}^{\mathcal{U}}X := \bigcup_i \mathcal{S}U_i \hookrightarrow \mathcal{S}X \tag{3.6}$$

induces a chain homotopy equivalence. (We have already used this in Section 2.9.) Hence the Mayer–Vietoris sequence applies to finite open covers of topological spaces. (One can generalise everything done so far to countable covers, but we will not need that.)

Now let $X = X_\Sigma$ be the toric variety associated to a fan $\Sigma \subset N \cong \mathbf{Z}^r$. The algebraic torus $T_N \cong (\mathbf{C}^*)^r$ acts on X. We want to determine the T_N-equivariant cohomology of X.

Cover Σ by (the fans generated by the elements of) some subset $\mathcal{U} \subset \Sigma$, for example by the set of maximal cones in Σ. (And any other cover is a superset of this.) This induces an open cover of X by the toric subvarieties X_σ, $\sigma \in \mathcal{U}$, which we continue to call $\mathcal{U}$. The inclusion (3.6) is equivariant, hence we may substitute $\mathcal{S}^{\mathcal{U}}X$ for $\mathcal{S}X$ in what follows by Proposition 2.9.2 (1). We also drop the functor $\mathcal{S}$ from our notation.

The spaces $\boldsymbol{t}X_\sigma$ form a cover, again called $\mathcal{U}$, of the Borel construction $\boldsymbol{t}\mathcal{S}^{\mathcal{U}}X$. The motivation for this procedure is that the equivariant cohomology of an affine toric variety X_σ is simple to describe: Choose a lattice complement to $N_\sigma = \sigma - \sigma$ in N. This corresponds to a decomposition of X_σ into a non-degenerate toric variety Y_σ and a torus T', compatible with the decomposition of T into T' and the torus T_σ associated to the lattice N_σ. The torus T_σ acts on Y_σ, and the canonical inclusion $Y_\sigma \hookrightarrow X_\sigma$ induces an isomorphism in equivariant cohomology because

$$(X_\sigma \times ET)/T = (Y_\sigma \times T' \times ET_\sigma \times ET')/(T_\sigma \times T') \cong (Y_\sigma \times ET_\sigma)/T_\sigma \times ET',$$

and the latter factor is contractible. The fact that any non-degenerate affine toric variety is equivariantly contractible to its unique fixed point [**Fu**, Sec. 2.3] implies that $(Y_\sigma \times ET_\sigma)/T_\sigma$ is homotopy equivalent to BT_σ over BT_σ. Summarising, the sequence of equivariant maps

$$X_\sigma \longleftarrow Y_\sigma \longrightarrow *$$

induces the c-equivalence

$$\boldsymbol{t}_{T_N} X_\sigma \longleftarrow \boldsymbol{t}_{T_\sigma} Y_\sigma \longrightarrow \boldsymbol{t}_{T_\sigma} * = BT_\sigma \tag{3.7}$$

in ***MapCl***. As a consequence we deduce an isomorphism of algebras

$$H_T^*(X_\sigma) \cong R[\sigma] := S(N_\sigma^* \otimes R) \tag{3.8}$$

between the T-equivariant cohomology of X_σ and the symmetric algebra $R[\sigma]$ of the tensor product $N_\sigma^* \otimes R$ (over $\mathbf{Z}$). This isomorphism is natural with respect to morphisms between such toric varieties. In particular, the module structure of $H_T^*(X_\sigma)$ over $\mathbf{S}^* = H_T^*(X_0)$ corresponds to the obvious multiplication in $R[\sigma]$ by polynomials defined on all of N.

All this shows that the term $E_1(\mathcal{U})$ of the Mayer–Vietoris spectral sequence associated to the cover $\mathcal{U}$ of $\boldsymbol{t}\mathcal{S}^{\mathcal{U}}X$ is very simple: It consists of a bunch of polynomial algebras on certain subspaces of N with differential induced by the various inclusions. But there is another way to think about $E_1(\mathcal{U})$: Assume $R = \mathbf{Q}$ for the moment. Then a polynomial on $N_\sigma \otimes R$ is essentially the same as its restriction to (the rational points in) σ. Consider the vector space of piecewise polynomial functions on the support

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$$

of Σ, i. e., all functions on $|\Sigma|$ that are polynomial on each cone σ. The complex $E_1(\mathcal{U})$ looks like the Mayer–Vietoris double complex of this vector space associated to the cover $\mathcal{U}$ of $|\Sigma|$. If one wants to formalise this idea, it is more convenient to take a purely algebraic approach:

Assume that the fan Σ is ***R*-regular**. This means that for each cone $\sigma \in \Sigma$ the images in $N_\sigma \otimes R$ of the generators of the rays of σ form a basis. For $R = \mathbf{Z}$ this is the usual notion of a regular fan, made up of regular cones, and a $\mathbf{Q}$-regular fan contains only simplicial cones. The **Stanley–Reisner ring** $R[\Sigma]$ of Σ is the quotient of the evenly graded polynomial algebra $R[\{\rho : \rho \text{ ray in } \Sigma\}]$ by all monomials $\rho_1 \cdots \rho_s$ such that the rays appearing in this product do not all belong to a single cone in Σ. We call the minimal cone containing all rays of a remaining monomial the "cone referred to by this monomial." The ring $R[\Sigma]$ is an $\mathbf{S}^*$-module via the map of algebras

$$\mathbf{S}^* \to R[\Sigma], \qquad \mathbf{S}^2 \ni \xi \mapsto \sum_\rho \langle \rho, \xi \rangle \rho,$$

where $\langle \rho, \xi \rangle$ denotes the value of ξ on the minimal generator of ρ. Note that the Stanley–Reisner ring of an R-regular cone σ is isomorphic to $R[\sigma]$ as defined above. Moreover, the Stanley–Reisner ring of a, say, $\mathbf{Q}$-regular fan can be identified with the piecewise polynomial functions on $|\Sigma|$. Any inclusion $j\colon \Sigma' \hookrightarrow \Sigma$ of a subfan induces a projection of algebras $j^*\colon R[\Sigma] \to R[\Sigma']$ annihilating all monomials referring to cones in $\Sigma \setminus \Sigma'$. Choosing as Σ' a cone in $\mathcal{U}$, this leads in analogy with equation (3.5) to a map of $\mathbf{S}^*$-algebras

$$\tilde{\varepsilon}\colon R[\Sigma] \to E_1(\mathcal{U}).$$

Lemma 3.3.1. *If Σ is R-regular, then $\tilde{\varepsilon}$ is a c-equivalence of $\mathbf{S}^*$-algebras.*

Proof. We adapt the standard proof of the exactness of the Mayer–Vietoris double complex for singular (co)homology [**BT**, §15] and proceed by induction on the number of cones in the covering $\mathcal{U}$. For $|\mathcal{U}| = 1$ there is nothing to prove.

Consider a covering $\mathcal{U}$ of a fan Σ by more than one cone. Removing the according to the chosen total ordering greatest cone σ', we arrive at a covering $\mathcal{U}' = \mathcal{U} \setminus \sigma'$ of some subfan $\Sigma' \subset \Sigma$ by $|\mathcal{U}| - 1$ cones. The inclusion $j\colon \Sigma' \hookrightarrow \Sigma$ induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker j^* & \longrightarrow & R[\sigma'] & \longrightarrow & \bigoplus_{\sigma \in \mathcal{U}'} R[\sigma \cap \sigma'] & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R[\Sigma] & \longrightarrow & \bigoplus_{\sigma \in \mathcal{U}} R[\sigma] & \longrightarrow & \bigoplus_{\sigma,\tau \in \mathcal{U}} R[\sigma \cap \tau] & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle j^*} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R[\Sigma'] & \longrightarrow & \bigoplus_{\sigma \in \mathcal{U}'} R[\sigma] & \longrightarrow & \bigoplus_{\sigma,\tau \in \mathcal{U}'} R[\sigma \cap \tau] & \longrightarrow & \cdots .
\end{array}$$

Here the middle terms in all but the first column are the direct sums of the top and bottom terms, and the maps are the corresponding inclusions and projections, respectively. In particular, these columns are exact, as is the leftmost one. The bottom row is exact by induction hypothesis. If the top row is exact as well, then so is the middle one by the associated long exact homology sequence.

Note that the top row is, up to the first two terms, just the double complex associated to the cover of the fan $\{\sigma \in \Sigma' : \sigma \subset \sigma'\}$ induced by $\mathcal{U}'$, hence by induction exact from the third term on. The kernel of j^* is generated by all monomials referring to faces of σ' not in Σ'. This shows $\ker j^* \subset R[\sigma']$ and gives exactness of the top row at the remaining positions. □

THEOREM 3.3.2. *Let Σ be an R-regular fan. Then the equivariant cochain complex of X_Σ is c-equivalent to the Stanley–Reisner ring of Σ, including all multiplicative structure. In particular, there are isomorphisms of $\mathbf{S}^*$-algebras and $\mathbf{\Lambda}$-algebras, respectively,*

$$H_T^*(X_\Sigma) = R[\Sigma] \qquad \textit{and} \qquad H^*(X_\Sigma) = H(\mathbf{h}R[\Sigma]) = H(\mathbf{\Lambda}^* \tilde{\otimes} R[\Sigma]).$$

The ordinary cohomology of smooth projective toric varieties has been determined by Jurkiewicz. Danilov has extended this result to the compact case, cf. [**Fu**, Sec. 5.2]. The identification of the Stanley–Reisner ring as the equivariant cohomology is due to Davis–Januszkiewicz [**DJ**] and Brion [**Bri**]. Since smooth compact toric varieties are equivariantly formal, this implies the Jurkiewicz–Danilov result by Proposition 3.2.1. Buchstaber and Panov have shown (using a different method) that the equivariant cochain complex of a smooth toric variety is c-equivalent to the Stanley–Reisner ring [**BP**, Prop. 3.4.3]. This gives the ordinary cohomology by the Eilenberg–Moore theorem mentioned in the introduction, cf. [**BP**, Thms. 4.2.1 & 4.4.11]. Since this "simplified version" of Koszul duality does not account for the dual module structure, one only gets the cohomology as algebra. The description of the $\mathbf{\Lambda}$-module structure seems to be new.

