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Discs in Hulls of Real Immersions into Stein Manifolds

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Abstract—We obtain results on existence of complex discs in plurisubharmonically convex hulls of Lagrangian and totally real immersions to Stein manifolds.

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1. INTRODUCTION

Denote by J_{st} the standard complex structure of $\mathbb{C}^n$; the value of n will be clear from the context. Let $\mathbb{D} = \{\zeta \in \mathbb{C}\colon |\zeta| < 1\}$ be the unit disc in $\mathbb{C}$ equipped with J_{st} and (M, J) be an (almost) complex manifold with an (almost) complex structure J. A *J-complex* (or *J-holomorphic*) *disc* in M is a map $f\colon \mathbb{D} \to M$ holomorphic with respect to J_{st} and J; following tradition, we often identify f with its image. When the complex structure J is fixed, we simply say that f is a complex (or holomorphic) disc.

If a disc f is continuous on $\overline{\mathbb{D}}$, the restriction $f|_{\partial\mathbb{D}}$ is called the *boundary of* f. We say that the boundary of f is *attached* or *glued* to a subset $K \subset M$ if $f(\partial\mathbb{D}) \subset K$. Construction of complex discs with boundaries on a prescribed (compact) subset of M is an old and fundamental problem in complex geometry. It plays a major role in the theory of polynomially, holomorphically, or plurisubharmonically (psh) convex hulls (see [9, 22]).

The seminal paper [10] by Gromov reveals a profound connection between the hull problems in complex geometry and symplectic and contact geometry. One of his most striking results states that a smooth compact Lagrangian submanifold E of $\mathbb{C}^n$ contains the boundary of a nonconstant complex disc. In [11] Gromov suggested that his proof must also work for the case of arbitrary Lagrangian immersions to $\mathbb{C}^n$. This could be a very natural extension of this result, since the existence of a Lagrangian immersion is a topologically much less restrictive condition on E than that of a Lagrangian embedding (see [3, 9]). Nevertheless, later on it became clear that some technical difficulties occur. A complex disc with the boundary glued to E essentially arises in Gromov's method as a disc-bubble which is smooth on $\overline{\mathbb{D}} \setminus \{1\}$. When E is smooth, Gromov's removable boundary singularity theorem allows one to extend the map to the whole boundary $\partial\mathbb{D}$, since in the Lagrangian case the area of a bubble is bounded. The difficulty is to prove an analogous removable singularity theorem for discs attached to immersed manifolds. In the present study we propose an approach inspired by the work of Alexander [1], who adapted Gromov's method to the case of totally real manifolds.

A *nearly smooth* complex disc of class C^m is a bounded complex disc $f\colon \mathbb{D} \to M$ which extends C^m-smoothly to $\partial\mathbb{D} \setminus \{1\}$. We say that a nearly smooth complex disc f is *attached* to a compact subset $K \subset M$ if $f(\partial\mathbb{D} \setminus \{1\}) \subset K$. If additionally f is *nonconstant*, we call it an *A-disc* of class C^m, after Herbert Alexander, who proved the existence of such discs for totally real (not necessarily Lagrangian) manifolds in $\mathbb{C}^n$ (see [1]). We simply call it A-disc if it is of class C^∞. Alexander's proof combines Gromov's general method with the standard complex analytic tools avoiding application of Gromov's compactness theorem. For this reason his approach relies heavily

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on the affine structure of $\mathbb{C}^n$. In [20] we extended this result to the case of certain totally real immersions to $\mathbb{C}^n$. The goal of the present paper is to extend Alexander's result and the results in [20] to the case of totally real immersions to some Stein manifolds (in fact, integrability of the complex structure is not needed for some of our results). Here we use the general approach of Gromov. It turns out that Alexander's version of Gromov's result can indeed be generalized to immersions by almost literally following Gromov's method (Theorem 2.4). As a consequence we find that a totally real immersion of dimension n to a complex n-dimensional manifold of type $\mathbb{C} \times X$, where X is Stein, is not psh convex (Corollary 2.5).

In Section 3 we consider hulls of Lagrangian immersions into Stein manifolds. Our main observation is that removal of the boundary singularity is connected with the "complex" convexity properties of the singular set of the immersed manifold. In [20] we used the polynomial convexity working in $\mathbb{C}^n$; the notion of plurisubharmonic convexity is suitable in the Stein case. We prove the removable boundary singularity property (and hence the existence of a nonconstant complex disc with boundary glued to E) for a Lagrangian immersion E with isolated locally psh convex singularities (Theorem 3.4). This condition always holds for transverse double intersections (Proposition 3.2).

2. GLUING DISCS TO TOTALLY REAL EMBEDDINGS AND IMMERSIONS

The study of symplectic properties of Stein manifolds started in the foundational work of Eliashberg and Gromov [8] and was continued by many authors. Recall that an (almost) complex manifold is called a *Stein manifold* if it admits a smooth strictly plurisubharmonic exhaustion function. Let (X, J_X) be a Stein manifold of complex dimension $n-1$ with a complex structure J_X. Fix a symplectic form ω_X taming J_X on X, i.e., $\omega(v, J_X v) > 0$ for every nonzero tangent vector v (see [3, 16, 21]). We use the notation (X, ω_X, J_X) for a complex manifold equipped with a taming symplectic form and a complex structure. Denote by $\omega_{\mathrm{st}} = (i/2)\sum_{j=1}^n dz_j \wedge d\overline{z}_j$ the standard symplectic form on $\mathbb{C}^n$; the value of n will be clear from the context. The product $M = \mathbb{C} \times X$ is also a Stein manifold with the complex structure $J = J_{\mathrm{st}} \otimes J_X$ and the taming symplectic form $\omega = \omega_{\mathrm{st}} \oplus \omega_X$. We call such an M a *reducible* Stein manifold. This class of Stein manifolds is our main object of study.

