THE MOD 2 COHOMOLOGY RING OF REAL MOMENT-ANGLE COMPLEXES

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ABSTRACT. We give a short proof for Cai's description of the cohomology ring of a real moment-angle complex with coefficients in $\mathbb{Z}_2$. We use it to confirm the counterexample to de Longueville's claim found by Gitler and López de Medrano.

1. INTRODUCTION

Let Σ be a simplicial complex on $[m] = \{ 1, \dots, m \}$ and let

$$\mathbb{R}\mathcal{Z}_\Sigma = (D^1, S^0)^\Sigma \tag{1.1}$$

be the associated real moment-angle complex. In this note we give a short proof of Cai's description [3, Thm. 5.1] of the cohomology ring $H^*(\mathbb{R}\mathcal{Z}_\Sigma; \Bbbk)$ for $\Bbbk = \mathbb{Z}_2$, noting that Cai's methods actually work for arbitrary coefficients. Our approach is modelled on that in [4, Thm. 1.2], where the integer cohomology ring of a (complex) moment-angle complex was computed for the first time. The categorical viewpoint used in the proof below has been advocated in [7].

Let $\mathcal{R}$ be the differential graded $\Bbbk$-algebra generated by indeterminates $t_1, \dots, t_m$ of degree 0 and $u_1, \dots, u_m$ of degree 1 subject to the relations

$$u_i u_j = u_j u_i, \quad u_\sigma = 0 \quad \text{if } \sigma \notin \Sigma, \quad du_i = 0, \tag{1.2}$$

$$t_i t_j = t_j t_i, \quad t_i^2 = t_i, \quad t_i u_j = u_j t_i, \quad t_i u_i = u_i t_i + u_i, \quad dt_i = u_i. \tag{1.3}$$

for i, $j \in [m]$, $i \neq j$. Here we have written $u_\sigma = \prod_{i\in\sigma} u_i$ for $\sigma \subset [m]$, and we define t_σ similarly.

Theorem 1.1. *There is an isomorphism of graded algebras*

$$H^*(\mathbb{R}\mathcal{Z}_\Sigma) \cong H^*(\mathcal{R}).$$

In particular, there is an isomorphism of graded vector spaces

$$H^*(\mathbb{R}\mathcal{Z}_\Sigma) \cong \operatorname{Tor}^{\mathbf{S}^*}(\Bbbk[\Sigma], \Bbbk)$$

where $\Bbbk[\Sigma]$ is the Stanley–Reisner algebra, $\mathbf{S}^* = \Bbbk[u_1, \dots, u_m]$ and the torsion product is considered with the total grading.

Consider the subspace $\mathcal{M} \subset \mathcal{R}$ spanned by the monomials $t_\sigma u_\tau$ for all disjoint subsets σ, $\tau \subset [m]$. Then $\mathcal{M}$ is a subcomplex of $\mathcal{R}$ since

$$d(t_\sigma u_\tau) = \sum_{i\in\sigma} t_{\sigma\setminus i}\, u_{\tau\cup i}. \tag{1.4}$$

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We equip $\mathcal{M}$ with the product

$$t_\sigma u_\tau * t_{\sigma'} u_{\tau'} = \begin{cases} t_{\sigma\cup(\sigma'\setminus\tau)}\, u_{\tau\cup\tau'} & \text{if } (\sigma\cup\tau)\cap\tau'=\emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{1.5}$$

cf. [3, eq. (25)]. Note that σ and $\sigma'\setminus\tau$ need not be disjoint, but their union is disjoint from $\tau\cup\tau'$ under the condition stated above.