PROOF. We construct the claimed c-equivalence in several steps. For the moment being, we will neglect all additional structure and take $C^*(\boldsymbol{t}X_\Sigma)$ as a complex.

Let $Q\colon (N', \Sigma') \to (N, \Sigma)$ be the Cox construction. More precisely: Let N' be the free $\mathbf{Z}$-module over the rays in Σ. For each cone in Σ define a cone in N' spanned by the same extremal rays, now considered as elements of N'. The collection of all these cones forms a fan Σ', isomorphic to Σ as partially ordered set because Σ is simplicial. The map Q sending each $\rho \in N'$ to its generator in N is a morphism of fans, hence induces a morphism $q\colon X_{\Sigma'} \to X_\Sigma$ of toric varieties.

1. We cover Σ by some subset $\mathcal{U}$ and replace $C^*(\boldsymbol{t}X_\Sigma)$ by the c-equivalent Mayer–Vietoris double complex $E^*(\mathcal{U})$. This has been discussed above.
2. Cover Σ' by the corresponding collection $\mathcal{U}'$. This covering being compatible with $\mathcal{U}$ by construction, the map $\boldsymbol{t}q\colon \boldsymbol{t}X_{\Sigma'} \to \boldsymbol{t}X_\Sigma$ passes to a map $E^*(\boldsymbol{t}q)\colon E^*(\mathcal{U}) \to E^*(\mathcal{U}')$ between their Mayer–Vietoris double complexes. The map Q induces an isomorphism $N_{\sigma'} \otimes R \to N_\sigma \otimes R$ for each $\sigma \in \Sigma$ with preimage σ' because it maps the canonical basis of $N_{\sigma'} \otimes R$ by hypothesis to a basis of $N_\sigma \otimes R$. This gives by equation (3.8) a c-equivalence
$$\boldsymbol{t}_{N'} X_{\sigma'} \to \boldsymbol{t}_N X_\sigma.$$
Since the E_1 terms of the Mayer–Vietoris spectral sequences are made up of these terms, we conclude that $E^*(\boldsymbol{t}q)$ is a c-equivalence.
3. Using the c-equivalence (3.7), we now replace each space $\boldsymbol{t}_{N'} X_{\sigma'}$ appearing in $E^*(\mathcal{U}')$ by $BT_{\sigma'}$, and the inclusions between them by those induced by the inclusions between the $T_{\sigma'}$. This gives a new complex E'' because the composition of maps is functorial, i. e., the composition
$$BT_{\rho'} \longrightarrow BT_{\sigma'} \longrightarrow BT_{\tau'}$$

does only depend on $\rho' \subset \tau' \in \Sigma'$. Since all c-equivalences in (3.7) are natural in σ, the complex E'' is c-equivalent to $E^*(\mathcal{U}')$.

4. Replace each cochain complex $C^*(BT_{\sigma'})$ in E'' by its homology with the help of the maps $\bar{f}^*$ constructed in Section 2.11. This is compatible with the inclusions between them because the latter maps are componentwise with respect to the canonical decomposition of $BT_{N'}$ given by the distinguished basis of N'. (This was the reason to pass to N'.) This thus yields a c-equivalence between E'' and the complex $E_1(\mathcal{U}') \cong E_1(\mathcal{U})$.
5. The complex $E_1(\mathcal{U})$ finally is c-equivalent to $R[\Sigma]$ by Lemma 3.3.1.

Now choose representatives for some basis of $\mathbf{S}^* = H^*(BT_N)$. This gives rise to a twisting cochain, hence to a differential and a product on $\mathbf{h}C^*(\boldsymbol{t}X_\Sigma)$. Due to the commutative diagram

$$\begin{array}{ccccccc} BT_{N'} & \longleftarrow & \boldsymbol{t}_{T_{N'}} X_{\Sigma'} & \longleftarrow & \mathcal{S}^{\mathcal{U}} \boldsymbol{t}_{T_{N'}} X_{\Sigma'} & \longleftarrow & \boldsymbol{t}_{T_{N'}} X_{\sigma'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ BT_N & \longleftarrow & \boldsymbol{t}_{T_N} X_\Sigma & \longleftarrow & \mathcal{S}^{\mathcal{U}} \boldsymbol{t}_{T_N} X_\Sigma & \longleftarrow & \boldsymbol{t}_{T_N} X_\sigma \end{array}$$

all replacements made in the first two steps are compatible with these structures. The third step is structure-preserving because of the commutative diagram

$$\begin{array}{ccc} BT_{\sigma'} & \longleftarrow & \boldsymbol{t}_{T_{\sigma'}} Y_{\sigma'} \\ \downarrow & & \downarrow \\ BT_{N'} & \longleftarrow & \boldsymbol{t}_{T_{N'}} X_{\sigma'}. \end{array}$$

The c-equivalence in the last step is one of $\mathbf{S}^*$-algebras. Here all Steenrod–Hirsch products are zero. So we only have to examine the fourth step in more detail: Since the maps $C^*(BT_{\sigma'}) \to H^*(BT_{\sigma'})$ are c-equivalences of algebras, natural with respect to the maps $T_{\sigma'} \hookrightarrow T_{N'}$, we conclude that multiplication by ξ_i' in $C^*(BT_{\sigma'})$ corresponds to multiplication by ξ_i in $H^*(BT_{\sigma'})$. We also need that all cup products by the ξ_π, $|\pi| \geq 2$, and the cup_1 products by all ξ_π are mapped to zero. But this is guaranteed by Proposition 2.11.2.

Hence the weak $\mathbf{S}^*$-module structure and the product on $R[\Sigma]$ are the right ones. By Koszul duality the formula for $H^*(X_\Sigma)$ follows. □

We finally exhibit a class of equivariantly formal toric varieties. Call a sequence $(\sigma_1, \ldots, \sigma_k)$ of full-dimensional simplicial cones generating a fan Σ a **shelling** of Σ if, for each i, the intersection of σ_i with $\sigma_1 \cup \cdots \cup \sigma_{i-1}$ is a union of facets of σ_i. A fan possessing a shelling is called **shellable**. (See for instance [**Z**, §8.1] for more about shellings. Our definition is there Remark 8.3 (ii).)

PROPOSITION 3.3.3. *Let R be a principal ideal domain and Σ a shellable R-regular fan. Then X_Σ is equivariantly formal.*

PROOF. By Proposition 3.2.1 we have to show that $H_T^*(X_\Sigma) = R[\Sigma]$ is a free $\mathbf{S}^*$-module. Let $(\sigma_1, \ldots, \sigma_k)$ be a shelling of Σ. We proceed by induction on k. For $k = 1$ we have $R[\Sigma] = \mathbf{S}^*$ because σ_1 is R-regular and full-dimensional. For $k > 1$ denote by Σ' the (shellable) fan generated by $(\sigma_1, \ldots, \sigma_{k-1})$. Then the inclusion $\Sigma' \hookrightarrow \Sigma$ induces the projection $p\colon R[\Sigma] \to R[\Sigma']$ annihilating all monomials referring to cones not in Σ'. By [**Z**, Exercise 8.2] or [**K**, Lemma 18.2 (1)], the set of these cones is just the star of some face $\tau \subset \sigma_k$, i. e., all faces of σ_k containing τ. Since σ_k is R-regular, we may assume that the rays of σ_k, considered as elements of $R[\Sigma]$, are the images of the generators $(\xi_1, \ldots, \xi_r)$ of $\mathbf{S}^*$. If ξ_1, ... , ξ_s are the rays of τ, then the kernel of p is generated by the monomial $\xi_1 \cdots \xi_s$. Therefore the kernel of p is free, as is by induction the image. Hence so is $R[\Sigma]$. □

This observation is well-known. Since the fans associated to projective toric varieties are shellable [**Z**, Thm. 8.11], this proves in particular Jurkiewicz's theorem. An exhaustive study of free Stanley–Reisner rings appears in [**BR**].

APPENDIX

The gory details

This appendix contains the simple, but lengthy verification of quite a few claims made in the main text. For those who think that mathematics is the formal manipulation of collections of symbols according to some list of transformation laws the following sections will be heaven.

1. Proof of Lemma 1.3.1

Let $p\colon B \otimes C \to B \otimes_\Lambda C$ be the projection. We have to verify that the composition

$$(p \otimes p)\Delta_{B\otimes C}\colon B \otimes C \to (B \otimes C) \otimes (B \otimes C) \to (B \underset{\Lambda}{\otimes} C) \otimes (B \underset{\Lambda}{\otimes} C)$$

factors through $B \otimes_\Lambda C$, and likewise for the augmentation $\varepsilon_{B\otimes C} = \varepsilon_B \otimes \varepsilon_C$. Let $b \in B$ and $c \in C$ have the images

$$\Delta_B b = \sum_i b'_i \otimes b''_i \qquad \text{and} \qquad \Delta_C c = \sum_j c'_j \otimes c''_j$$

under the diagonals. Then for a generator $x \in \Lambda$

$$\Delta_B(bx) = \sum_i \bigl(b'_i \otimes b''_i x + \{b''_i\}\, b'_i x \otimes b''_i\bigr),$$
$$\Delta_C(xc) = \sum_j \bigl(xc'_j \otimes c''_j + \{c'_j\}\, c'_j \otimes xc''_j\bigr),$$

hence

$$\Delta_{B\otimes C}(bx \otimes c)$$
$$= \sum_{i,j} \{b''_i x, c'_j\}\, (b'_i \otimes c'_j) \otimes (b''_i x \otimes c''_j) + \{b''_i\}\{b''_i, c'_j\}\, (b'_i x \otimes c'_j) \otimes (b''_i \otimes c''_j)$$

and

$$\Delta_{B\otimes C}(b \otimes xc)$$
$$= \sum_{i,j} \{c'_j\}\{b''_i, c'_j\}\, (b'_i \otimes c'_j) \otimes (b''_i \otimes xc''_j) + \{b''_i, xc'_j\}\, (b'_i \otimes xc'_j) \otimes (b''_i \otimes c''_j)$$

are equal in $B \otimes_\Lambda C$.