According to the classical theory of Stein manifolds, a complex manifold M is a Stein manifold if and only if it admits a smooth strictly plurisubharmonic exhaustion function ρ. In the almost complex case the existence of such a function is required by the definition of an almost complex Stein structure (see [7, 8]). For every positive integer k the sublevel set $M_k = \{p \in M \colon \rho(p) < k\}$ is a relatively compact domain in M, and the increasing sequence (M_k) is an exhaustion sequence for M. Consider a J-complex disc $f\colon \mathbb{D} \to M$ continuous on $\overline{\mathbb{D}}$ and such that $f(\partial\mathbb{D}) \subset \partial M_k$. Applying the maximum principle to the subharmonic function $\rho \circ f$ on the unit disc, we conclude that $f(\mathbb{D})$ is contained in $\overline{M_k}$. This means that every Stein manifold is *convex at infinity* in the sense of [10] and Gromov's compactness theorem can be applied on this class of manifolds. We will use this fact in the present paper.

Recall that ω and J canonically define the metric h by $h(u,v) = (\omega(u, Jv) + \omega(v, Ju))/2$. In what follows we use the norms and distances on M induced by h.

For a J-complex disc $f\colon \mathbb{D} \to (M, \omega, J)$, $\mathbb{D} \ni \zeta = \xi + i\eta \mapsto f(\zeta)$, its (symplectic) area is defined by

$$\operatorname{area}(f) = \int\limits_{\mathbb{D}} f^*\omega. \tag{2.1}$$

The quantity

$$E(f) := \frac{1}{2}\int\limits_{\mathbb{D}} \left(\left\| \frac{\partial f}{\partial \xi} \right\|_h^2 + \left\| \frac{\partial f}{\partial \eta} \right\|_h^2 \right) d\xi \wedge d\eta, \tag{2.2}$$

where the norm $\|\cdot\|_h$ is taken with respect to h, is called the *energy* of f. It coincides with the area defined by the metric h:

$$E(f) = \operatorname{area}(f). \tag{2.3}$$

This fundamental equality is called the energy identity (see, for instance, [16]). Similar notions still make sense for holomorphic maps $f\colon (\Omega, J_{\rm st}) \to (M, \omega, J)$, where Ω is a domain in $\mathbb{C}$. Of course, in this case the unit disc $\mathbb{D}$ must be replaced with Ω in (2.1)–(2.3).

Recall that a submanifold E in (M, ω, J) is called *totally real* if $T_pE \cap JT_pE = \{0\}$ for every point $p \in E$, and is called *Lagrangian* if $\omega|_E = 0$ and $\dim E = n$. It is well known that every Lagrangian manifold is totally real if J is tamed by ω; the converse is in general not true. *In the present paper we consider only totally real submanifolds of maximal possible dimension n in a manifold (M, ω, J) of complex dimension n.*

We begin with the embedded case. Our first result is the following theorem.

Theorem 2.1. *Let E be a compact totally real C^∞-smooth submanifold of real dimension n in a reducible Stein manifold M of complex dimension n. Then there exists an A-disc attached to E.*

As mentioned in the Introduction, Alexander proved this result for $M = \mathbb{C}^n$. We present a proof which does not use the integrability of the complex structure J. In fact, even if J is integrable, Gromov's method requires the use of almost complex structures.

Proof of Theorem 2.1. First we define suitable manifolds of discs.

Fix a point $p \in E$ and fix also a non-integer $r > 1$. Consider the set of maps

$$\mathcal{F} = \bigl\{ f \in C^{r+1}(\mathbb{D}, M)\colon\ f(\partial\mathbb{D}) \subset E,\ f(1) = p \bigr\}. \tag{2.4}$$

Note that f is defined on the boundary of $\mathbb{D}$, because it satisfies the Hölder condition in the disc.

Denote by F an open subset of $\mathcal{F}$ which consists of f homotopic to a constant map $f^0 \equiv p$. It is well known that F is a C^∞-smooth complex Banach manifold. A disc f is holomorphic if and only if it satisfies the Cauchy–Riemann equation

$$J \circ df = df \circ J_{\rm st}. \tag{2.5}$$

Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ be local coordinates on M (not necessarily holomorphic with respect to J) in a neighbourhood U of a point $p \in M$. That is, $z\colon U \to \mathbb{C}^n$ is a smooth local diffeomorphism and $z(p) = 0$. The direct image $z_*(J) = dz \circ J \circ dz^{-1}$ of J can be viewed as a complex structure in a neighbourhood of the origin. One can always choose z such that the structure $z_*(J)$ coincides with $J_{\rm st}$ at the origin.