Theorem 1.2. *The complex $\mathcal{M}$ together with the product $*$ and the unit 1 is a differential graded algebra. Moreover, there is an isomorphism of graded algebras*

$$H^*(\mathcal{M}) \cong H^*(\mathbb{R}\mathcal{Z}_\Sigma).$$

We finally note that analogously to the moment-angle complex $\mathcal{Z}_\Sigma = (D^2, S^1)^\Sigma$ one has a Hochster decomposition of $H^*(\mathbb{R}\mathcal{Z}_\Sigma)$ into the reduced cohomology of full subcomplexes,

$$H^i(\mathbb{R}\mathcal{Z}_\Sigma) = \bigoplus_{\omega\in\Sigma} \tilde{H}^{i-1}(\Sigma_\omega), \tag{1.6}$$

cf. [2, Thms. 3.2.4, 4.5.7]. This follows from (1.4) because the subcomplex spanned by the monomials $t_\sigma u_\tau$ with $\sigma\cup\tau=\omega$ is isomorphic to the reduced simplicial cochain complex of Σ_ω. Notice that this in particular shows that $\mathbb{R}\mathcal{Z}_\Sigma$ and $\mathcal{Z}_\Sigma$ have the same mod 2 Betti sum.

2. Preliminaries

All vector spaces are over $\Bbbk = \mathbb{Z}_2$, and all (co)homology is taken with coefficients in $\Bbbk$. Moreover, all complexes are assumed to be cohomological and bounded below. All differential graded algebras (dgas) will be associative with unit. Graded algebras are considered as dgas with trivial differential.

We think of Σ as a category with inclusions of faces as morphisms. We also write $\hat\sigma = [m]\setminus\sigma$ for $\sigma\in\Sigma$.

Let $G = \{\pm1\}^m$. We point out that unlike in the case of tori, the homology and the cohomology of G (not of BG!) are not isomorphic as algebras: Since we work in characteristic 2, any element squares to either 0 or 1 in the group algebra $H_*(G)$, while in $H^*(G)$, which is the algebra of functions $G\to\Bbbk$ with pointwise multiplication, every element is idempotent. It will be essential for us that $H_*(G)$ is an exterior algebra on generators $c_i = 1 + g_i$, where g_i for the canonical i-th generator of G for $i\in[m]$.

For $\sigma\subset[m]$, we write G_σ for the subgroup $\{\pm1\}^\sigma\subset G$ and $\mathbf{A}^*_\sigma = H^*(G_\sigma)$. Let $\mathbf{A}^* = \mathbf{A}^*_{[m]}$, and let $t_i\in\mathbf{A}^*$ be the function $G=\{\pm1\}^m\to\{\pm1\}\cong\Bbbk$ such that $t_i(g)=1$ if and only if the i-th coordinate of g equals -1. The products t_σ with $\sigma\subset[m]$ form a basis for $\mathbf{A}^*$, and

$$c_i\cdot t_\sigma = \begin{cases} t_{\sigma\setminus\{i\}} & \text{if } i\in\sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Moreover, we write $\mathbf{S}^*_\sigma$ for the polynomial algebra generated by indeterminates u_i, $i\in\sigma$, of degree 1, and $\mathbf{S}^* = \mathbf{S}^*_{[m]}$.

A *G-algebra* is a dga on which G acts by dga automorphisms. The G-action on such a dga extends to one of $H_*(G)$ (which is not by dga maps anymore). Note that if both B and C are G-algebras, then so is $B\otimes C$ with the diagonal G-action.

Example 2.1. The singular cochain complex of a G-space is canonically a G-algebra, and any morphism of G-spaces $X \to Y$ induces a morphism of G-algebras $C^*(Y) \to C^*(X)$.

By looking at $G_{\hat\sigma} = G/G_\sigma$, we in particular get a contravariant functor $\sigma \mapsto \mathbf{A}^*_{\hat\sigma}$ from Σ to the category of G-algebras. The map $\mathbf{A}^*_\sigma \to \mathbf{A}^*_\emptyset = \Bbbk$ is called the *augmentation* and denoted by ε.

Example 2.2. The assignment $\sigma \mapsto \mathbf{S}^*_\sigma$ is a contravariant functor from Σ to the category of dgas. Through the canonical projection $\pi\colon \mathbf{S}^* \to \mathbf{S}^*_\sigma$, each $\mathbf{S}^*_\sigma$ becomes additionally an $\mathbf{S}^*$-module. We again call the map $\mathbf{S}^*_\sigma \to \mathbf{S}^*_\emptyset = \Bbbk$ the *augmentation* and denote it by ε.

Example 2.3. The relations (1.2) define the *Stanley–Reisner algebra* $\Bbbk[\Sigma]$ associated to Σ. It is a quotient of $\mathbf{S}^*$.