Since $\varepsilon(bx \otimes c) = \varepsilon(b \otimes xc) = 0$ by the $\mathbf{\Lambda}$-equivariance of the augmentations, we also get an induced augmentation $\varepsilon_{B\otimes_\Lambda C}$.

2. Proof of Lemma 1.5.1

It is clear that the map $f\colon \mathbf{S}^* \tilde{\otimes} N^* \to (N \tilde{\otimes} \mathbf{S})^*$ given in the lemma is bijective and $\mathbf{S}^*$-equivariant, so we just have to show that it commutes with differentials. We have

$$d(\sigma \otimes \nu) = -\sum_{i=1}^{r} \xi_i \sigma \otimes x_i \nu + \sigma \otimes d\nu,$$

in $\mathbf{S}^* \tilde{\otimes} N^*$ and

$$d(n \otimes s) = dn \otimes s + \{n\} \sum_{i=1}^{r} n x_i \otimes s \cap \xi_i$$

in $N \tilde{\otimes} \mathbf{S}$, hence

$$\begin{aligned}\big\langle n \otimes s, f(d(\sigma \otimes \nu))\big\rangle &= -\sum_i \langle n, x_i \nu\rangle \langle s, \xi_i \sigma\rangle + \langle n, d\nu\rangle\langle s, \sigma\rangle \\ &= -\sum_i \langle n x_i, \nu\rangle\langle s \cap \xi_i, \sigma\rangle - \{n\}\langle dn, \nu\rangle\langle s, \sigma\rangle \\ &= -\{n\}\big\langle d(n \otimes s), f(\sigma \otimes \nu)\big\rangle = \big\langle n \otimes s, d(f(\sigma \otimes \nu))\big\rangle,\end{aligned}$$

which was to be shown.

3. Proof of Proposition 1.6.1

Let $h\colon P' \to P$ be given by the matrix A_{ij} relative to the chosen bases of P' and P, respectively, i. e.,

$$h(x'_j) = \sum_i A_{ij} x_i \qquad \text{and} \qquad h^*(\xi_i) = \sum_j A_{ij} \xi'_j.$$

Denote the map associated with $f\colon N \to N'$ as in the lemma by $\mathbf{t}f$. If f is a chain map, then

$$\begin{aligned}\mathbf{t}f d(\sigma \otimes n) &= \mathbf{t}f\Big(-\sum_i \xi_i \sigma \otimes x_i n + \sigma \otimes dn\Big) \\ &= -\sum_i h^*(\xi_i \sigma) \otimes f(x_i n) + \sigma \otimes f(dn) \\ &= -\sum_i h^*(\xi_i) h^*(\sigma) \otimes f(x_i n) + \sigma \otimes f(dn) \\ &= -\sum_{i,j} A_{ij} \xi'_j h^*(\sigma) \otimes f(x_i n) + \sigma \otimes f(dn) \\ &= -\sum_j \xi'_j h^*(\sigma) \otimes f(h(x'_j) n) + \sigma \otimes df(n) \\ &= -\sum_j \xi'_j h^*(\sigma) \otimes x'_j f(n) + \sigma \otimes df(n) \\ &= d\mathbf{t}f(\sigma \otimes n),\end{aligned}$$

hence $\mathbf{t}f$ is a chain map, too. If f is a chain homotopy instead, say from f' to f'', we may continue from the middle line above

$$\begin{aligned}\mathbf{t}fd(\sigma\otimes n) &= -\sum_{i,j} A_{ij}\xi'_j h^*(\sigma)\otimes f(x_i n) + \sigma\otimes f(dn)\\ &= \sum_j \xi'_j h^*(\sigma)\otimes f(h(x'_j)n) - \sigma\otimes df(n) + \sigma\otimes f''(n) - \sigma\otimes f'(n)\\ &= -\sum_j \xi'_j h^*(\sigma)\otimes x'_j f(n) - \sigma\otimes df(n) + \sigma\otimes f''(n) - \sigma\otimes f'(n)\\ &= -d\mathbf{t}f(\sigma\otimes n) + \mathbf{t}f''(\sigma\otimes n) - \mathbf{t}f'(\sigma\otimes n),\end{aligned}$$

as it should be. The proof for the functor $\mathbf{h}$ is analogous.

4. Proof of Lemma 1.6.2

We first show that the composition $b_{(r)}\circ\cdots\circ b_{(1)}$ equals the map b from the statement of the lemma. Assume that we have already done that for $i-1$. Then, taking into account the way x_i acts and the signs due to the reordering

$$\mathbf{ht}\cong \mathbf{h}_{(x_i)}\circ\cdots\circ\mathbf{h}_{(x_1)}\circ\mathbf{t}_{(x_i)}\circ\cdots\circ\mathbf{t}_{(x_1)}\cong \mathbf{h}_{(x_i)}\circ\mathbf{t}_{(x_i)}\circ\cdots\circ\mathbf{h}_{(x_1)}\circ\mathbf{t}_{(x_1)},$$

we find

$$\begin{aligned}&(b_{(i)}\circ\cdots\circ b_{(1)})(m\otimes n)\\ &\quad= b_{(i)}\big((b_{(i-1)}\circ\cdots\circ b_{(1)})(m\otimes n)\big)\\ &\quad= \sum_{\pi\subset[i-1]}\{\pi\}\, m\otimes\xi_\pi\otimes 1\otimes x_\pi n - \sum_{\pi\subset[i-1]}\{\pi\}\{\pi\}\, m\otimes\xi_i\wedge\xi_\pi\otimes 1\otimes(x_i\wedge x_\pi)n\\ &\quad= \sum_{\pi\subset[i-1]}\{\pi\}\, m\otimes\xi_\pi\otimes 1\otimes x_\pi n - \sum_{\pi\subset[i-1]}\{\pi\}\, m\otimes\xi_i\wedge\xi_\pi\otimes 1\otimes(x_\pi\wedge x_i)n\\ &\quad= \sum_{\pi\subset[i]}\{\pi\}\, m\otimes\xi_\pi\otimes 1\otimes x_\pi n,\end{aligned}$$

which was to be shown.

We now consider a single pair $a_{(i)}$ and $b_{(i)}$, which we write from now on as a and b for simplicity. Moreover, we write x and ξ for x_i and ξ_i, respectively. Recall from equation (1.15b) the identity $x\cdot\xi=-1$.

The map a is a chain map: The differential on $M\tilde{\otimes}N_{(i)} = M\tilde{\otimes}\mathbf{h}_{(x_i)}\mathbf{t}_{(x_i)}N_{(i-1)}$ is determined by

$$\begin{aligned}d(m\otimes 1\otimes\xi^l\otimes n') &= dm\otimes 1\otimes\xi^l\otimes n' - \{m\}\sum_{k\neq i}\xi_k m\otimes 1\otimes\xi^l\otimes x_k n'\\ &\qquad - \{m\}\, m\otimes 1\otimes\xi^{l+1}\otimes xn' + \{m\}\, m\otimes 1\otimes\xi^l\otimes dn',\\ d(m\otimes\xi\otimes\xi^l\otimes n') &= dm\otimes\xi\otimes\xi^l\otimes n'\\ &\qquad + \{m\}\,\xi m\otimes 1\otimes\xi^l\otimes n' + \{m\}\sum_{k\neq i}\xi_k m\otimes\xi\otimes\xi^l\otimes x_k n'\\ &\qquad - \{m\}\, m\otimes 1\otimes\xi^{l+1}\otimes n' + \{m\}\, m\otimes\xi\otimes\xi^{l+1}\otimes xn'\\ &\qquad - \{m\}\, m\otimes\xi\otimes\xi^l\otimes dn'.\end{aligned}$$

Comparison with the differential on $M \tilde{\otimes} N_{(i-1)}$ shows that the additional terms cancel out after applying a, or are mapped to zero. The fact that b is a chain map can be checked directly, too, but it is also a consequence of the following calculations: We show that $H = H_{(i)}$ is a homotopy from the identity to ba. This implies that ba is a chain map, hence also b because a is surjective.

