Math. Z. (2017) 286:1145–1171
DOI 10.1007/s00209-016-1796-5

Mathematische Zeitschrift

Distributional boundary values of holomorphic functions on product domains

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Received: 20 December 2015 / Accepted: 15 August 2016 / Published online: 15 November 2016

Abstract We show that holomorphic functions of polynomial growth on domains with corners have distributional boundary values in an appropriate sense, provided the corners are generic CR manifolds. We also prove an analog of the Bochner–Hartogs theorem for these boundary values for the simplest such domains, the product domains.

Mathematics Subject Classification 32A40 · 46F20

Contents

This work was partially supported by a grant from the Simons Foundation (#316632 to Debraj Chakrabarti). Debraj Chakrabarti was also partially supported by an Early Career internal grant from Central Michigan University. Rasul Shafikov was partially supported by an NSERC grant.

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1 Introduction

1.1 Distributional boundary values

The study of boundary values of holomorphic functions as generalized functions has a long history, going back to the theory of analytic functionals developed by Fantappiè in the 1920's and 30's. The development of the theory of topological vector spaces and distributions led to significant progress in this problem in the 1950's and 60's by Köthe, Silva, Grothendieck, Sato, Martineau, Tillmann and many others. See the book review [12] for a short history of the topic, and [16] for an annotated bibliography of the early contributions till 1964. Such generalized boundary values may be studied either as distributions in the sense of Sobolev and Schwartz or as hyperfunctions in the sense of Sato. The former approach, which is adopted in this paper, allows $\mathcal{C}^{\infty}$-smooth boundaries, but requires the holomorphic functions to grow at most polynomially as one approaches the boundary. If we use hyperfunctions, we can obtain boundary values of all holomorphic functions, but we must restrict the boundary to be real analytic (see [9,20]). We note however that many of the considerations of this paper apply to hyperfunction boundary values as well, and this aspect will be discussed in detail in a forthcoming paper. It is also possible to obtain boundary values of other holomorphic objects (p-forms, sections of vector bundles) by routine extensions of the methods of this paper.

Our motivation is to try to generalize the notion of distributional boundary value to holomorphic functions defined on domains which have *piecewise smooth* boundaries. Recall that there exist distributional boundary values of holomorphic functions of polynomial growth defined on a "wedge" attached to a generic "edge" (see [4]). This rightly suggests that we can define boundary values of holomorphic functions of polynomial growth on the class of piecewise smooth domains whose corners are generic CR manifolds. The boundary value is realized as a de Rham current in the ambient complex manifold, a well-known formalism in complex analysis (see [11]).

The main thrust of this paper is to study global holomorphic extension properties of "CR" boundary currents on piecewise smooth domains, generalizing the classical Bochner–Hartogs theorem. We give a complete solution of this problem for products of smoothly bounded domains. This may be considered a generalization of one of the earliest results in several complex variables due to Hartogs, the extension of a holomorphic function from a neighborhood of the boundary of a polydisc to the whole polydisc (see [19, Theorem 1, page 12], and also [14]). The invariant nature of the de Rham currents allows us

to state our results for domains in general complex manifolds, without any cohomological constraints.

1.2 Main results

Let Ω be a relatively compact domain in a complex manifold $\mathfrak{M}$ that may be written as

$$\Omega = \bigcap_{j=1}^{N} \Omega_j, \tag{1.1}$$

where each $\Omega_j \subset \mathfrak{M}$ is a smoothly bounded domain. If for each subset $S \subset \{1, \dots, N\}$ the intersection $B_S = \bigcap_{j \in S} b\Omega_j$, if nonempty, is a CR manifold of CR-dimension $n - |S|$, we say that Ω is a *domain with generic corners*. Domains with generic corners are significant in many areas of complex analysis, see [5,8,10,26]. The most important examples of domains with generic corners are the *product domains*.