Let R be a quotient of $\mathbf{S}^*$ and let B be a G-algebra. The *twisted tensor product* $R \mathbin{\tilde\otimes} B$ is the tensor product of graded vector spaces $R \otimes B$ with operations

$$d(v \otimes b) = v \otimes db + \sum_{i=1}^{m} u_i v \otimes c_i \cdot b, \quad g \cdot (v \otimes b) = v \otimes g \cdot b, \quad b u_j = u_j (g_j \cdot b)$$

for all $v \in R$, $b \in B$, $g \in G$ and $j \in [m]$.

Lemma 2.4. *With these definitions, $R \mathbin{\tilde\otimes} B$ becomes a G-algebra.*

Proof. This is a direct computation. □

Lemma 2.5. *Let R be a quotient of $\mathbf{S}^*$ and let $B \to C$ be a quasi-isomorphism of G-algebras. Then the induced morphism $R \mathbin{\tilde\otimes} B \to R \mathbin{\tilde\otimes} C$ is a quasi-isomorphism of G-algebras.*

Proof. It is clear that we get a morphism of G-algebras. That it induces an isomorphism in cohomology is a standard spectral sequence argument. □

Note that for any differential graded (dg) $\mathbf{S}^*$-module M we can similarly form the complex

$$M \mathbin{\tilde\otimes} \mathbf{A}^*, \quad d(v \otimes a) = dv \otimes a + \sum_{i=1}^{m} u_i v \otimes c_i \cdot a. \tag{2.2}$$

Lemma 2.6. *Let $M \to N$ be a quasi-isomorphism of dg $\mathbf{S}^*$-modules. Then the induced morphism $M \mathbin{\tilde\otimes} \mathbf{A}^* \to N \mathbin{\tilde\otimes} \mathbf{A}^*$ is again a quasi-isomorphism.*

Proof. The algebra $\mathbf{A}^*$ has an additional grading with t_σ having degree $|\sigma|$, and this induces a grading on $M \mathbin{\tilde\otimes} \mathbf{A}^*$. The part $dv \otimes a$ of $d(v \otimes a)$ preserves this grading while the part $\sum_i u_i v \otimes c_i \cdot a$ decreases it by 1. Filtering $M \mathbin{\tilde\otimes} \mathbf{A}^*$ and $N \mathbin{\tilde\otimes} \mathbf{A}^*$ accordingly leads to a map of spectral sequences which by assumption is an isomorphism between the E_1 pages. The claim follows. □

Lemma 2.7. *For any G-algebra B, the map*

$$\varphi\colon \mathbf{S}^* \mathbin{\tilde\otimes} (\mathbf{A}^* \otimes B) \to B, \quad v \otimes a \otimes b \mapsto \varepsilon(v)\, \varepsilon(a)\, b$$

is a quasi-isomorphism of dgas.

The augmentations have been defined in Examples 2.1 and 2.2.

Proof. The map ψ clearly is a morphism of dgas, and it is also a homotopy equivalence. We prove the latter claim by induction on m and can therefore assume $m = 1$. In this case an (even multiplicative) strict right homotopy inverse to φ is given by

$$\psi\colon B \to \mathbf{S}^* \mathbin{\tilde{\otimes}} (\mathbf{A}^* \otimes B), \quad b \mapsto 1 \otimes 1 \otimes b + 1 \otimes t \otimes c \cdot b, \tag{2.3}$$

where we have written $u = u_1$, $c = c_1$ and $t = t_1$. A homotopy between $\psi\varphi$ and the identity of $\mathbf{S}^* \mathbin{\tilde{\otimes}} B \otimes \mathbf{A}^*$ is given by

$$h(u^k \otimes a \otimes b) = \begin{cases} 0 & \text{if } k = 0 \text{ or } a = t, \\ u^{k-1} \otimes t \otimes (c+1) \cdot b & \text{if } k > 0 \text{ and } a = 1. \end{cases} \tag{2.4}$$

□

3. Proof of Theorem 1.1

Morally, we are going to show that the G-equivariant cohomology of $\mathbb{R}\mathcal{Z}_\Sigma$ is given by the Stanley–Reisner algebra $\Bbbk[\Sigma]$ and that the non-equivariant cohomology can be recovered from it as the cohomology of the complex $\mathcal{R} = \Bbbk[\Sigma] \mathbin{\tilde{\otimes}} \mathbf{A}^*$.