The map H assumes the values

$$\begin{aligned} H(m \otimes 1 \otimes \xi^l \otimes n') &= \{m\} \sum_{p+q=l-1} \xi^p m \otimes \xi \otimes \xi^q \otimes n', \\ H(m \otimes \xi \otimes \xi^l \otimes n') &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} dH(m \otimes 1 \otimes \xi^l \otimes n') = \sum_{p+q=l-1} \Big(& \{m\}\, \xi^p dm \otimes \xi \otimes \xi^q \otimes n' + \xi^{p+1} m \otimes 1 \otimes \xi^q \otimes n' \\ & + \sum_{k \neq i} \xi_k \xi^p m \otimes \xi \otimes \xi^q \otimes x_k n' \\ & - \xi^p m \otimes 1 \otimes \xi^{q+1} \otimes n' + \xi^p m \otimes \xi \otimes \xi^{q+1} \otimes xn' \\ & - \xi^p m \otimes \xi \otimes \xi^q \otimes dn' \Big), \\ Hd(m \otimes 1 \otimes \xi^l \otimes n') = \sum_{p+q=l-1} \Big(& -\{m\}\, \xi^p dm \otimes \xi \otimes \xi^q \otimes n' + \xi^p m \otimes \xi \otimes \xi^q \otimes dn' \\ & - \sum_{k \neq i} \xi^p \xi_k m \otimes \xi \otimes \xi^q \otimes x_k n' \Big) \\ & - \sum_{p+q=l} \xi^p m \otimes \xi \otimes \xi^q \otimes xn', \end{aligned}$$

whose sum is

$$\begin{aligned} \xi^l m \otimes 1 \otimes 1 \otimes n' - \xi^l \otimes \xi \otimes 1 \otimes xn' - m \otimes 1 \otimes \xi^l \otimes n' \\ = ba(m \otimes 1 \otimes \xi^l \otimes n') - m \otimes 1 \otimes \xi^l \otimes n', \end{aligned}$$

as it should be. Since

$$dH(m \otimes \xi \otimes \xi^l \otimes n') = 0$$

and

$$\begin{aligned} Hd(m \otimes \xi \otimes \xi^l \otimes n') &= \sum_{p+q=l-1} \xi^{p+1} m \otimes \xi \otimes \xi^q \otimes n' - \sum_{p+q=l} \xi^p m \otimes \xi \otimes \xi^q \otimes n' \\ &= -m \otimes \xi \otimes \xi^l \otimes n', \end{aligned}$$

we see that H is a homotopy from the identity to ba, as claimed.

5. Proof of Proposition 1.8.2

We first check formula (1.36). We may assume $\alpha = 1$ and $m' = 1$, and even $\alpha = \omega$, because this product is $\mathbf{\Lambda}$-equivariant as well: We have (trivially for $\mu \cap \pi \neq \emptyset$)

$$\begin{aligned} &x_\mu \cdot \big(\{\pi\}\{\pi,\omega\}\, x_\pi \cdot \omega \otimes \xi'_\pi * m\big) \\ &\quad = \{\pi\}\{\pi,\omega\}\{\pi,\mu\}(x_\pi \wedge x_\mu) \cdot \omega \otimes \xi'_\pi * m \\ &\quad = \{\pi\}\{\pi, x_\mu \cdot \omega\}\, x_\pi \cdot (x_\mu \cdot \omega) \otimes \xi'_\pi * m, \end{aligned}$$

whence

$$x_\mu \cdot \big((1 \otimes m)(\omega \otimes 1)\big) = \{\mu, m\}(1 \otimes m)(x_\mu \cdot \omega \otimes 1).$$

Here we have used that the first term on the right hand side of (1.36) is obviously equivariant.

This first summand is furthermore just the first term of (1.35), so to check the explicit form we just have to determine $D_t(\omega \otimes m)$ for $m \in M$. A careful reading of formula (1.28a) reveals

$$(1 \otimes t)\Delta(\omega) = \sum_{\pi \neq \emptyset} \{\xi'_\pi, \omega\}\, x_\pi \cdot \omega \otimes \xi'_\pi = \sum_{\pi \neq \emptyset} \{\pi,\omega\}\{\omega\}\, x_\pi \cdot \omega \otimes \xi'_\pi,$$

hence

$$D'_t(\omega \otimes m) = \sum_{\pi \neq \emptyset} \{\pi\}\{\pi,\omega\}\, x_\pi \cdot \omega \otimes \xi'_\pi * m,$$

and

$$(1 \otimes m)(\omega \otimes 1) = \{m,\omega\}\left(\omega \otimes m - D'_t(\omega \otimes m)\right) \tag{A.1}$$

has the correct form.

We finally show that the new product is a chain map. By what we already know it suffices to test this on a product of the form (A.1). (Use the bimodule structure.)

$$\begin{aligned} &\{m,\omega\}\, d\big((1 \otimes m)(\omega \otimes 1)\big) \\ &\quad = \sum_{\pi \neq \emptyset} \{\xi'_\pi, \omega\}\, x_\pi \cdot \omega \otimes \xi'_\pi m + \{\omega\}\, \omega \otimes dm \\ &\qquad - \sum_{\substack{\mu \cap \nu = \emptyset \\ \mu \neq \emptyset,\, \mu \neq \emptyset}} \{\mu\}\{\mu,\omega\}\{\xi'_\nu,\omega\}\{\mu\}\,(x_\mu \wedge x_\nu) \cdot \omega \otimes \xi'_\nu(\xi'_\mu * m) \\ &\qquad - \sum_{\pi \neq \emptyset} \{\pi\}\{\pi,\omega\}\{x_\pi \cdot \omega\}\, x_\pi \cdot \omega \otimes d(\xi'_\pi * m) \\ &\quad = \{\omega\} \sum_{\pi \neq \emptyset} \{\pi,\omega\}\, x_\pi \cdot \omega \otimes \xi'_\pi m + \{\omega\}\, \omega \otimes dm \\ &\qquad - \{\omega\} \sum_{\pi \neq \emptyset} \{\pi,\omega\}\, x_\pi \cdot \omega \otimes \sum_{(\mu,\nu) \vDash \pi} \{(\mu,\nu)\}\, \xi'_\nu(\xi'_\mu * m) \\ &\qquad - \{\omega\} \sum_{\pi \neq \emptyset} \{\pi,\omega\}\, x_\pi \cdot \omega \otimes d(\xi'_\pi * m). \end{aligned}$$

The properties (1.29) of the Steenrod–Hirsch product together with the known expression (1.25) of the differential of ξ'_π give

$$\begin{aligned} d(\xi'_\pi * m) &= \xi'_\pi m - \{\xi'_\pi, m\}\, m\xi'_\pi - d\xi'_\pi * m - \{\xi'_\pi\}\, \xi'_\pi * dm \\ &= \xi'_\pi m - \{\pi, m\}\{m\}\, m\xi'_\pi + \sum_{(\mu,\nu)\vDash\pi} \{\nu\}\{(\mu,\nu)\}\, \xi'_\nu\xi'_\mu * m + \{\pi\}\, \xi'_\pi * dm \\ &= \xi'_\pi m - \{\pi, m\}\{m\}\, m\xi'_\pi - \sum_{(\mu,\nu)\vDash\pi} \{(\mu,\nu)\}\, \xi'_\nu(\xi'_\mu * m) \\ &\quad + \sum_{(\mu,\nu)\vDash\pi} \{\nu\}\{(\mu,\nu)\}\{\mu, m\}\{m\}\, (\xi'_\nu * m)\xi'_\mu + \{\pi\}\, \xi'_\pi * dm \end{aligned}$$

The first term on the right hand side, if substituted into the preceding expression, cancels against the first term of that sum. The same is true for the respective third terms. Reordering the remaining summands yields

$$\begin{aligned} &\{m,\omega\}\, d\big((1\otimes m)(\omega\otimes 1)\big) \\ &\quad = \{\omega\}\, \omega\otimes dm - \{\omega\} \sum_{\pi\neq\emptyset} \{\pi\}\{\pi,\omega\}\, x_\pi\cdot\omega\otimes\xi'_\pi * dm \\ &\qquad + \sum_{\pi\neq\emptyset} \{\xi'_\pi,\omega\}\{\pi,m\}\{m\}\, x_\pi\cdot\omega\otimes m\xi'_\pi \\ &\qquad - \{\omega\} \sum_{\substack{\mu\cap\nu=\emptyset \\ \mu\neq\emptyset,\, \mu\neq\emptyset}} \{\mu\cup\nu,\omega\}\{\nu\}\{\nu,\mu\}\{\mu,m\}\{m\}(x_\nu\wedge x_\mu)\cdot\omega\otimes(\xi'_\nu * m)\xi'_\mu. \end{aligned}$$

Therefore,

$$\begin{aligned} &d\big((1\otimes m)(\omega\otimes 1)\big) \\ &\quad = \{dm,\omega\}\, \omega\otimes dm - \{dm,\omega\} \sum_{\pi\neq\emptyset} \{\pi\}\{\pi,\omega\}\, x_\pi\cdot\omega\otimes\xi'_\pi * dm \\ &\qquad + \{m\} \sum_{\pi\neq\emptyset} \{\xi'_\pi,\omega\}\{m, x_\pi\cdot\omega\}\, x_\pi\cdot\omega\otimes m\xi'_\pi \\ &\qquad - \{m\} \sum_{\substack{\mu\cap\nu=\emptyset \\ \mu\neq\emptyset,\, \mu\neq\emptyset}} \{\xi'_\mu,\omega\}\{m, x_\mu\cdot\omega\}\{\nu\}\{\nu, x_\mu\cdot\omega\}\, x_\nu\cdot(x_\mu\cdot\omega)\otimes(\xi'_\nu * m)\xi'_\mu \\ &\quad = (1\otimes dm)(1\otimes\omega) + \{m\}\, (1\otimes m) \sum_{\mu\neq\emptyset} \{\xi'_\mu,\omega\}\, x_\mu\cdot\omega\otimes\xi'_\mu \\ &\quad = d(1\otimes m)(1\otimes\omega) + \{m\}\, (1\otimes m) d(\omega\otimes 1), \end{aligned}$$

which was to be shown.

