# LAGRANGIAN INCLUSION WITH AN OPEN WHITNEY UMBRELLA IS RATIONALLY CONVEX

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Abstract. It is shown that a Lagrangian inclusion of a real surface in $\mathbb{C}^2$ with a standard open Whitney umbrella and double transverse self-intersections is rationally convex.

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## 1. Introduction

This paper is concerned with the study of rational convexity of compact real surfaces in $\mathbb{C}^2$. A compact set $X$ in $\mathbb{C}^n$ is *rationally convex* if for every point $p$ in the complement of $X$ there exists a complex algebraic hypersurface passing through $p$ and avoiding $X$. See Stout [9] for a comprehensive treatment of this fundamental notion.

A nondegenerate closed 2-form $\omega$ on $\mathbb{C}^2$ is called a *symplectic form.* By Darboux's theorem every symplectic form is locally equivalent to the standard form

$$\omega_{\rm st} = \frac{i}{2}(dz\wedge d\bar z + dw \wedge d\bar w) = dd^c\,\phi_{\rm st}, \quad \phi_{\rm st} = |z|^2+|w|^2,$$

where $(z,w)$, $z=x+iy$, $w=u+iv$ are complex coordinates in $\mathbb{C}^2$, and $d^c = i(\overline{\partial}-\partial)$. If a symplectic form $\omega$ is of bidegree $(1,1)$ and strictly positive, it is called a *Kähler form.* A strictly plurisubharmonic function $\phi$ is called a potential of $\omega$ if $dd^c\phi=\omega$. A real $n$-dimensional submanifold $S\subset\mathbb{C}^n$ is called *Lagrangian* for $\omega$ if $\omega|_S=0$. According to a theorem of Duval and Sibony [2], a compact $n$-dimensional submanifold of $\mathbb{C}^n$ is rationally convex if and only if it is Lagrangian for some Kähler form. This result displays a connection between rational convexity and symplectic properties of real submanifolds.

Being Lagrangian imposes certain topological restrictions on a submanifold, for example, the only compact orientable surface that admits a Lagrangian embedding into $(\mathbb{C}^2,\omega_{\rm st})$ is a torus. On the other hand, according to the result of Givental [4], any compact surface (orientable or not) admits a *Lagrangian inclusion* into $\mathbb{C}^2$, i.e., a smooth map $\iota: S\to\mathbb{C}^2$ which is a local Lagrangian embedding except a finite set of singular points that are either transverse double self-intersections or the so-called *open Whitney umbrellas.* The *standard* open Whitney umbrella is a map

$$\pi:\mathbb{R}^2_{(t,s)}\ni(t,s)\mapsto\left(ts,\frac{2t^3}{3},t^2,s\right)\in\mathbb{R}^4_{(x,u,y,v)}. \tag{1}$$

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The open Whitney umbrella is then defined as the image of the standard umbrella under a local symplectomorphism, i.e., a local diffeomorphism that preserves the form $\omega_{\text{st}}$. It was proved by Gayet [3] that an immersed Lagrangian (with respect to some Kähler form) submanifold in $\mathbb{C}^n$ with transverse double self-intersections is also rationally convex. This was generalized to certain nontransverse self-intersections by Duval and Gayet [1].

The goal of this paper is show how the technique of [2], [3], and [1] can be adapted to prove rational convexity of a Lagrangian inclusion with one standard open Whitney umbrella. More precisely, we prove the following.

**Theorem 1.** *Let $\iota : S \mapsto (\mathbb{C}^2, \omega_{\text{st}})$ be a Lagrangian inclusion of a compact surface $S$. Suppose that the singularities of $\iota$ consist of transverse double self-intersections and one standard open Whitney umbrella. Then $\iota(S)$ is rationally convex in $\mathbb{C}^2$.*

We remark that the standard open Whitney umbrella can be replaced by its image under a complex affine map that preserves the symplectic form $\omega_{\text{st}}$. The existence of Lagrangian inclusions satisfying the conditions of Theorem 1 follows from a recent result of Nemirovski and Siegel [6].

## 2. Proof of Theorem 1.

We will identify $S$ and $\iota(S)$ as a slight abuse of notation. The ball of radius $\varepsilon$ centred at a point $p$ is denoted by $\mathbb{B}(p, \varepsilon)$, and the standard Euclidean distance between a point $p \in \mathbb{C}^n$ and a set $Y \subset \mathbb{C}^n$ is denoted by $\text{dist}(p, Y)$. Our approach is a modification of the method of Duval-Sibony and Gayet. The main tool here is the following result.