We write $X = \mathbb{R}\mathcal{Z}_\Sigma$; it comes with an obvious G-action. For $\sigma \in \Sigma$ we define

$$X_\sigma = (D^1)^\sigma \times (S^0)^{\hat\sigma} = (D^1)^\sigma \times G_{\hat\sigma}; \tag{3.1}$$

it is a G-equivariant neighbourhood deformation retract in X.

Let $\mathcal{U} = \{ \sigma_1, \dots, \sigma_l \}$ be the set of maximal simplices in Σ. For any contravariant functor Φ from Σ to the category of dgas we can consider the Mayer–Vietoris double complex $E(\Phi)$ associated to Φ and the covering $\mathcal{U}$, *cf.* [1, §15]. It is a dga, and any natural transformation $F\colon \Phi \to \Psi$ induces a dga morphism $E(\Phi) \to E(\Psi)$. If $\Phi(\sigma) \to \Psi(\sigma)$ is a quasi-isomorphism for each $\sigma \in \Sigma$, then so is $E(\Phi) \to E(\Psi)$ by a standard spectral sequence argument.

For the Mayer–Vietoris double complex $E(C^*(X_\sigma))$ defined by the contravariant functor $\sigma \mapsto C^*(X_\sigma)$ we have a quasi-isomorphism of G-algebras

$$C^*(X) \to E(C^*(X_\sigma)) \tag{3.2}$$

given by the restriction of any $\gamma \in C^*(X)$ to the $C^*(X_\sigma)$ for $\sigma \in \mathcal{U}$.

The natural transformation $\mathbf{A}^*_{\hat\sigma} \to C^*(X_\sigma)$ induced by the canonical projection $X_\sigma \to G_{\hat\sigma}$ is a quasi-isomorphism for each $\sigma \in \Sigma$. We thus get a quasi-isomorphism of G-algebras

$$E(\mathbf{A}^*_{\hat\sigma}) \to E(C^*(X_\sigma)). \tag{3.3}$$

Tensoring with $\mathbf{A}^*$ gives a quasi-isomorphism of G-algebras

$$E(\mathbf{A}^*_{\hat\sigma} \otimes \mathbf{A}^*) = E(\mathbf{A}^*_{\hat\sigma}) \otimes \mathbf{A}^* \to E(C^*(X_\sigma)) \otimes \mathbf{A}^* = E(C^*(X_\sigma) \otimes \mathbf{A}^*). \tag{3.4}$$

Passing to twisted tensor products, we get a zigzag of dga morphisms

$$\mathbf{S}^* \mathbin{\tilde{\otimes}} (C^*(X) \otimes \mathbf{A}^*) \to E(\mathbf{S}^* \mathbin{\tilde{\otimes}} (C^*(X_\sigma) \otimes \mathbf{A}^*)) \leftarrow E(\mathbf{S}^* \mathbin{\tilde{\otimes}} (\mathbf{A}^*_{\hat\sigma} \otimes \mathbf{A}^*)) \tag{3.5}$$

which are quasi-isomorphisms by Lemma 2.5. Moreover, the dga on the left-hand side is quasi-isomorphic to $C^*(X)$ by Lemma 2.7.

Lemma 3.1. *For $\sigma \subset [m]$, we have a natural quasi-isomorphism of dgas*

$$\mathbf{S}^* \mathbin{\tilde{\otimes}} (\mathbf{A}^*_{\hat\sigma} \otimes \mathbf{A}^*) \to \mathbf{S}^*_\sigma \mathbin{\tilde{\otimes}} \mathbf{A}^*, \quad v \otimes b \otimes a \mapsto \varepsilon(b)\, \pi(v) \otimes a.$$

The projection π has been defined in Example 2.2.