In these coordinates a disc f can in turn be viewed as a map $z\colon \mathbb{D} \to \mathbb{C}^n$, $\zeta \mapsto z(\zeta)$, and the Cauchy–Riemann equations can be written in the form convenient for the usual analytic tools:

$$z_{\overline{\zeta}} - A(z)\overline{z}_{\overline{\zeta}} = 0, \tag{2.6}$$

where the complex $n \times n$ matrix function $A = A(z)$ satisfies the condition $\|A\| < 1$; we use the matrix norm induced by the Euclidean inner product. More precisely, A is uniquely determined by J as the matrix representation of the operator $(J_{\rm st} + z_*(J))^{-1}(J_{\rm st} - z_*(J))$, which is complex antilinear with respect to $J_{\rm st}$. In particular, $A(0) = 0$, since $z_*(J)$ coincides with $J_{\rm st}$ at the origin. The integrability of J means that the local coordinates can be chosen to be holomorphic, which is equivalent to A vanishing identically in a neighbourhood of the origin (see more details in [17]).

Denote by V the bundle $\overline{\mathbb{D}} \times TM$ over $\overline{\mathbb{D}} \times M$. For every disc f, consider $V_f = f^*TM$, the pull-back by f of the tangent bundle TM. It can be viewed as the restriction of V to the graph of f in $\overline{\mathbb{D}} \times M$. Denote by $\Omega^{0,1}\mathbb{D}$ the bundle of $(0,1)$-forms on $\mathbb{D}$. Extend this bundle to $\overline{\mathbb{D}} \times M$ keeping the same notation $\Omega^{0,1}\mathbb{D}$. Then we obtain the bundle $\Omega^{0,1}\mathbb{D} \otimes V$ over $\overline{\mathbb{D}} \times M$.

We introduce the operator $\overline{\partial}_J$ by setting

$$\overline{\partial}_J f = \frac{1}{2}(df + J \circ df \circ J_{\mathrm{st}}). \tag{2.7}$$

This is just the complex antilinear part of df with respect to J. This operator takes its values in the bundle $\Omega^{0,1}\mathbb{D} \otimes V$. More precisely, for every $\zeta \in \mathbb{D}$ the expression $\overline{\partial}_J f(\zeta)$ belongs to $\Omega^{0,1}_\zeta \mathbb{D} \otimes V_{(\zeta, f(\zeta))}$, the fibre of $\Omega^{0,1}\mathbb{D} \otimes V$ over $(\zeta, f(\zeta))$. Conversely, for every continuous section $g = g(\zeta, z) \in \Gamma(\overline{\mathbb{D}} \times M, \Omega^{0,1}\mathbb{D} \otimes V)$ we can consider the nonhomogeneous Cauchy–Riemann equation

$$\overline{\partial}_J f(\zeta) = g(\zeta, f(\zeta)). \tag{2.8}$$

A more detailed discussion can be found in [10, 12, 16, 17].

An observation of Gromov [10] allows us to interpret the nonhomogeneous equation (2.8) as the usual Cauchy–Riemann equation (2.5) for a suitable almost complex structure determined by J and g.

Consider the product $\mathbb{D} \times M$ and define there an almost complex structure J_g by

$$J_g = \begin{pmatrix} J_{\mathrm{st}} & 0 \\ g & J \end{pmatrix}. \tag{2.9}$$

Note that $J_g|_{TM} = J$.

Lemma 2.2 (see [10])**.** *A disc $f\colon \mathbb{D} \to M$ satisfies* (2.8) *if and only if the map $\widehat{f}\colon \zeta \mapsto (\zeta, f(\zeta))$ is J_g-complex, i.e., satisfies equation* (2.5) *with $J = J_g$. Furthermore, there exists a constant $C_0 = C_0(M, \omega, J)$ such that for every $g \in C^0(\overline{\mathbb{D}} \times M, \Omega^{0,1}\mathbb{D} \otimes V)$ with $\|g\|_{L^\infty(\overline{\mathbb{D}} \times M)} \le C_1 < \infty$ the structure J_g is tamed by the symplectic form $\widehat{\omega} = C_0 C_1 \omega_{\mathrm{st}} \oplus \omega$.*

This construction can easily be viewed in local coordinates quite similarly to the equivalence between the coordinate-free version of the homogeneous Cauchy–Riemann equations (2.5) and their coordinate representation (2.6). Indeed, consider the lift $\widehat{f}\colon \zeta \mapsto (\zeta, f(\zeta))$ of f to $\mathbb{C}^{n+1} = \mathbb{C}_w \times \mathbb{C}^n_z$. In coordinates on $\mathbb{C}^n$ a section g can be viewed as a "vector-valued" form, i.e., a $(0,1)$-form on $\mathbb{D}$ with values in $\mathbb{C}^n$. Hence, it can be identified with a map $g\colon \mathbb{D} \to \mathbb{C}^n$ (we denote it again by g). Then in coordinates the nonhomogeneous $\overline{\partial}_J$-equation (2.8) can be written in the form

$$z_{\overline{\zeta}}(\zeta) - A(z(\zeta))\overline{z(\zeta)}_{\overline{\zeta}} = g(\zeta), \tag{2.10}$$

which is equivalent to the following PDE system for the lift $\widehat{f}$:

$$\begin{cases} w_{\overline{\zeta}} = 0, \\ z_{\overline{\zeta}} - g(w)\overline{w}_{\overline{\zeta}} - A(z)\overline{z}_{\overline{\zeta}} = 0. \end{cases} \tag{2.11}$$

This is precisely the (homogeneous) Cauchy–Riemann equations (2.6) for the almost complex structure J_g on $\mathbb{D} \times \mathbb{C}^n$.