6. Some identities between face and degeneracy maps

We introduce for a set $\mu\subset\mathbf{N}$ and i, $j\in\mathbf{N}$ the abbreviations

$$\begin{aligned} \mu^i &= \mu^{<i} = \{k : k\in\mu,\, k<i\}, & \mu+i &= \{k+i : k\in\mu\}, \\ \mu_j &= \mu_{\geq j} - j, & \mu_{\geq j} &= \{k : k\in\mu,\, j\leq k\}, \\ \mu_i^j &= \mu_{\geq i}^{<j} - i, & \mu_{\geq i}^{<j} &= \mu^{<j}\cap\mu_{\geq i}. \end{aligned}$$

LEMMA A.6.1. *Let x be an n-simplex, $\mu \subset \mathbf{N}$ with $\mu \subset \{0,\dots,n+|\mu|-1\}$, $0 \le i < j \le n$, and $0 \le k \le n-j+i+1$. Then:*

$$\partial_{k+1}^{n-j+i+1}\partial_{i+1}^{j-1}x = \begin{cases} \partial_{k+1}^{n}x & \text{if } k \le i, \\ \partial_{i+1}^{j-1}\partial_{k+j-i}^{n}x & \text{if } i \le k, \end{cases}$$

$$\partial_0^{k-1}\partial_{i+1}^{j-1}x = \begin{cases} \partial_{i-k+1}^{j-k-1}\partial_0^{k-1}x & \text{if } k \le i+1, \\ \partial_0^{k+j-i-2}x & \text{if } i+1 \le k, \end{cases}$$

$$\partial_{i+1}^{n+|\mu|}s_\mu x = s_{\mu^i}\partial_{i+1-|\mu^i|}^{n}x,$$

$$\partial_0^{j-1}s_\mu x = s_{\mu_j}\partial_0^{j-1-|\mu^j|}x,$$

$$\partial_{i+1}^{j-1}s_\mu x = \begin{cases} s_{\mu_j+i+1}s_{\mu^{i+1}}x & \text{if } \{i,\dots,j-1\} \subset \mu, \\ s_{\mu_j+i+1}s_{\mu^i}\partial_{i+1-|\mu^i|}^{j-1-|\mu^j|}x & \text{otherwise}, \end{cases}$$

$$\partial_0^{i-1}\partial_{j+1}^{n+|\mu|}s_\mu x = s_{\mu_i^j}\partial_0^{i-1-|\mu^i|}\partial_{j+1-|\mu^j|}^{n}x.$$

PROOF. This is a repeated application of the commutation relations (2.1):

$$\partial_{k+1}^{n-j+i+1}\partial_{i+1}^{j-1}x = \begin{cases} \partial_{k+1}^{i}\partial_{i+1}^{n-j+i+1}\partial_{i+1}^{j-1}x = \partial_{k+1}^{i}\partial_{i+1}^{n} = \partial_{k+1}^{n}x & \text{if } k \le i, \\ \partial_{i+1}^{j-1}\partial_{k+j-i}^{n}x & \text{if } i \le k, \end{cases}$$

$$\partial_0^{k-1}\partial_{i+1}^{j-1}x = \begin{cases} \partial_{i-k+1}^{j-k-1}\partial_0^{k-1}x & \text{if } k-1 \le i, \\ \partial_0^{i}\partial_{i+1}^{k-1}\partial_{i+1}^{j-1}x = \partial_0^{i}\partial_{i+1}^{k+j-i-2}x = \partial_0^{k+j-i-2}x & \text{if } i+1 \le k. \end{cases}$$

To proceed, we need the formula

$$\partial_{i+1}^{j}s_k x = \begin{cases} s_k\partial_i^{j-1}x & \text{if } k < i \text{ or } i = j, \\ \partial_{i+1}^{j-1}x & \text{if } i \le k \le j \text{ and } i \ne j, \\ s_{k-j+i}\partial_{i+1}^{j}x & \text{if } j < k \text{ or } i = j. \end{cases}$$

Here the middle alternative is for $i \le k < j$ due to the identity

$$\partial_{i+1}^{j}s_k x = \partial_{i+1}^{k}\partial_{k+1}^{j}s_k x = \partial_{i+1}^{k}\partial_{k+1}s_k\partial_{k+1}^{j-1}x = \partial_{i+1}^{j-1}x.$$

This gives

$$\begin{aligned} \partial_{i+1}^{n+|\mu|}s_\mu x &= \partial_{i+1}^{n+|\mu|}s_{\mu_{\ge i}}s_{\mu^{<i}}x = \partial_{i+1}^{n+|\mu|-|\mu_{\ge i}|}s_{\mu^{<i}}x = s_{\mu_i}\partial_{i+1-|\mu^i|}^{n}x, \\ \partial_0^{j-1}s_\mu x &= \partial_0^{j-1}s_{\mu_{\ge j}}s_{\mu^{<j}}x = s_{\mu_j}\partial_0^{j-1}s_{\mu^{<j}}x = s_{\mu_j}\partial_0^{j-1-|\mu^j|}x, \\ \partial_{i+1}^{j-1}s_\mu x &= \partial_{i+1}^{j-1}s_{\mu_{\ge j}}s_{\mu_{\ge i}^{<j}}s_{\mu^{<i}}x = s_{\mu_j+i+1}\partial_{i+1}^{j-1}s_{\mu_{\ge i}^{<j}}s_{\mu^{<i}}x \\ &= \begin{cases} s_{\mu_j+i+1}s_i s_{\mu^{<i}}x = s_{\mu_j+i+1}s_{\mu^{i+1}}x & \text{if } |\mu_i^j| = j-i, \\ s_{\mu_j+i+1}\partial_{i+1}^{j-1-|\mu_{\ge i}^{<j}|}s_{\mu^{<i}}x = s_{\mu_j+i+1}s_{\mu^i}\partial_{i+1-|\mu^i|}^{j-1-|\mu^j|}x & \text{otherwise.} \end{cases} \end{aligned}$$

Combining the first two of the last identities, we finally get

$$\partial_0^{i-1}\partial_{j+1}^{n+|\mu|}s_\mu x = \partial_0^{i-1}s_{\mu^j}\partial_{j+1-|\mu^j|}^{n}x = s_{\mu_i^j}\partial_0^{i-1-|\mu^i|}\partial_{j+1-|\mu^j|}^{n}x.$$

This concludes the proof. □

7. Proof of Proposition 2.2.1

For any $(x,y) \in (X \times Y)_p$ and $(z,w) \in (Z \times W)_q$ we have by Lemma A.6.1

$$\begin{aligned}
&AW(\mathrm{id},\tau,\mathrm{id})_*\nabla\big((x,y)\otimes(z,w)\big)\\
&= \sum_{\substack{0\le i\le p+q\\(\mu,\nu)\vdash(p,q)}} \{(\mu,\nu)\}\,\partial_{i+1}^{p+q}(s_\nu x, s_\mu z)\otimes\partial_0^{i-1}(s_\nu y, s_\mu w)\\
&= \sum_{\substack{0\le i\le p+q\\(\mu,\nu)\vdash(p,q)}} \{(\mu,\nu)\}\,(s_{\nu^i}\partial^p_{i+1-|\nu^i|}x, s_{\mu^i}\partial^q_{i+1-|\mu^i|}z)\otimes(s_{\nu_i}\partial_0^{i-1-|\nu^i|}y, s_{\mu_i}\partial_0^{i-1-|\mu^i|}w).
\end{aligned}$$

Now introduce $\alpha=\mu^i$, $\beta=\nu^i$, $\kappa=\mu_i$, and $\lambda=\nu_i$ and note that by equations (1.1) and (1.2)

$$\begin{aligned}
\{(\mu,\nu)\} &= \{(\mu^{<i}\cup\mu_{\ge i}, \nu^{<i}\cup\nu_{\ge i})\} = \{(\mu^{<i},\mu_{\ge i},\nu^{<i},\nu_{\ge i})\}\\
&= \{\mu_{\ge i},\nu^{<i}\}\{(\mu^{<i},\nu^{<i},\mu_{\ge i},\nu_{\ge i})\} = \{\mu_{\ge i},\nu^{<i}\}\{(\mu^{<i},\nu^{<i})\}\{(\mu_{\ge i},\nu_{\ge i})\}\\
&= \{\mu_i,\nu^i\}\{(\mu^i,\nu^i)\}\{(\mu_i,\nu_i)\} = \{\kappa,\beta\}\{(\alpha,\beta)\}\{(\kappa,\lambda)\}.
\end{aligned}$$

Replacing i by $|\alpha|+|\beta|$ and introducing new $i=|\alpha|$ and $j=|\beta|$, we may proceed writing

$$\begin{aligned}
&AW(\mathrm{id},\tau,\mathrm{id})_*\nabla\big((x,y)\otimes(z,w)\big)\\
&= \sum_{\substack{0\le i\le p,\ 0\le j\le q\\(\alpha,\beta)\vdash(i,j)\\(\kappa,\lambda)\vdash(p-i,p-j)}} \{\kappa,\beta\}\{(\alpha,\beta)\}\{(\kappa,\lambda)\}(s_\beta\partial^p_{i+1}x, s_\alpha\partial^q_{j+1}z)\otimes(s_\lambda\partial_0^{i-1}y, s_\kappa\partial_0^{j-1}w)\\
&= (\nabla\otimes\nabla)\sum_{i=0}^{p}\sum_{j=0}^{q}(-1)^{(p-i)j}\partial^p_{i+1}x\otimes\partial^q_{j+1}z\otimes\partial_0^{i-1}y\otimes\partial_0^{j-1}w\\
&= (\nabla\otimes\nabla)(1\otimes T\otimes 1)(AW\otimes AW)\big((x,y)\otimes(z,w)\big).
\end{aligned}$$

Hence the diagram commutes already before normalisation.