Proof. Let $B = \mathbf{S}^*_\sigma \mathbin{\tilde{\otimes}} \mathbf{A}^*$. Since g_i acts trivially on $\mathbf{A}^*_{\hat\sigma}$ for $i \in \sigma$, we have

$$\mathbf{S}^* \mathbin{\tilde{\otimes}} (\mathbf{A}^*_{\hat\sigma} \otimes \mathbf{A}^*) = \mathbf{S}^*_{\hat\sigma} \mathbin{\tilde{\otimes}} (\mathbf{A}^*_{\hat\sigma} \otimes B) \tag{3.6}$$

as dgas, and applying the augmentations of $\mathbf{S}^*_{\hat\sigma}$ and $\mathbf{A}^*_{\hat\sigma}$ takes the right-hand side to B. By Lemma 2.7, this map is a quasi-isomorphism of dgas. □

As a consequence, we have a quasi-isomorphism of dgas

$$E(\mathbf{S}^* \mathbin{\tilde{\otimes}} (\mathbf{A}^*_{\hat\sigma} \otimes \mathbf{A}^*)) \to E(\mathbf{S}^*_\sigma \mathbin{\tilde{\otimes}} \mathbf{A}^*) \tag{3.7}$$

Lemma 3.2. *The map that sends $v \in \Bbbk[\Sigma]$ to its restrictions to all $\mathbf{S}^*_\sigma$ with $\sigma \in \mathcal{U}$ is a quasi-isomorphism $\Bbbk[\Sigma] \to E(\mathbf{S}^*_\sigma)$ of dgas and of dg $\mathbf{S}^*$-modules.*

Proof. The map is a morphism of dgas and of dg $\mathbf{S}^*$-modules. It is a quasi-isomorphism because the sheaf $\sigma \mapsto \mathbf{S}^*_\sigma$ on Σ is flabby, *cf.* [2, Thm. 3.5.6, Prop. 8.1.1]. □

Observing that $E(\mathbf{S}^*_\sigma \mathbin{\tilde{\otimes}} \mathbf{A}^*) = E(\mathbf{S}^*_\sigma) \mathbin{\tilde{\otimes}} \mathbf{A}^*$ as complexes, we get from the above result and Lemma 2.6 a quasi-isomorphism of $\mathbf{S}^*$-algebras

$$\mathcal{R} = \Bbbk[\Sigma] \mathbin{\tilde{\otimes}} \mathbf{A}^* \to E(\mathbf{S}^*_\sigma \mathbin{\tilde{\otimes}} \mathbf{A}^*). \tag{3.8}$$

This concludes the proof of the first part of Theorem 1.1. For the second part, we note that $\mathbf{S}^* \mathbin{\tilde{\otimes}} \mathbf{A}^*$ is the total complex of the Koszul resolution of $\Bbbk$ over $\mathbf{S}^*$ by (2.1) and (2.2).

4. Proof of Theorem 1.2

The inclusion $\iota\colon \mathcal{M} \hookrightarrow \mathcal{R}$ is a quasi-isomorphism [4, Lemma 6.1], [2, proof of Thm. 3.2.9]. Being a chain map, the projection

$$\pi\colon \mathcal{R} \to \mathcal{M}, \quad t_\sigma u^\alpha = \begin{cases} t_\sigma u^\alpha & \text{if } u^\alpha = u_\tau \text{ for some } \tau \text{ disjoint from } \sigma, \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

is a quasi-inverse. Note that the condition $u^\alpha = u_\tau$ simply means that no component α_i of the multi-index $\alpha \in \mathbb{N}^m$ is larger than 1, and then τ is the set of indices where it equals 1.

We can use the maps ι and π to transfer the product from $\mathcal{R}$ to $\mathcal{M}$,

$$\mathcal{M} \otimes \mathcal{M} \xrightarrow{\iota\otimes\iota} \mathcal{R} \otimes \mathcal{R} \longrightarrow \mathcal{R} \xrightarrow{\pi} \mathcal{M}. \tag{4.2}$$

We obviously get a chain map this way, and it is easy to check that it acts as given in (1.5).