Denote by G the complex Banach space of all sections $g \in C^r(\overline{\mathbb{D}} \times M, \Omega^{0,1}\mathbb{D} \otimes V)$. Set

$$H = \{(f, g) \in F \times G\colon\ \overline{\partial}_J f = g\}. \tag{2.12}$$

Then H is a connected submanifold of $F \times G$.

We need the following

Lemma 2.3. *Suppose that a sequence (f_k) in $\mathcal{F}$ converges to a continuous mapping $f\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, E)$ uniformly on $\overline{\mathbb{D}}$ and the sequence (g_k), $g_k := \overline{\partial}_J f_k$, converges in G to $g \in G$.*

Assume also that the energies $E(f_k)$ are uniformly bounded. Then $f \in C^{r+1}(\mathbb{D})$ and (f_k) converges to f in $\mathcal{F}$ after possibly passing to a subsequence.

This is quite a special case of Gromov's compactness theorem [10] (see details in [21, Proposition 5.1.2]). Indeed, the lifts $\widehat{f}_k$ are attached to the manifold $\widehat{E} := \partial\mathbb{D} \times E$, which is totally real with respect to the almost complex structure J_{g_k}. By the hypothesis of the lemma, the areas of $\widehat{f}_k$ are uniformly bounded. Since the sequence $(\widehat{f}_k)$ converges uniformly, bubbles cannot occur and the lemma follows by Gromov's compactness. Note that the simplest version of Gromov's compactness theorem is used here. The proof is based on standard elliptic estimates in the interior of the disc and near the boundary, where the reflection principle can be used (for the refection principle and elliptic estimates for J-complex curves with totally real boundary data, see, for example, [13]). Technically all elliptic estimates follow from the classical regularity properties of the integral transform

$$T_j f(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathbb{D}} \frac{f(\tau)\, d\tau \wedge d\overline{\tau}}{(\tau - \zeta)^j}, \qquad j = 1, 2.$$

For $j = 1$ this is the ordinary Cauchy transform; for $j = 2$ this is its formal derivative (the Beurling transform), which is defined as a singular integral operator (i.e., in the sense of the Cauchy principal value). In the case of $(\mathbb{C}^n, J_{\mathrm{st}})$ considered by Alexander, we have $A = 0$ in equations (2.6) and the proof becomes particularly transparent (in particular, the Beurling transform is not needed). We point out that all estimates are purely local and can be obtained near each interior point in $\mathbb{D}$ or a point in $\partial\mathbb{D}$, and then globalized using finite open coverings.

Recall that a linear bounded map $u\colon L \to L'$ between two Banach spaces is called a *Fredholm operator* if $\ker u$ and $\operatorname{coker} u$ have finite dimension; the *Fredholm index* $\dim\ker u - \dim\operatorname{coker} u$ is homotopy invariant. A C^1-map $\phi\colon M_1 \to M_2$ between two Banach manifolds is called a *Fredholm map* if for every point $q \in M_1$ the tangent map $d\phi_q\colon T_qM_1 \to T_{\phi(q)}M_2$ is a Fredholm operator; the index of the tangent map is independent of q and is called the *index* of ϕ. A point $q \in M_1$ is called a *regular point* if $d\phi_q$ is surjective. A point $p \in M_2$ is called a *regular value* if $\phi^{-1}(p)$ does not contain nonregular points (in particular, $\phi^{-1}(p)$ can be empty).

Consider the canonical projection $\pi\colon H \to G$ given by $\pi(f, g) = g$. The following properties of π are well known [1, 10, 12]:

(i) π is a map of class C^1 between two Banach manifolds;

(ii) π is a Fredholm map of index 0;

(iii) the constant map f^0 is a regular point for π.

The crucial property of π is that the *map* $\pi\colon H \to G$ *is not surjective* [1, 10, 12]. More precisely, in our case it follows from the argument of [12, p. 104]. Note that this argument requires that $M = \mathbb{C} \times X$; this is why we consider reducible Stein manifolds rather than arbitrary Stein manifolds.

Arguing by contradiction, suppose that an A-disc of class $C^{r+1}(\mathbb{D})$ for E does not exist. In particular, $\pi^{-1}(0) = \{f^0\}$. Then $0 \in G$ is a regular value of π. If π is proper, then Gromov's argument based on the Sard–Smale theorem implies the surjectivity of π (see [1, 10, 12]), which is a contradiction. Thus, it remains to show that $\pi\colon H \to G$ is proper.

All we need is a well-known description of bubbling; we follow [21]. Arguing by contradiction, suppose that π is not proper. Then there exists a sequence $\{(f_k, g_k)\} \subset H$ such that $g_k \to g$ in G but f_k diverge in F. Consider the lifts $\widehat{f}_k(\zeta) = (\zeta, f_k(\zeta))$ and $\widehat{f}\colon \mathbb{D} \to \mathbb{C} \times M$ as in Lemma 2.2. Every $\widehat{f}_k$ is holomorphic with respect to the almost complex structure J_{g_k} tamed by the symplectic form $\widehat{\omega}$ as in Lemma 2.2. We measure norms and distances using the metric h_k defined by $\widehat{\omega}$ and J_{g_k}. Set $M_k = \sup_{\overline{\mathbb{D}}}\|d\widehat{f}_k(\zeta)\|$. There exists $\lambda_k \in \overline{\mathbb{D}}$ such that $M_k = \|d\widehat{f}_k(\lambda_k)\|$.