**Lemma 2** ([2], [3])**.** *Let $\phi$ be a plurisubharmonic $C^\infty$-smooth function on $\mathbb{C}^n$, and let $h$ be a $C^\infty$-smooth function on $\mathbb{C}^n$ such that*

(1) $|h| \le e^\phi$, *and* $X := \{|h| = e^\phi\}$ *is compact;*
(2) $\overline{\partial} h = O(\text{dist}(\cdot, S)^{\frac{3n+5}{2}})$*;*
(3) $|h| = e^\phi$ *with order 1 on $S$;*
(4) *For any point $p \in X$ at least one of the following conditions hold: (i) $h$ is holomorphic in a neighbourhood of $p$, or (ii) $p$ is a smooth point of $S$, and $\phi$ is strictly plurisubharmonic at $p$.*

*Then $X$ is rationally convex.*

The proof of Theorem 1 consists of finding the functions $\phi$ and $h$ that satisfy Lemma 2 and such that the set $X$ contains $S$ and is contained in the union of $S$ with the balls of arbitrarily small radius centred at singular points of $S$. This will be achieved in three steps: first we construct a closed $(1,1)$-form $\omega$ that vanishes near singular points of $S$ and such that $\omega|_S = 0$. This is done is Section 2.1. The form $\omega$ is a modification of the standard symplectic form $\omega_{\text{st}}$ in $\mathbb{C}^2$ near singular points of $S$. Near self-intersection points this is done in the paper of Gayet [3], and so we will deal with the umbrella point. Secondly, from $\omega$ and its potential $\phi$ we construct the required function $h$. This is done in Section 2.2. In the last step, in Section 2.3, we replace $\phi$ with a function $\phi + \rho$, for a suitable $\rho$, so that the pair $\{\phi + \rho, h\}$ satisfies all the conditions of Lemma 2.

2.1. **The form $\omega$.** Near the umbrella point the Lagrangian inclusion map $\iota$ coincides with $\pi$ given by (1). For a function $f$ we have

$$d^c f = -f_y dx + f_x dy - f_v du + f_u dv.$$

Direct computations show that $\pi^* d^c \phi_{\rm st} = -2t^2 s dt - \frac{2}{3} t^3 ds$. Consider the pluriharmonic function $\zeta = \frac{v^2}{2} - \frac{u^2}{2}$. Then $\pi^* d^c \zeta = \pi^* d^c \phi_{\rm st}$. The function

$$\phi = \phi_{\rm st} - \zeta$$

is strictly plurisubharmonic and satisfies

$$\pi^* d^c \phi = 0. \tag{2}$$

Let $r : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth increasing convex function such that $r(t) = 0$ when $t \le \varepsilon_1$ and $r(t) = t - c$ when $t > \varepsilon_2$, for some suitably chosen $c > 0$ and $0 < \varepsilon_1 < \varepsilon_2$. We choose $\varepsilon_2 > 0$ so small that the set $\{\phi < \varepsilon_2\}$ does not contain singular points of $S$ except the origin. Let

$$\omega = dd^c(r \circ \phi). \tag{3}$$

Then $\pi^*\omega = 0$ by (2). Therefore, the surface $S$ remains Lagrangian with respect to the form $\omega$. This gives us the required modification of $\omega_{\rm st}$. By construction there exist two neighbourhoods $U \Subset U'$ of the origin such that $\omega|_U = 0$ and $\omega = \omega_{\rm st}$ in $\mathbb{C}^2 \setminus U'$, while the potential changed globally.

Denote by $p_1, \dots, p_N$ the points of self-intersection of $S$, and by $p_0$ the standard umbrella. Then [3, Prop. 1] gives further modification $\tilde\omega$ of the form $\omega$ in (3), near the self-intersection points. Combining everything together yields the following result.

**Lemma 3.** *Given $\varepsilon > 0$ sufficiently small, there exists a $(1,1)$-form $\tilde\omega$ and $\varepsilon_1 > 0$, such that*

(i) $\tilde\omega|_S = 0$;
(ii) $\tilde\omega = \omega$ *on* $\mathbb{C}^2 \setminus \cup_{j=0}^N \mathbb{B}(p_j, \varepsilon)$.
(iii) $\tilde\omega$ *vanishes on* $\mathbb{B}(p_j, \varepsilon_1)$, $j = 0, \dots, N$.

*Furthermore, there exists a smooth function $\tilde\phi$ on $\mathbb{C}^2$ such that $dd^c\tilde\phi = \tilde\omega$. The function $\tilde\phi$ is plurisubharmonic on $\mathbb{C}^2$, and strictly plurisubharmonic on $\mathbb{C}^2 \setminus \cup_{j=0}^N \mathbb{B}(p_j, \varepsilon)$.*

2.2. **The function $h$.** Let $\iota : S \to \mathbb{C}^2$ be a Lagrangian inclusion, and $\tilde\phi$ be the potential of the form $\tilde\omega$ given by Lemma 3. For simplicity we drop tilde from the notation. In this subsection we recall the construction in [2] and [3] of a smooth function $h$ on $\mathbb{C}^2$ such that $|h|\,\big|_S = e^\phi$ and $\overline\partial h(z) = O(\mathrm{dist}(z,S)^6)$. The two conditions, that $\overline\partial h$ vanishes on $S$ and that $\phi - \log|h|$ vanishes on $S$ with order 1 imply that $\iota^*(d^c\phi - d(\arg h)) = 0$. The latter condition can be met by further perturbation of $\phi$.