To see that the product is associative, we observe that $(t_\sigma u_\tau * t_{\sigma'} u_{\tau'}) * t_{\sigma''} u_{\tau''}$ and $t_\sigma u_\tau * (t_{\sigma'} u_{\tau'} * t_{\sigma''} u_{\tau''})$ both equal

$$t_{\sigma \cup (\sigma' \setminus \tau) \cup (\sigma'' \setminus (\tau \cup \tau'))}\, u_{\tau \cup \tau' \cup \tau''} \tag{4.3}$$

if

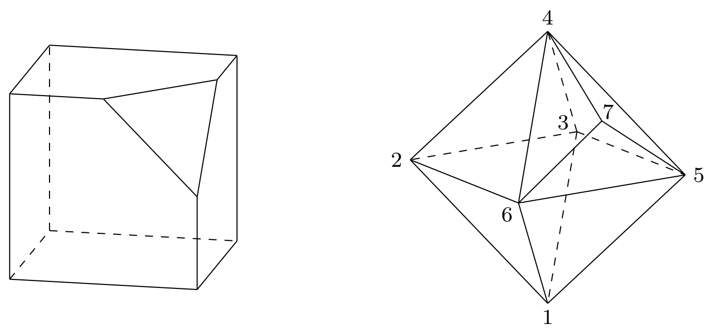
$$\tau \cap \tau' = \tau \cap \tau'' = \tau' \cap \tau'' = \sigma \cap (\tau' \cup \tau'') = \sigma' \cap \tau'' = \emptyset \tag{4.4}$$

and 0 otherwise.

The projection π is multiplicative, hence a quasi-isomorphism of dgas. It induces the isomorphism of graded algebras claimed in Theorem 1.2.

5. The truncated cube

We consider the truncated cube discussed in [5, Sec. 3], see also [3, Ex. 5.4]. This example shows that de Longueville's description [6, Thm. 1.1] of the integral cohomology of a complement of a real coordinate subspace arrangement is incorrect: According to de Longueville, $H^*(\mathcal{Z}_\Sigma;\mathbb{Z})$ and $H^*(\mathbb{R}\mathcal{Z}_\Sigma;\mathbb{Z})$ are isomorphic as ungraded rings modulo 2 for any Σ. In the case at hand, both spaces have torsion-free integral cohomology, so that the tensor product with $\mathbb{Z}_2$ gives the mod 2 cohomology rings. Let us determine these rings.



The truncated cube is shown on the left, and its dual together with the vertex labelling we are going to use on the right. By Hochster's formula, a "common" basis for $H^*(\mathcal{Z}_\Sigma)$ and $H^*(\mathbb{R}\mathcal{Z}_\Sigma)$ as well as representatives are given by

(5.1) $$1 = [1],$$

(5.2) $$x_{17} = [t_1u_7], \qquad x_{27} = [t_2u_7], \qquad x_{37} = [t_3u_7],$$

(5.3) $$x_{14} = [t_1u_4], \qquad x_{25} = [t_2u_5], \qquad x_{36} = [t_3u_6],$$

(5.4) $$x_{127} = [t_1t_2u_7], \qquad x_{137} = [t_1t_3u_7], \qquad x_{237} = [t_2t_3u_7],$$

(5.5) $$x_{147} = [t_4t_7u_1], \qquad x_{257} = [t_5t_7u_2], \qquad x_{367} = [t_6t_7u_3],$$

(5.6) $$x_{456} = [t_4u_5u_6],$$

(5.7) $$x_{1237} = [t_1t_2t_3u_7],$$

(5.8) $$x_{1245} = [t_4t_5u_1u_2], \qquad x_{1346} = [t_4t_6u_1u_3], \qquad x_{2356} = [t_5t_6u_2u_3],$$

(5.9) $$x_{1456} = [t_1t_6u_4u_5], \qquad x_{2456} = [t_2t_4u_5u_6], \qquad x_{3456} = [t_3t_5u_4u_6],$$

(5.10) $$x_{12456} = [t_1t_2t_6u_4u_5], \qquad x_{13456} = [t_1t_3t_5u_4u_6], \qquad x_{23456} = [t_2t_3t_4u_5u_6],$$

(5.11) $$x_{12457} = [t_4t_5t_7u_1u_2], \qquad x_{13467} = [t_4t_6t_7u_1u_3], \qquad x_{23567} = [t_5t_6t_7u_2u_3],$$

(5.12) $$x_{1234567} = [t_4t_5t_6t_7u_1u_2u_3].$$

Note that the degrees are different in the two cases. For example, the fundamental class $x_{1234567}$ has degree 10 in $H^*(\mathcal{Z}_\Sigma)$ and degree 3 in $H^*(\mathbb{R}\mathcal{Z}_\Sigma)$.