8. Proof of Lemma 2.3.1

We start by verifying formula 2.3.1 (3). Lemma A.6.1 yields for all simplices $(x,y,z)\in(X\times Y\times Z)_n$

$$\begin{aligned}
AW\partial_{i+1}^{j-1}(y,z) &= \sum_{k=0}^{n-j+i+1}\partial^n_{k+1}\partial^{j-1}_{i+1}y\otimes\partial_0^{k-1}\partial_{i+1}^{j-1}z\\
&= \sum_{k=0}^{i}\partial^n_{k+1}y\otimes\partial^{j-k-1}_{i-k+1}\partial_0^{k-1}z + \sum_{k=i+1}^{n-j+i+1}\partial^{j-1}_{i+1}\partial^n_{k+j-i}y\otimes\partial_0^{k+j-i-2}z
\end{aligned}$$

and

$$\partial_0^{i-1}\partial^n_{j+1}x = \begin{cases}\partial_0^{i-k-1}\partial_0^{k-1}\partial^n_{j+1}x = \partial_0^{i-k-1}\partial^{n-k}_{j-k+1}\partial_0^{k-1}x & \text{if } k\le i,\\ \partial_0^{i-1}\partial^{j+k-i-1}_{j+1}\partial^n_{j+k-i}x & \text{if } i<k.\end{cases}$$

Hence

$$(1\otimes AW)ST\big(x,(y,x)\big) = -\sum_{0\le i<j\le n}(-1)^{(i+1)(j+1)}\partial_0^{i-1}\partial_{j+1}^{n}x\otimes AW\partial_{i+1}^{j-1}(y,z)$$

$$= -\sum_{0\le k\le i<j\le n}(-1)^{(i+1)(j+1)}\partial_0^{i-k-1}\partial_{j-k+1}^{n-k}\partial_0^{k-1}x\otimes\partial_{k+1}^{n}y\otimes\partial_{i-k+1}^{j-k-1}\partial_0^{k-1}z$$

$$-\sum_{\substack{0\le i<j\le n\\ i<k\le n-j+i+1}}(-1)^{(i+1)(j+1)}\partial_0^{i-1}\partial_{j+1}^{j+k-i-1}\partial_{k+j-i}^{n}x\otimes\partial_{i+1}^{j-1}\partial_{k+j-i}^{n}y\otimes\partial_0^{k+j-i-2}z$$

Now replace i and j in the first sum by $i+k$ and $j+k$, respectively, and k by $k-j+i+1$ in the second. This gives

$$= -\sum_{\substack{0\le k\le n\\ 0\le i<j\le n-k}}(-1)^{(i+k+1)(j+k+1)}\partial_0^{i-1}\partial_{j+1}^{n-k}\partial_0^{k-1}x\otimes\partial_{k+1}^{n}y\otimes\partial_{i+1}^{j-1}\partial_0^{k-1}z$$

$$-\sum_{\substack{0\le k\le n\\ 0\le i<j\le k}}(-1)^{(i+1)(j+1)}\partial_0^{i-1}\partial_{j+1}^{k}\partial_{k+1}^{n}x\otimes\partial_{i+1}^{j-1}\partial_{k+1}^{n}y\otimes\partial_0^{k-1}z$$

Now the second term on the right hand side is equal to $(ST\otimes 1)AW\big((x,y),z\big)$. As to the first, note that modulo 2

$$\text{(A.2)}\qquad \begin{aligned}(i+k+1)(j+k+1) &\equiv (i+1)(j+1)+k(i+j+2)+k^2\\ &\equiv (i+1)(j+1)+k(1+j-i),\end{aligned}$$

whence

$$(1\otimes AW)ST\big(x,(y,x)\big)$$

$$= -\sum_{\substack{0\le k\le n\\ 0\le i<j\le k}}(-1)^{(i+1)(j+1)+k(1+j-i)}\partial_0^{i-1}\partial_{j+1}^{n-k}\partial_0^{k-1}x\otimes\partial_{k+1}^{n}y\otimes\partial_{i+1}^{j-1}\partial_0^{k-1}z$$

$$= -(T\otimes 1)\sum_{k=0}^{n}(-1)^k\sum_{0\le i<j\le k}(-1)^{(i+1)(j+1)}\partial_{k+1}^{n}y\otimes\partial_0^{i-1}\partial_{j+1}^{n-k}\partial_0^{k-1}x\otimes\partial_{i+1}^{j-1}\partial_0^{k-1}z$$

because the degrees of the transposed factors are k and $j-i$, respectively,

$$= (T\otimes 1)(1\otimes ST)\sum_{k=0}^{n}\partial_{k+1}^{n}y\otimes\partial_0^{k-1}(x,z)$$

by definition (1.7) of $1\otimes ST$,

$$= (T\otimes 1)(1\otimes ST)AW\big(y,(x,z)\big) = (T\otimes 1)(1\otimes ST)AW(\tau,\mathrm{id})_*(x,y,z).$$

We now prove the commutativity of diagram 2.3.1 (4). Let $x\in X_p$ and $(y,z)\in(Y\times Z)_q$. Then

$$ST\nabla\big(x\otimes(y,z)\big) = ST\sum_{(\mu,\nu)\vdash(p,q)}\{(\mu,\nu)\}\,(s_\nu x,s_\mu y,s_\mu z)$$

$$= -\sum_{\substack{(\mu,\nu)\vdash(p,q)\\ 0\le i<j\le p+q}}(-1)^{(i+1)(j+1)}\{(\mu,\nu)\}\,\partial_0^{i-1}\partial_{j+1}^{p+q}(s_\nu x,s_\mu y)\otimes\partial_{i+1}^{j-1}s_\mu z$$

For the last component not to be degenerate, it is necessary by Lemma A.6.1 that $\mu^i = \mu_j = \emptyset$ and $\{i, \dots, j-1\} \not\subset \mu$, i. e., that μ be a proper subset of $\{i, \dots, j-1\}$. This implies in particular $i < j-p$. Then $|\nu^i| = i$, $|\nu^j| = j-p$, and $|\nu_j| = p+q-j$. We may therefore continue

$$= -\sum_{\substack{(\mu,\nu)\vdash(p,q)\\ 0\le i<j-p\le q\\ \mu^i=\mu_j=\emptyset}} (-1)^{(i+1)(j+1)}\{(\mu,\nu)\}\big(s_{\nu_i^j}x, s_{\mu-i}\partial_0^{i-1}\partial_{j+1-p}^q y\big) \otimes \partial_{i+1}^{j-1-p} z$$

Now define κ and λ by $\mu = \kappa + i$ and $\nu = \{0, \dots, i-1\} \mathbin{\dot\cup} (\lambda + i) \mathbin{\dot\cup} \{j, \dots, p+q-1\}$ and replace j by $j+p$. Then the above is equal to

$$\begin{aligned}
&= -\sum_{\substack{(\kappa,\lambda)\vdash(p,j-i)\\ 0\le i<j\le q}} (-1)^{(i+1)(j+p+1)+ip}\{(\kappa,\lambda)\}\big(s_\lambda x, s_\kappa\partial_0^{i-1}\partial_{j+1}^q y\big) \otimes \partial_{i+1}^{j-1} z\\
&= -(-1)^p(\nabla\otimes 1)\sum_{0\le i<j\le q}(-1)^{(i+1)(j+1)} x\otimes\partial_0^{i-1}\partial_{j+1}^q y\otimes\partial_{i+1}^{j-1} z\\
&= (\nabla\otimes 1)(1\otimes ST)\big(x\otimes(y,z)\big),
\end{aligned}$$

where the factor $(-1)^p$ has disappeared due to definition (1.7).

We finally check identity 2.3.1 (5), which is trivial if z is a vertex. So let (x, y) be a p-simplex in $X \times Y$ and $z \in Z_1$. We have

$$ST_{X,Y\times Z}\nabla_{X\times Y,Z}\big((x,y)\otimes z\big) = \sum_{k=0}^{p}(-1)^{p-k} ST(s_k x, s_k y, s_{p\setminus k} z)$$

with the abbreviation $p \setminus k = \{0, \dots, p\} \setminus k$. By Lemma A.6.1 we must have $k < i$ or $k \ge j$ in order for the first component in formula (2.4) not to be degenerate. Then in particular $\{i, \dots, j-1\} \subset p \setminus k$, and

$$\begin{aligned}
&= -\sum_{0\le k<i<j\le p+1}(-1)^{p-k+(i+1)(j+1)}\partial_0^{i-2}\partial_j^p x\otimes(s_k\partial_i^{j-2}y, s_{(p-j+i+1)\setminus k}z)\\
&\quad-\sum_{0\le i<j\le k\le p}(-1)^{p-k+(i+1)(j+1)}\partial_0^{i-1}\partial_{j+1}^p x\otimes(s_{k-j+i+1}\partial_{i+1}^{j-1}y, s_{(p-j+i+1)\setminus(k-j+i+1)}z)
\end{aligned}$$

Now substitute in the first sum $i+1$ and $j+1$ for i and j, respectively, and $k+j-i-1$ for k in the second. This gives

$$\begin{aligned}
&= -\sum_{\substack{0\le i<j\le p\\ 0\le k\le p-j+i+1}}(-1)^{p-(k+j-i-1)+(i+1)(j+1)}\partial_0^{i-1}\partial_{j+1}^p x\otimes(s_k\partial_{i+1}^{j-1}y, s_{(p-j+i+1)\setminus k}z)\\
&= (1\otimes\nabla_{YZ})(ST_{XY}\otimes 1)\big((x,y)\otimes z\big),
\end{aligned}$$

as claimed.