If the sequence (M_k) is bounded, then by Lemma 2.3 the sequence (f_k) converges, so we can assume that $M_k \to +\infty$.

Case 1: *the sequence* (λ_k) *converges to a point in* $\mathbb{D}$. Without loss of generality assume that it converges to 0. Consider the renormalized sequence $F_k(\zeta) := \widehat{f}_k(\lambda_k + \zeta/M_k)$. Then the gradients of the maps in this sequence are uniformly bounded on every compact subset of $\mathbb{C}$ and we can assume that it converges uniformly to some J_g-holomorphic map $\widehat{f}\colon \mathbb{C} \to \mathbb{D} \times M$. This map is bounded since the sequence $(\widehat{f}_k)$ is. Furthermore, $\widehat{f} = (0, f)$ and the map f is holomorphic with respect to J_X (see equations (2.11)). Since (X, J_X) is a Stein manifold, it admits a strictly plurisubharmonic function u. Then the composition $u \circ f$ is a subharmonic function bounded on $\mathbb{C} \setminus \{0\}$. Therefore, it extends as a bounded subharmonic function on $\mathbb{C}$. Hence $u \circ f$ is constant and f is constant. However, it is easy to check that $\|d\widehat{F}_k(0)\| = 1$ (see [21, p. 184, case (a)]). This is a contradiction.

Case 2: *the sequence* (λ_k) *converges to a point in* $\partial\mathbb{D}$. Let $\delta_k = 1 - |\lambda_k|$. If $(M_k\delta_k)$ is an unbounded sequence, then, arguing as in [21, p. 184, case (b)], we reduce the situation to case 1. Hence, the only possibility is that the sequence $(M_k\delta_k)$ is bounded. Then the standard renormalization argument [21, p. 184, case (c)] produces a noncompact sequence (ϕ_k) of automorphisms of $\mathbb{D}$ such that (ϕ_k) converges uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$ to a constant map and such that the maps $\widehat{f}_k \circ \phi_k$ have uniformly bounded gradients on every compact subset of $\overline{\mathbb{D}} \setminus \{1\}$. One can assume that the sequence $(\widehat{f}_k \circ \phi_k)$ converges uniformly on every compact subset of $\overline{\mathbb{D}} \setminus \{1\}$. By Lemma 2.3 the convergence will be in the C^m-norm on every compact subset of $\overline{\mathbb{D}} \setminus \{1\}$ for every m to a J_g-complex disc. The limit map is nonconstant, as shown in [21, p. 184, case (c)]. Hence, the limit is an A-disc for $\widehat{E}$ of the form (const, f). Then f is an A-disc for E. This is a contradiction, which proves that π is proper and completes the proof of Theorem 2.1. $\square$

Note that in the above argument the discs in renormalized sequences have uniformly bounded gradients (hence uniformly bounded areas) only on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$. Therefore, in general the whole area of a constructed A-disc can be infinite. If E is a Lagrangian manifold, then the areas of compact sets in $\overline{\mathbb{D}}$ are uniformly bounded and the constructed A-disc f has a bounded area, so it is just the usual bubble. By Gromov's removable singularity theorem, f extends to the point 1 as a map of class $C^\infty(\overline{\mathbb{D}})$, and we obtain Gromov's theorem on the existence of a nonconstant holomorphic disc attached to a Lagrangian submanifold in $\mathbb{C}^n$.

Consider now the case of totally real immersions. Only minor modifications of the above argument are needed. Let $E = (\widetilde{E}, \iota)$ be a pair which consists of a compact smooth manifold $\widetilde{E}$ of dimension n and a C^∞-smooth totally real immersion $\iota\colon \widetilde{E} \to M$. We will identify it with the image $\iota(\widetilde{E})$ and simply say that E is an immersed totally real manifold in M. We say that an A-disc f is *adapted* for the immersion E if for every point $\zeta \in \partial\mathbb{D} \setminus \{1\}$ there exists an open arc $\gamma \subset \partial\mathbb{D}$ containing ζ and a smooth map $f_{\mathrm{b}}\colon \gamma \to \widetilde{E}$ satisfying $\iota \circ f_{\mathrm{b}} = f|_\gamma$. In other words, in a neighbourhood of every self-intersection point p of E, the values of f belong to a smooth component of E through p. By the cluster set $C(f, \partial\mathbb{D})$ of a complex disc we mean the set of partial limits of the sequences $f(\zeta_k)$ for all sequences (ζ_k) in $\mathbb{D}$ converging to $\partial\mathbb{D}$, i.e., such that $\mathrm{dist}(\zeta_k, \partial\mathbb{D}) \to 0$.

Theorem 2.4. *Let $E = (\widetilde{E}, \iota)$ be an immersed totally real manifold in a reducible Stein manifold M. Then*

(i) *there exists an adapted A-disc $f \in C(\overline{\mathbb{D}} \setminus \{1\})$ for E;*

(ii) *if in addition E is Lagrangian, then f is of bounded area with the cluster set $C(f, \partial\mathbb{D})$ contained in E; its image $\Sigma = f(\mathbb{D})$ is a holomorphic curve of bounded area with the boundary $\partial\Sigma := \overline{\Sigma} \setminus \Sigma$ contained in E.*

Proof. We begin with assertion (i). Fix a point $p = \iota(\widetilde{p}) \in E$ which is not a self-intersection point and also fix a non-integer $r > 1$. Consider the set of pairs

$$\mathcal{F} = \big\{(f, f_{\mathrm{b}}) \in C^{r+1}(\mathbb{D}, M) \times C^{r+1}(\partial\mathbb{D}, \widetilde{E})\colon\ f(\partial\mathbb{D}) \subset E,\ f(1) = p,\ \iota \circ f_{\mathrm{b}} = f|_{\partial\mathbb{D}}\big\}. \tag{2.13}$$

In other words, together with a (not necessarily complex) disc f we specify a lift of its boundary to the manifold $\widetilde{E}$. For brevity we write f instead of (f, f_{b}).