Let $\tilde S$ be the deformation retract of $S$. Note that it exists because near the umbrella point the surface $S$ is the graph of a continuous vector-function. Let $\gamma_k$, $k = 1, \dots, m$, be the basis in $H_1(\tilde S, \mathbb{Z}) \cong H_1(S, \mathbb{Z})$ supported on $S$. Using de Rham's theorem and an argument similar to that of Lemma 3 one can find smooth functions $\psi_k$ with compact support in $\mathbb{C}^2$ that vanish on $S \cup (\cup_j B(p_j, \varepsilon))$, where $B(p_j, \varepsilon)$ are the balls around the singular points on $S$ as in Lemma 3, such that $\int_{\gamma_k} \iota^* d^c \psi_l = \delta_{kl}$. Further, one can find small rational numbers $\lambda_k$ and an integer $M$, such that for the function

$$\tilde\phi = M\left(\phi + \sum_{j=1}^m \lambda_k \psi_k\right) \tag{4}$$

the form $\iota^* d^c \tilde\phi$ is closed on $S$ and has periods which are multiples of $2\pi$. Then there exists a $C^\infty$-smooth function $\mu : S \to \mathbb{R}/2\pi\mathbb{Z}$ that vanishes on the intersection of $S$ with $B(p_j, \varepsilon)$, $j = 0, \dots, N$, and such that $\iota^* d^c \tilde\phi = d\mu$. By [5], there exists a function $h$ defined on $\mathbb{C}^2$ such that

$$h|_S = e^{\tilde\phi + i\mu}|_S$$

and $\overline{\partial}h(z) = O(\mathrm{dist}(z,S)^6)$. It follows that $\tilde{\phi} - \log|h|$ vanishes to order 1 on $S$. Note that $h$ is constant near singular points of $S$.

2.3. **The function $\phi$.** Again, for simplicity of notation we denote by $\phi$ the function (4) constructed in Section 2.2. It does not yet satisfy the conditions of Lemma 2 because there are still some smooth points on $S$ where the function $h$ is not holomorphic and $\phi$ is not strictly plurisubharmonic. For this we will replace $\phi$ by a function $\tilde{\phi} = \phi + c\cdot\rho$, where the function $\rho$ will be constructed using local polynomial convexity of $S$, and $c > 0$ will be a suitable constant.

We recall our result from [7, 8].

**Lemma 4.** *Let $S$ be a Lagrangian inclusion in $\mathbb{C}^2$, and let $p_0,\ldots,p_N$ be its singular points. Suppose that $S$ is locally polynomially convex near every singular point. Then there exists a neighbourhood $\Omega$ of $S$ in $\mathbb{C}^2$ and a continuous non-negative plurisubharmonic function $\rho$ on $\Omega$ such that $S\cap\Omega = \{p\in\Omega : \rho(p) = 0\}$. Furthermore, for every $\delta > 0$ one can choose $\rho = (\mathrm{dist}(z,S))^2$ on $\Omega\setminus\cup_{j=0}^N\mathbb{B}(p_j,\delta)$; in particular, it is smooth and strictly plurisubharmonic there.*

The standard open Whitney umbrella is locally polynomially convex by [7], and $S$ is locally polynomially convex near transverse double self-intersection points by [8]. For the proof of the lemma we refer the reader to [8].

To complete the construction of the function $\phi$, we choose the function $\rho$ in Lemma 4 with $\delta > 0$ so small that the balls $\mathbb{B}(p_j,\delta)$ are contained in balls $\mathbb{B}(p_j,\varepsilon_1/2)$ given by Lemma 3. Note that $\rho$ is defined only in a neighbourhood $\Omega$ of $S$, but we can extend it as a smooth function with compact support in $\mathbb{C}^2$. Consider now the function

$$\tilde{\phi} = \phi + c\cdot\rho.$$

We choose the constant $c > 0$ so small that the function $\tilde{\phi}$ remains to be plurisubharmonic on $\mathbb{C}^2$. At the same time, since $c > 0$ and $\rho$ is strictly plurisubharmonic on $S$ outside small neighbourhoods of singular points, we conclude that the function $\tilde{\phi}$ is strictly plurisubharmonic outside the balls $\mathbb{B}(p_j,\delta)$.

The pair $\tilde{\phi}$ and $h$ now satisfies all the conditions of Lemma 2. This completes the proof of Theorem 1.

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