9. Proof of Proposition 2.7.1

Using formulas (2.10) one finds for $e \in EG_n$ and $0 \le i \le j \le n$

$$\partial_0^{i-1} Se = \begin{cases} Se & \text{if } i = 0, \\ \partial_0^{i-2} e & \text{otherwise,} \end{cases}$$
$$\partial_{j+1}^{n+1} Se = \begin{cases} e_0 & \text{if } j = 0, \\ S\partial_j^n e & \text{otherwise,} \end{cases}$$

and for $i < j$ also

$$\partial_0^{i-1} \partial_{j+1}^{n+1} Se = \partial_0^{i-1} S\partial_j^n e = \begin{cases} S\partial_j^n e & \text{if } i = 0, \\ \partial_0^{i-2}\partial_j^n e & \text{otherwise,} \end{cases}$$
$$\partial_{i+1}^{j-1} Se = S\partial_i^{j-2} e.$$

Hence for $(e, e') \in (EG \times EH)_n$,

$$\begin{aligned} AWS(e,e') &= \sum_{i=0}^{n+1} \partial_{i+1}^{n+1} Se \otimes \partial_0^{i-1} Se' = e_0 \otimes Se' + \sum_{i=1}^{n+1} S\partial_i^n e \otimes \partial_0^{i-2} e' \\ &= e_0 \otimes Se' + \sum_{i=0}^{n} S\partial_{i+1}^n e \otimes \partial_0^{i-1} e' = e_0 \otimes Se' + (S \otimes 1)AW(e,e'); \\ STS(e,e') &= -\sum_{0 \le i < j \le n+1} (-1)^{(i+1)(j+1)}\, \partial_0^{i-1}\partial_{j+1}^{n+1} Se \otimes \partial_{i+1}^{j-1} Se' \\ &= -\sum_{0 < j \le n+1} (-1)^{j+1} S\partial_j^n e \otimes S\partial_0^{j-2} e' \\ &\quad - \sum_{0 < i < j \le n+1} (-1)^{(i+1)(j+1)}\, \partial_0^{i-2}\partial_j^n e \otimes S\partial_i^{j-2} e' \\ &= -\sum_{0 \le j \le n} (-1)^j S\partial_{j+1}^n e \otimes S\partial_0^{j-1} e' \\ &\quad - \sum_{0 \le i < j \le n} (-1)^{j-i+1+(i+1)(j+1)}\, \partial_0^{i-1}\partial_{j+1}^n e \otimes S\partial_{i+1}^{j-1} e', \end{aligned}$$

where we have used equation (A.2), and by definition (1.7) finally

$$= -(S \otimes S)AW(e,e') - (1 \otimes S)ST(e,e').$$

For the remaining identity let e be in EG_p and $e' \in EH_q$. We split the sum

$$\nabla(S \otimes S)(e \otimes e') = (-1)^p \sum_{(\mu,\nu) \vdash (p+1,q+1)} \{(\mu,\nu)\}(s_\nu Se, s_\mu Se')$$

into the shuffles with $0 \in \mu$ and those with $0 \in \nu$ and apply equations (2.10): If $0 \in \mu$, then $s_\mu Se' = s_{\mu\setminus 0} s_0 Se' = s_{\mu\setminus 0} SSe' = Ss_{(\mu\setminus 0)-1} Se'$, hence we may continue

$$\begin{aligned}
&= (-1)^p \sum_{0\in\mu} \{(\mu,\nu)\}(Ss_{\nu-1}e, Ss_{(\mu\setminus 0)-1}Se') \\
&\qquad + (-1)^p \sum_{0\in\nu} \{(\mu,\nu)\}(Ss_{(\nu\setminus 0)-1}Se, Ss_{\mu-1}e') \\
&= (-1)^p \sum_{(\mu,\nu)\vdash(p,q+1)} \{(\mu,\nu)\} S(s_\nu e, s_\mu Se') - \sum_{(\mu,\nu)\vdash(p+1,q)} \{(\mu,\nu)\} S(s_\nu Se, s_\mu e') \\
&= (-1)^p S\nabla(e\otimes Se') - S\nabla(Se\otimes e') \\
&= S\nabla(1\otimes S - S\otimes 1)(e\otimes e').
\end{aligned}$$

The proposition is proven.

10. Proof of Theorem 2.8.2

We first verify that (2.15) defines simplicial maps. Since the maps are obviously inverse to each other, it suffices to examine one of them. Furthermore, we just have to check that it commutes with ∂_0 because the other face maps as well as the degeneracy maps on a fibre bundle do not differ from those on an ordinary Cartesian product. Now for $(g, xg, b) \in \boldsymbol{ht}X$,

$$\partial_0(g, xg, b) = (\partial_0 g\tau_G(b), \partial_0 x\partial_0 g\tau_G(b), \partial_0 b),$$

which is the image of

$$\partial_0(x, g, b) = (\partial_0 x, \partial_0 g\tau_G(b), \partial_0 b)$$

in $X \times EG$, as claimed.

We now consider the map h from the second part of the proof. The restriction of h to $(1) \times \boldsymbol{th}Y$ is obviously the identity. Let us show by induction that the restriction to $(0) \times \boldsymbol{th}Y$ equals $q_Y \circ p_Y$: This is trivially true in degree 0 because BG has only one vertex. Assume therefore that (x, g, b, y) is of degree greater than zero and that x consists only of zeros. Then the first component of $\partial_0(x, g, b, y)$ is still a sequence of zeros, and the last component is $\partial_0 y$. This gives

$$\begin{aligned}
h(\partial_0(x, g, b, y)) &= (1, p_Y(\partial_0 y), \partial_0 y) \\
&= (1, \partial_0 p_Y(y), \partial_0 y)
\end{aligned}$$

by induction, hence

$$\begin{aligned}
h(x, g, b, y) &= (S(\tau_Y(y), \partial_0 p_Y(y)), y) \\
&= (1, p_Y(y), y)
\end{aligned}$$

because the zeroth face map on BG just drops the leading component of $p_Y(y)$, which is $\tau_G(p_Y(y)) = \tau_Y(y)$.

We finally show that h is a simplicial map. Since the restrictions of h to $(0)\times\boldsymbol{th}Y$ and $(1)\times\boldsymbol{th}Y$ are trivially simplicial, we may assume $|x| > 0$ and $x_0 = 0$. Then the leading component of $\partial_i x$ is still zero for $i > 0$. In what follows, we will make free use of the identities (2.5) and (2.10).

$$\begin{aligned}
\partial_0 h(x,g,b,y) &= \partial_0\big(S(\tau_Y(y)g',b'),y\big)\\
&= \big(\tau_Y(y)^{-1}\partial_0 S(\tau_Y(y)g',b'),\partial_0 y\big)\\
&= (g',b',\partial_0 y)\\
&= h(\partial_0(x,g,b,y)),\\
\partial_1 h(x,g,b,y) &= \big(\partial_1 S(\tau_Y(y)g',b'),\partial_1 y\big)\\
&= \big(S\partial_0(\tau_Y(y)g',b'),\partial_1 y\big)\\
&= \big(S(\partial_0\tau_Y(y)\partial_0 g'\tau_G(b'),\partial_0 b'),\partial_1 y\big)\\
&= \big(S(\tau_Y(\partial_1 y)\tau_Y(\partial_0 y)^{-1}\partial_0 g'\tau_G(b'),\partial_0 b'),\partial_1 y\big)\\
&= h(\partial_1(x,g,b,y))
\end{aligned}$$

because by induction and by the choice of g' and b'

$$\begin{aligned}
h(\partial_0\partial_1(x,g,b,y)) &= h(\partial_0\partial_0(x,g,b,y))\\
&= \partial_0 h(\partial_0(x,g,b,y))\\
&= \partial_0(g',b',\partial_0 y)\\
&= (\tau_Y(\partial_0 y)^{-1}\partial_0 g'\tau_G(b'),\partial_0 b',\partial_0\partial_0 y)\\
&= (\tau_Y(\partial_0 y)^{-1}\partial_0 g'\tau_G(b'),\partial_0 b',\partial_0\partial_1 y).
\end{aligned}$$

For $i > 0$ one finds

$$\begin{aligned}
\partial_{i+1} h(x,g,b,y) &= \big(\partial_{i+1} S(\tau_Y(y)g',b'),\partial_{i+1} y\big)\\
&= \big(S(\partial_i\tau_Y(y)\partial_i g',\partial_i b'),\partial_{i+1} y\big)\\
&= \big(S(\tau_Y(\partial_{i+1} y)\partial_i g',\partial_i b'),\partial_{i+1} y\big)\\
&= h(\partial_{i+1}(x,g,b,y))
\end{aligned}$$

since

$$\begin{aligned}
(\partial_i g',\partial_i b',\partial_0\partial_{i+1} y) &= \partial_i(g',b',\partial_0 y)\\
&= \partial_i h(\partial_0(x,g,b,y))\\
&= h(\partial_0\partial_{i+1}(x,g,b,y)).
\end{aligned}$$

This shows that h commutes with face maps. We now turn to the degeneracy maps. Here the leading element of $s_i x$ is always zero in case $x_0 = 0$. Since

$$\begin{aligned}
h(\partial_0 s_0(x,b,g,y)) &= h(x,b,g,y)\\
&= (S(\tau_Y(y)g',b'),y)
\end{aligned}$$

and $\tau_Y(s_0 y) = 1$, one has

$$\begin{aligned}
s_0 h(x,g,b,y) &= (SS(\tau_Y(y)g',b'),s_0 y)\\
&= h(s_0(x,g,b,y)).
\end{aligned}$$

For $i \geq 0$,

$$\begin{aligned} s_{i+1}h(x,g,b,y) &= (Ss_i(\tau_Y(y)g', b'), s_{i+1}y) \\ &= (S(\tau_Y(s_{i+1}y)s_i g', s_i b'), s_{i+1}y) \\ &= h(s_{i+1}(x,g,b,y)) \end{aligned}$$

because by induction

$$\begin{aligned} (s_i g', s_i b', \partial_0 s_{i+1} y) &= s_i(g', b', \partial_0 y) \\ &= s_i h(\partial_0(g,b,y)) \\ &= h(s_i \partial_0(g,b,y)) \\ &= h(\partial_0 s_{i+1}(g,b,y)). \end{aligned}$$

This completes the proof.