In $H^*(\mathcal{Z}_\Sigma)$, the non-zero products of basis elements different from 1 whose degrees add up to less than 10 are given by

(5.13) $$x_{14}\cdot x_{25} = x_{1245}, \qquad x_{14}\cdot x_{36} = x_{1346}, \qquad x_{25}\cdot x_{36} = x_{2356},$$

(5.14) $$x_{14}\cdot x_{257} = x_{12457}, \qquad x_{14}\cdot x_{367} = x_{13467}, \qquad x_{25}\cdot x_{147} = x_{12457},$$

(5.15) $$x_{25}\cdot x_{367} = x_{23567}, \qquad x_{36}\cdot x_{147} = x_{13467}, \qquad x_{36}\cdot x_{257} = x_{23567}.$$

This agrees with the products given in [5, p. 1524] if one sets

$$a_1 = x_{14}, \qquad a_2 = x_{25}, \qquad a_3 = x_{36}, \tag{5.16}$$

$$a'_1 = x_{147}, \qquad a'_2 = x_{257}, \qquad a'_3 = x_{367}, \tag{5.17}$$

$$b_{12} = x_{1245}, \qquad b_{13} = x_{1346}, \qquad b_{23} = x_{2356}, \tag{5.18}$$

$$b'_{12} = x_{12457}, \qquad b'_{13} = x_{13467}, \qquad b'_{23} = x_{23567}. \tag{5.19}$$

In $H^*(\mathbb{R}\mathcal{Z}_\Sigma)$ we have identical multiplication rules for basis elements in degree 1 except for the additional non-zero products

$$x_{147} * x_{257} = x_{12457}, \qquad x_{147} * x_{367} = x_{13467}, \qquad x_{257} * x_{367} = x_{23567}. \tag{5.20}$$

Hence, if we set

$$a''_1 = x_{14} + x_{147}, \qquad a''_2 = x_{25} + x_{257}, \qquad a''_3 = x_{36} + x_{367}, \tag{5.21}$$

$$b''_{12} = x_{1245} + x_{12457}, \qquad b''_{13} = x_{1346} + x_{13467}, \qquad b''_{23} = x_{2356} + x_{23567}, \tag{5.22}$$

then all products $a_i * a''_j$ vanish, and $a''_i * a''_j = b''_{ij}$ for $i \neq j$. These are the products of degree-1 elements in the connected sum $(S^1 \times S^1 \times S^1)\#(S^1 \times S^1 \times S^1)$, as predicted by [5].

The analysis in [5] shows that $H^*(\mathcal{Z}_\Sigma)$ and $H^*(\mathbb{R}\mathcal{Z}_\Sigma)$ are not isomorphic as ungraded rings. A minor variation of their argument is the following: Let $\mathfrak{m}$ be the unique maximal ideal of $H^*(\mathcal{Z}_\Sigma)$ or $H^*(\mathbb{R}\mathcal{Z}_\Sigma)$ and set $V_k = \mathfrak{m}^k/\mathfrak{m}^{k+1}$. Then V_3 is spanned by $x_{1234567} = x_{14} \cdot x_{25} \cdot x_{367}$ in both cases.

Consider the multiplication

$$f \colon V_1 \times V_1 \to V_2, \tag{5.23}$$

which is a symmetric bilinear map. Let $W \subset V_1$ be the kernel of f and let $\bar{f}$ be the induced map

$$\bar{f} \colon V_1/W \times V_1/W \to V_2. \tag{5.24}$$

It follows from the multiplication tables that in both cases the quotient V_1/W is 6-dimensional and spanned by (the images of) the a_i and the a'_i. Now in the case of $\mathcal{Z}_\Sigma$ the elements a'_i have pairwise vanishing products whereas in the case of $\mathbb{R}\mathcal{Z}_\Sigma$ no three linearly independent elements with this property exist. Hence $H^*(\mathcal{Z}_\Sigma)$ and $H^*(\mathbb{R}\mathcal{Z}_\Sigma)$ are not isomorphic as ungraded rings.

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