Denote by F an open subset of $\mathcal{F}$ which consists of f homotopic to a constant map $f^0 \equiv p$ in $\mathcal{F}$. It is well known that F is a C^∞-smooth complex Banach manifold. Now we define G and H as above. Note that H is a connected submanifold of $F \times G$.

An immediate but crucial observation is that the proof of Lemma 2.3 is *purely local*, i.e., all estimates and the convergence are established in a neighbourhood of a given boundary point of a disc. This local character of Lemma 2.3 allows us to pass automatically from an embedded E to a globally immersed $E = (\widetilde{E}, \iota)$ in Lemma 2.3. Indeed, suppose that q is a self-intersection point of E and $f(\zeta_0) = q$ for some $\zeta_0 \in \partial\mathbb{D}$. It follows from the uniform convergence of the sequence (f_k) and the definition of the set $\mathcal{F}$ that there exists a neighbourhood U of ζ_0 such that $f(U \cap \partial\mathbb{D})$ and, after passing to a subsequence, all $f_k(U \cap \partial\mathbb{D})$ belong to the same smooth component through p of the immersed manifold E. This reduces the situation to the embedded case of Gromov's compactness theorem.

The canonical projection $\pi\colon H \to G$ has the same properties as in the embedded case (see [1, 12]). Arguing again by contradiction, assume that an adapted A-disc of class $C^{r+1}(\mathbb{D})$ for E does not exist. As above, in order to get a contradiction, we show that $\pi\colon H \to G$ is proper.

Suppose on the contrary that π is not proper and consider a sequence $\{(f_k, g_k)\} \subset H$ as above and the corresponding M_k and λ_k. If the sequence (M_k) is bounded, then by Lemma 2.3 the sequence (f_k) converges, so we may assume that $M_k \to +\infty$.

Case 1: *the sequence* (λ_k) *converges to a point of* $\mathbb{D}$. In this case we obtain a contradiction as in the previous theorem.

Case 2: *the sequence* (λ_k) *converges to a point of* $\partial\mathbb{D}$. Again, as in the previous proof, this case can be handled using a normalization. It provides a noncompact sequence (ϕ_k) of automorphisms of $\mathbb{D}$ such that (ϕ_k) converges uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$ to a constant map and such that $\widehat{f}_k \circ \phi_k$ have uniformly bounded gradients on every compact subset of $\overline{\mathbb{D}} \setminus \{1\}$. Hence we assume that the sequence $(\widehat{f}_k \circ \phi_k)$ converges uniformly there. Recall that we are dealing with adapted discs; locally their boundaries are attached (along every sufficiently small open arc) to a single regular branch of E, which is an embedded manifold. Since Lemma 2.3 is local, it applies in our situation, which gives the convergence also in the C^{r+1}-norm on every compact subset in $\overline{\mathbb{D}} \setminus \{1\}$ (the intersection $K \cap \partial\mathbb{D}$ can be covered by a finite number of open arcs such that every arc is taken by the maps to a single regular branch of E). This is the key observation that makes Alexander's construction valid in the immersed case. Since locally E is an embedding and the limit disc is adapted, it is C^∞-smooth on $\overline{\mathbb{D}} \setminus \{1\}$ by the boundary regularity theorem for complex discs with (embedded) totally real boundary value conditions (see, for example, [13, 21]). Therefore, the limit disc is an adapted A-disc of the form (const, f) for $\widehat{E}$. Then f is an adapted A-disc for E. This contradiction proves that π is proper and completes the proof of assertion (i).

To prove assertion (ii), note that the A-disc f constructed in assertion (i) is of bounded area. It follows from [14, Lemma A.4.1] that the cluster set $C(f, \partial\mathbb{D})$ is contained in E. This completes the proof. $\square$

Let K be a compact subset in a complex manifold M. Its psh convex hull is defined by

$$\widehat{K}_M^{\mathrm{psh}} = \left\{ p \in M \;\middle|\; u(p) \le \sup_K u \ \text{ for all continuous plurisubharmonic functions } \ u\colon M \to \mathbb{R} \right\}.$$

Such a K is called psh convex in M if $\widehat{K}_M^{\mathrm{psh}} = K$ (see, for example, [3, 22]).

Corollary 2.5. *Let E be a compact totally real immersion of dimension n in a reducible Stein manifold M of complex dimension n. Then E is not psh convex.*

This follows from Theorem 2.4(i), because by the maximum principle for subharmonic functions an A-disc is contained in the psh convex hull of E (of course, E does not contain nonconstant holomorphic curves since it is totally real).

Remarks and comments. 1. Theorems 2.1 and 2.4 remain true for an almost complex Stein manifold $(M, \omega, J) = \mathbb{C} \times X$ with a symplectic form ω taming J. Indeed, all proofs go through without modifications. The existence of a strictly plurisubharmonic function implies that a bounded holomorphic map from $\mathbb{C} \setminus \{0\}$ to X is constant; we used this in case 1 of the above proof.