11. Proof of Proposition 2.11.3

We start by deducing (2.24) from (2.25). The case $\pi = \emptyset$ is trivial. Suppose that we have already shown (2.24) for all proper subsets of $\emptyset \neq \pi \subset [r]$. Using definition (2.21) of ζ_π, one finds

$$x_i \cdot \zeta_\pi = -\sum_{\substack{(\mu,\nu)\vdash\pi \\ \pi^+\in\mu}} \Big(\{(\mu,\nu)\}\, x_i \cdot \zeta_\nu \cup \chi'_\mu + \{\nu\}\, \zeta_\nu \cup x_i \cdot \chi'_\mu\Big).$$

For $i = \pi^+$, this gives

$$x_i \cdot \zeta_\pi = -\{(i, \pi \setminus i)\}\{\pi \setminus i\}\, \zeta_{\pi\setminus i} = -\{(\pi \setminus i, i)\}\, \zeta_{\pi\setminus i}.$$

If $i \neq \pi^+$, we continue instead by induction hypothesis

$$\begin{aligned} x_i \cdot \zeta_\pi &= \sum_{\substack{(\mu,\nu)\vdash\pi \\ \pi^+\in\mu,\, i\in\nu}} \{(\mu,\nu)\}\{(\nu\setminus i, i)\}\, \zeta_{\nu\setminus i} \cup \chi'_\mu \\ &= \sum_{\substack{(\mu,\nu)\vdash\pi\setminus i \\ \pi^+\in\mu}} \{(\mu, \nu\cup i)\}\{(\nu, i)\}\, \zeta_\nu \cup \chi'_\mu \\ &= \{(\pi\setminus i, i)\} \sum_{\substack{(\mu,\nu)\vdash\pi\setminus i \\ \pi^+\in\mu}} \{(\mu,\nu)\}\, \zeta_\nu \cup \chi'_\mu = -\{(\pi\setminus i, i)\}\, \zeta_{\pi\setminus i}, \end{aligned}$$

where we have used in the penultimate step the identity (1.2).

It remains to show that ϕ is a chain map. By what we have just done it suffices to test this on $\omega \otimes 1$. Using the explicit description (1.28b) of the differential on $\mathbf{\Lambda}^* \tilde{\otimes} C^*(BT)$, we see that this means verifying equation (2.22) for $\pi = [r]$. We do so again by induction on π, noting that there is nothing to prove for $\pi = \emptyset$. Assuming equation (2.23) for the moment, we obtain for non-empty π from the recursive definition of ζ_π

$$
\begin{aligned}
d\zeta_\pi &= -\sum_{\substack{(\mu,\nu)\vdash\pi\\ \pi^+\in\mu}} \{(\mu,\nu)\}\Big(d\zeta_\nu \cup \chi'_\mu + \{\nu\}\,\zeta_\nu \cup d\chi'_\mu\Big)\\
&= -\sum_{\substack{(\mu,\nu)\vdash\pi\\ \pi^+\in\mu}} \{(\mu,\nu)\}\{\nu\} \sum_{\substack{(\kappa,\lambda)\vdash\nu\\ \kappa\neq\emptyset}} \{(\kappa,\lambda)\}\,\zeta_\lambda \cup p^*\xi'_\kappa \cup \chi'_\mu\\
&\quad - \sum_{\substack{(\mu,\nu)\vdash\pi\\ \pi^+\in\mu}} \{(\mu,\nu)\}\{\nu\}\{\mu\}\Big(-\zeta_\nu \cup p^*\xi'_\mu - \sum_{\substack{(\kappa,\lambda)\vDash\mu\\ \pi^+\in\kappa}} \{\kappa\}\{(\kappa,\lambda)\}\,\zeta_\nu \cup p^*\xi'_\lambda \cup \chi'_\kappa\\
&\qquad\qquad + \sum_{\substack{(\kappa,\lambda)\vDash\mu\\ \pi^+\in\lambda}} \{(\kappa,\lambda)\}\,\zeta_\nu \cup \chi'_\lambda \cup p^*\xi'_\kappa\Big)\\
&= -\sum_{\substack{(\mu,\kappa,\lambda)\vdash\pi\\ \pi^+\in\mu,\,\kappa\neq\emptyset}} \{\kappa\}\{\lambda\}\Big(\{(\mu,\kappa\cup\lambda)\}\{(\kappa,\lambda)\} - \{(\mu,\kappa)\}\{(\mu\cup\kappa,\lambda)\}\Big)\zeta_\lambda \cup p^*\xi'_\kappa \cup \chi'_\mu\\
&\quad + \{\pi\} \sum_{\substack{(\mu,\nu)\vdash\pi\\ \pi^+\in\mu}} \{(\mu,\nu)\}\,\zeta_\nu \cup p^*\xi'_\mu\\
&\quad - \{\pi\} \sum_{\substack{(\kappa,\lambda,\nu)\vdash\pi\\ \pi^+\in\lambda,\,\kappa\neq\emptyset}} \{(\kappa,\lambda)\}\{(\kappa\cup\lambda,\nu)\}\,\zeta_\nu \cup \chi'_\lambda \cup p^*\xi'_\kappa
\end{aligned}
$$

The big bracket above yields zero because both terms in it are equal to $\{(\mu,\kappa,\lambda)\}$ by equation (1.2). We may therefore continue

$$
\begin{aligned}
&= \{\pi\} \sum_{\substack{(\mu,\nu)\vdash\pi\\ \pi^+\in\mu}} \{(\mu,\nu)\}\,\zeta_\nu \cup p^*\xi'_\mu\\
&\quad - \{\pi\} \sum_{\substack{(\kappa,\rho)\vdash\pi\\ \pi^+\in\rho,\,\kappa\neq\emptyset}} \sum_{\substack{(\lambda,\nu)\vdash\rho\\ \pi^+\in\lambda}} \{(\kappa,\rho)\}\{(\lambda,\nu)\}\,\zeta_\nu \cup \chi'_\lambda \cup p^*\xi'_\kappa\\
&= \{\pi\} \sum_{\substack{(\mu,\nu)\vdash\pi\\ \pi^+\in\mu}} \{(\mu,\nu)\}\,\zeta_\nu \cup p^*\xi'_\mu + \{\pi\} \sum_{\substack{(\kappa,\rho)\vdash\pi\\ \pi^+\in\rho,\,\kappa\neq\emptyset}} \{(\kappa,\rho)\}\,\zeta_\rho \cup p^*\xi'_\kappa
\end{aligned}
$$

by the definition of ζ_ρ, hence

$$
= \{\pi\} \sum_{\substack{(\mu,\nu)\vdash\pi\\ \mu\neq\emptyset}} \{(\mu,\nu)\}\,\zeta_\nu \cup p^*\xi'_\mu.
$$

We finally prove equation (2.23), this time without induction. For $\pi = \{i\}$ this is just equation (2.18). Differentiating the recursive definition (2.20) gives by (1.29a) and (2.18) (note that $\{\xi'_{\pi'}\} = -\{\pi'\} = \{\pi\}$)

$$
\begin{aligned}
\{\pi\}\,d\chi'_\pi &= -\{\pi\}\,p^*\xi'_{\pi'} \cup \chi'_{\pi^+} + \chi'_{\pi^+} \cup p^*\xi'_{\pi'} + \{\pi\}\,p^*d\xi'_{\pi'} \cup_1 \chi'_{\pi^+} + p^*\xi'_{\pi'} \cup_1 p^*\xi'_{\pi^+}\\
&= -\{\pi\}\,p^*\xi'_{\pi'} \cup \chi'_{\pi^+} + \chi'_{\pi^+} \cup p^*\xi'_{\pi'} + \{\pi\}\,p^*d\xi'_{\pi'} \cup_1 \chi'_{\pi^+} - p^*\xi'_\pi
\end{aligned}
$$

by definition (1.30) of ξ'_π. Using expression (1.25) for the differential of $d\xi'_{\pi'}$, we obtain for the penultimate term above

$$\{\pi\}\, p^* d\xi'_{\pi'} \cup_1 \chi'_{\pi^+} = \sum_{(\mu,\nu)\vDash\pi'} \{\mu\}\{(\mu,\nu)\}\, p^*(\xi'_\nu \cup \xi'_\mu) \cup_1 \chi'_{\pi^+},$$

and by the Hirsch formula (1.29b) and the definition of χ'_π

$$\begin{aligned} &= \sum_{(\mu,\nu)\vDash\pi'} \{\mu\}\{(\mu,\nu)\}\Big(-\{\nu\}\, p^*\xi'_\nu \cup (p^*\xi'_\mu \cup_1 \chi'_{\pi^+}) - \{\mu\}\,(p^*\xi'_\nu \cup_1 \chi'_{\pi^+}) \cup p^*\xi'_\mu\Big) \\ &= \sum_{(\mu,\nu)\vDash\pi'} \Big(\{\mu\}\{(\mu\cup\pi^+,\nu)\}\, p^*\xi'_\nu \cup \chi'_{\mu\cup\pi^+} + \{(\mu,\nu\cup\pi^+)\}\, \chi'_{\nu\cup\pi^+} \cup p^*\xi'_\mu\Big) \\ &= -\sum_{\substack{(\mu,\nu)\vDash\pi \\ \{\pi^+\}\subsetneq\mu}} \{\mu\}\{(\mu,\nu)\}\, p^*\xi'_\nu \cup \chi'_\mu + \sum_{\substack{(\mu,\nu)\vDash\pi \\ \{\pi^+\}\subsetneq\nu}} \{(\mu,\nu)\}\, \chi'_\nu \cup p^*\xi'_\mu. \end{aligned}$$

Substituting this into the result of the preceding calculation gives (2.23). This finishes the proof.

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Symbols