Impose on $\mathfrak{M}$ any metric compatible with its topology (we assume that all manifolds appearing in this paper are countable at infinity). Throughout the paper we denote by $\operatorname{dist}(z, X)$ the distance from a point $z \in \mathfrak{M}$ to a set X induced by the chosen metric. If $\Omega \Subset \mathfrak{M}$ is a relatively compact domain, then a holomorphic $f \in \mathcal{O}(\Omega)$ is said to be of *polynomial growth* if there is a $C > 0$ and $k \geq 0$ such that for each $z \in \Omega$ we have

$$|f(z)| \leq \frac{C}{\operatorname{dist}(z, \partial\Omega)^k}.$$

We denote the space of holomorphic functions of polynomial growth on Ω by $\mathcal{A}^{-\infty}(\Omega)$. This space has a natural topology (cf. Sect. 2.1 below). We can show that holomorphic functions of polynomial growth on domains with generic corners have boundary values in the sense of distributions:

Theorem 1.1 *Let Ω be a domain with generic corners in a complex manifold $\mathfrak{M}$, and let $f \in \mathcal{A}^{-\infty}(\Omega)$. There is a $(0, 1)$-current $\mathsf{bc} f \in \mathcal{D}'_{0,1}(\mathfrak{M})$ such that the following holds. If U is a coordinate neighborhood of $\mathfrak{M}$, and $\psi \in \mathcal{D}^{n,n-1}(\mathfrak{M})$ is a smooth $(n, n-1)$ form which has support in U, and there is a vector $v \in \mathbb{C}^n$ such that in the coordinates on U, the vector v points outward from Ω along each $\partial\Omega_j$ inside U, then we have*

$$\langle \mathsf{bc} f, \psi \rangle = \lim_{\epsilon \downarrow 0} \int_{\partial\Omega} f_\epsilon \psi, \tag{1.2}$$

where $f_\epsilon(z) = f(z - \epsilon v)$.

In fact, the existence of the boundary value is purely local: if $p \in \partial\Omega$, $\mathbb{B}$ is a ball with respect to some coordinate system centered at p and f is a holomorphic function on $\mathbb{B} \cap \Omega$ with polynomial growth as one appoaches $\partial\Omega \cap \mathbb{B}$, then there exists a boundary current $\mathsf{bc} f \in \mathcal{D}'_{0,1}(\mathbb{B})$ defined on $\mathbb{B}$ again by (1.2), and supported on $\partial\Omega \cap \mathbb{B}$.

We refer to $\mathsf{bc} f$ as the *boundary current* induced by the holomorphic function f of polynomial growth. The condition that the boundary of Ω has generic corners is necessary, if we want all holomorphic functions of polynomial growth on Ω to have distributional boundary values in the above sense. For example, the function $z \mapsto z^{-2}$ on the unit square $\Omega = \{z \in \mathbb{C} \colon 0 < \operatorname{Re}(z), \operatorname{Im}(z) < 1\}$ fails to have a boundary current near the origin which is not a generic corner in $\partial\Omega$ (see [7] for a proof). This example can be generalized to higher dimensions.

A natural question is to characterize the range of the map bc, i.e., to describe the currents $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ that arise as boundary values of holomorphic functions of polynomial growth on $\Omega \Subset \mathfrak{M}$, where Ω is a domain with generic corners in $\mathfrak{M}$. There is a simple necessary condition on all boundary currents which we first note. We say that a $(0,1)$-current $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ on a complex manifold $\mathfrak{M}$ satisfies the *Weinstock orthogonality condition* (cf. [27]) with respect to a domain $\Omega \subset \mathfrak{M}$, or simply the Weinstock condition, if for $\omega \in \mathcal{D}^{n,n-1}(\mathfrak{M})$,

$$\overline{\partial}\omega = 0 \quad \text{on } \overline{\Omega} \implies \langle \gamma, \omega \rangle = 0. \tag{1.3}$$

This is a generalization of the usual tangential Cauchy–Riemann equations for the boundary values of holomorphic functions, and in fact, for domains in $\mathbb{C}^n$ with connected complement, the Weinstock condition is equivalent to γ being $\overline{\partial}$-closed (see the proof of Corollary 1.3 below). In Proposition 2.6 we show that the Weinstock orthogonality is a necessary condition for γ to be the boundary value of a function $f \in \mathcal{A}^{-\infty}(\Omega)$. However, it is not true that a current satifying the Weinstock condition is a boundary current: a simple example is given by the current $\overline{\partial}\delta$ on $\mathbb{C}^n$, where δ is a Dirac distribution supported at one point of the boundary of Ω. Below we will be able to state necessary and sufficient conditions for a current to be a boundary current for smoothly bounded domains and product domains. It is an open problem to give such a characterization for general domains with generic corners.