2. Corollary 2.5 is well known in the case when $M = \mathbb{C}^n$ and E is a smooth (or even topological) submanifold (see [3, 22]); for totally real immersions in $\mathbb{C}^n$ it is obtained in [20]. Note that the topological methods used in [3, 22] can be adapted to the situation considered in Corollary 2.5. This was pointed out to us by S. Nemirovski. We also point out that, as is well known, there exist compact totally real manifolds (for example, some n-tori in $\mathbb{C}^n$) which do not contain the whole boundary of a nonconstant complex disc (see [2, 5]). An A-disc for such a manifold necessarily has infinite area.

3. Ivashkovich and Shevchishin [12] proved the existence of a complex disc f attached to an immersed Lagrangian manifold E under the assumption of weak transversality of E; in particular, this assumption holds for transverse double intersections. Their approach follows the original work of Gromov. They proved a general version of the compactness theorem (including the reflection principle and the removal of singularities) for J-complex curves with boundaries glued to a Lagrangian immersion with weakly transverse self-intersections. Their method also works for some symplectic manifolds of the form $\mathbb{C} \times X$ with tamed almost complex structures satisfying the assumptions of Gromov's compactness theorem.

4. It seems quite possible that our results can be extended to a wider class of Stein manifolds than the one of reducible Stein manifolds. On the other hand, it is clear that some restrictions on a class of Stein manifolds are necessary. Indeed, let $E = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_j| = 1,\ j = 1, 2\}$ be the standard torus in $\mathbb{C}^2$. The function $\rho(z) = \operatorname{dist}(z, E)^2$ (the usual Euclidean distance) is strictly plurisubharmonic in a neighbourhood of E and $M = \{z\colon \rho(z) < \varepsilon\}$ is a Stein manifold for $\varepsilon > 0$ small enough. It follows by the maximum principle that every nearly smooth complex disc in M with boundary attached to E is constant.

5. Since the complex curve Σ constructed in Theorem 2.4(ii) has finite area, it defines the current of integration $[\Sigma]$, which acts on a $(1, 1)$ test form ϕ by

$$\langle [\Sigma], \phi \rangle = \int_{\Sigma} \phi.$$

This is a positive current of finite mass and of bidimension $(1, 1)$ such that its support is contained in the hull of a Lagrangian immersion E and the support of the boundary current $d[\Sigma]$ is contained in E. If E is only totally real, an A-disc can have an infinite area and the current $[\Sigma]$ is not defined. Nevertheless, Duval and Sibony [6] showed how to use an A-disc in order to construct a positive current of bidimension $(1, 1)$ and of finite mass with the support contained in the polynomially convex hull of a totally real submanifold of $\mathbb{C}^n$ and with the boundary contained in E; their result also holds for totally real immersions [20] in $\mathbb{C}^n$ (of course, in general such a current is not a current of integration over a complex curve). Using Theorems 2.1 and 2.4, one can easily extend these results to the case of Stein manifolds. Applying methods of symplectic topology, Viterbo [23] proved that a totally real submanifold in an n-dimensional Stein manifold admitting an exhaustion strictly plurisubharmonic function with critical points of Morse index $< n$ contains the boundary of a complex curve.

6. Let E be a compact subset of $\mathbb{C}^n$ and p be a point in the polynomially convex hull of E. A number of papers are devoted to the construction of a holomorphic disc f centred at p with (a part of) the boundary contained in a prescribed neighbourhood of E. The first result of this type is due to Poletsky [18]. It was extended in several directions by Lárusson and Sigurdsson [15], Rosay [19], Drinovec Drnovšek and Forstnerič [4], and other authors.

7. If E is a smooth Lagrangian embedding in $\mathbb{C}^n$, the existence of a non-constant complex disc glued to E implies that E cannot be exact. This follows by a simple application of the Stokes formula: the symplectic area of such a disc is strictly positive and this immediately gives a contradiction to the exactness. However, this argument uses the high boundary regularity of a complex disc. In the case of Lagrangian immersion, the attached complex discs are just continuous up to the boundary. In general, they do not form an obstacle to the exactness. For example, Alexander [2] studied the hull of an exact Lagrangian immersion of the sphere S^2 to $\mathbb{C}^2$. The hull is filled with a one-parameter family of complex discs whose boundaries contain the only self-intersection point of the sphere. These discs are not regular enough, and the immersion is exact.

3. LAGRANGIAN IMMERSIONS TO STEIN MANIFOLDS

We begin with

Definition 3.1. A closed subset S of a complex manifold M is called *locally psh convex near a point* $p \in X$ if there exists a Stein neighbourhood U of p such that for every sufficiently small $\varepsilon > 0$ the intersection $S \cap \overline{\mathbb{B}(p,\varepsilon)}$ is psh convex in U.

Our next result establishes the local plurisubharmonic convexity near transverse self-intersection of Lagrangian immersions.

Theorem 3.2. *Let (M,ω,J) be a Stein manifold of complex dimension n. Assume that L_1 and L_2 are smooth totally real submanifolds such that their tangent spaces T_pL_1 and T_pL_2 are Lagrangian and intersect transversely at a point p. Then the union $L_1 \cup L_2$ is locally psh convex near p.*

Proof. The proof can be reduced to the case of $\mathbb{C}^n$ considered in [20]. In local holomorphic coordinates, we can identify p with the origin and view M as an open ball $\mathbb{B}(0,\varepsilon)$ equipped with the standard complex structure J_{st}, where $\varepsilon > 0$ is small enough. Consider the tangent spaces $E_j = T_0L_j$, $j = 1, 2$.

Lemma 3.3. *The union $E_1 \cup E_2$ is polynomially convex in $\mathbb{C}^n$.*

This result was proved in [20] for the case when E_j are Lagrangian spaces with respect to the standard symplectic structure ω_{st}. The same argument holds in our case of general ω taming J_{st}.

Proof of Lemma 3.3. If the union $E_1 \cup E_2$ is not polynomially convex in $\mathbb{C}^n$, there exists a nonconstant holomorphic annulus f with the boundary attached to $E_1 \cup E_2$ (see [20] for details). This is just a nonconstant map $f\colon \Omega \to \mathbb{C}^n$ holomorphic on the closed annulus $\Omega = \{\zeta \in \mathbb{C} \mid r_1 \le |\zeta| \le r_2\}$ and such that $f(r_j\partial\mathbb{D}) \subset E_j$, $j = 1, 2$; here $0 < r_1 < r_2$. For every $\delta > 0$, the annulus δf also is glued to $E_1 \cup E_2$. Choosing δ small enough, we can assume that δf is contained in $\mathbb{B}(0,\varepsilon)$. Since J_{st} is tamed by ω, the symplectic area of δf defined by (2.1) (with Ω instead of $\mathbb{D}$) is strictly positive. Let a 1-form λ be a primitive of ω in $\mathbb{B}(0,\varepsilon)$. Since E_j are Lagrangian spaces, the restrictions $\lambda|_{E_j}$, $j = 1, 2$, are exact. Then by Stokes' formula the area of δf is independent of δ and therefore is equal to zero. This is a contradiction. $\square$

Then, by [20], for every $\varepsilon > 0$ small enough the set $(L_1 \cup L_2) \cap \overline{\mathbb{B}(0,\varepsilon)}$ is polynomially convex in $\mathbb{C}^n$. Hence, there exists a smooth nonnegative plurisubharmonic function ρ on $\mathbb{C}^n$ which is strictly plurisubharmonic on $\mathbb{C}^n \setminus (L_1 \cup L_2 \cap \overline{\mathbb{B}(0,\varepsilon)})$ and is such that $(L_1 \cup L_2) \cap \overline{\mathbb{B}(0,\varepsilon)} = \rho^{-1}(0)$ (see [22, Theorem 1.3.8]). Transporting this function to a neighbourhood of p in M by a local holomorphic chart, we obtain a function with similar properties near p on M. This is equivalent to the local plurisubharmonic convexity of $L_1 \cup L_2$ (see [3, Proposition 5.13]). $\square$

Remark. In [7] Eliashberg obtained the following relevant result. Let L_1 and L_2 be two totally real submanifolds in $\mathbb{C}^n$ intersecting transversely at the origin. Suppose that the union of tangent spaces $T_0L_1 \cup T_0L_2$ is invariant with respect to J_{st}. Then $L_1 \cup L_2$ is locally polynomially convex near the origin. This result is a special case of Theorem 3.2. Indeed, after a $\mathbb{C}$-linear change of coordinates we have $T_0L_1 = \mathbb{R}^n$; therefore $T_0L_2 = i\mathbb{R}^n$. These spaces are Lagrangian with respect to the standard symplectic form ω_{st}, and the result follows from Theorem 3.2.

Now, arguing literally as in [20], we obtain the following result.

Theorem 3.4. *Suppose that a smooth compact Lagrangian immersion L to a reducible Stein manifold M has a finite number of self-intersection points and is locally psh convex near every self-intersection point. Then there exists a nonconstant complex disc continuous on $\overline{\mathbb{D}}$ with boundary attached to L.*

Indeed, by [3, Proposition 5.13], for every self-intersection point there exists a neighbourhood U and a smooth positive plurisubharmonic function ρ on U which is strictly plurisubharmonic on $U \setminus L$ and is such that $L \cap U = \rho^{-1}(0)$. Then similarly to [20, Sect. 5], these functions can be glued to a global plurisubharmonic function in a neighbourhood of L. Together with Theorem 2.4(ii) this implies the continuity of a complex disc up to the boundary. In the case of Lagrangian embeddings we again recover the result of Gromov [10]. In view of Theorem 3.2 we have the following

Corollary 3.5. *Let L be a smooth compact Lagrangian immersion to a reducible Stein manifold M with a finite number of double transverse self-intersection points. Then there exists a nonconstant complex disc continuous on $\overline{\mathbb{D}}$ with the boundary attached to L.*

Note that this result is also a consequence of Ivashkovich and Shevchishin's results [12]. In general Theorem 3.4 works in some cases when the Ivashkovich–Shevchishin result cannot be applied. Indeed, a Lagrangian immersion can be locally psh convex but not weakly transversal in the sense of [12] (see examples in [20]). It remains an open question whether Corollary 3.5 holds without any assumption on the set of self-intersection points (as Gromov suggested in [11]). Nevertheless, Theorem 2.4(ii) gives the existence of a non-constant Riemann surface Σ with the boundary contained in L without any assumptions on the set of self-intersection points.

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