First we consider the case of smoothly bounded domains. If $\Omega \Subset \mathfrak{M}$ is a smoothly bounded domain, we define a subspace $\mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$ of $\mathcal{D}'_{0,1}(\mathfrak{M})$ as follows. A current $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ belongs to $\mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$ if and only if γ satisfies the following two conditions:

(1) γ satisfies the Weinstock condition with respect to Ω.
(2) There is a *face distribution* $\alpha \in \mathcal{D}'_0(\partial\Omega)$ which induces γ in the following way: if $\iota : \partial\Omega \to \mathfrak{M}$ is the inclusion map, we have

$$\gamma = \iota_*(\alpha)^{0,1}. \tag{1.4}$$

Here ι_* is the pushforward operation on currents by ι [see (2.3) below] and for a 1-current θ on a complex manifold, we write

$$\theta = \theta^{0,1} + \theta^{1,0},$$

the decomposition of θ into parts of bidegree $(0,1)$ and $(1,0)$. The space $\mathcal{A}^{-\infty}(\Omega)$ has a natural topology, which is defined formally in Sect. 2.1. Further $\mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$ is a closed subspace of the space $\mathcal{D}'_{0,1}(\mathfrak{M})$, and therefore carries the subspace topology. We have the following:

Theorem 1.2 *Let $\Omega \Subset \mathfrak{M}$ be a domain with $\mathcal{C}^\infty$-smooth boundary. Then the map*

$$\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$$

is an isomorphism of topological vector spaces.

When $\mathfrak{M} = \mathbb{C}^n$, Theorem 1.2, allows us to obtain the distributional version of the Bochner–Hartogs phenomenon, thus recapturing a result of Straube (see [24, Thm 2.2]).

Corollary 1.3 *Let $n \geq 2$, and let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded domain such that $\mathbb{C}^n \setminus \overline{\Omega}$ is connected. Suppose that $\gamma \in \mathcal{D}^{0,1}(\mathbb{C}^n)$ is $\overline{\partial}$-closed and (1.4) holds. Then there is a holomorphic f on Ω of polynomial growth, such that $\gamma = \mathsf{bc} f$.*

See [21] for the history of the global holomorphic extension theorem for CR functions. Distributional analogs of other classical global holomorphic extension results (cf. [13]) can also be deduced from Theorem 1.2, using approximation properties of forms in particular manifolds.

While the problem of identifying boundary currents for an arbitrary domain with generic corners remains open, we are able to give a complete characterization of the range of the operator bc in the case when Ω is a *product domain*. Let $\mathfrak{M}_1, \dots, \mathfrak{M}_N$ be complex manifolds, and let

$$\mathfrak{M} = \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_N \tag{1.5}$$

be their product as a complex manifold. For $j = 1, \dots, N$, let $D_j \Subset \mathfrak{M}_j$ be be a domain with $\mathcal{C}^\infty$-smooth boundary. By a product domain, we mean a domain Ω of the form

$$\Omega = D_1 \times \cdots \times D_N. \tag{1.6}$$

For $j = 1, \dots, N$, we define a smoothly bounded domain $\Omega_j \subset \mathfrak{M}$ by setting

$$\Omega_j = \mathfrak{M}_1 \times \cdots \times D_j \times \cdots \times \mathfrak{M}_N, \tag{1.7}$$

where the jth factor is D_j and all other factors are $\mathfrak{M}_k$. Then we can represent the domain Ω as an intersection (1.1), and it is easy to see that each corner is a CR manifold, so product domains have generic corners.

With Ω and $\mathfrak{M}$ as above, we introduce a subspace $\mathcal{Y}^{0,1}_\Omega(\mathfrak{M})$ of $\mathcal{D}'_{0,1}(\mathfrak{M})$. A current $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ belongs to $\mathcal{Y}^{0,1}_\Omega(\mathfrak{M})$ if the following conditions are satisfied:

(1) γ satisfies the Weinstock condition (1.3) with respect to Ω.
(2) Suppose that the piecewise smooth domain Ω is represented as an intersection of smoothly bounded domains as in (1.1). For each $j = 1, \dots, N$, let

$$\iota^j : \partial\Omega_j \to \mathfrak{M} \tag{1.8}$$

denote the inclusion map. There are distributions $\alpha_j \in \mathcal{D}'_0(\partial\Omega_j)$ with support in $\partial\Omega_j \cap \overline{\Omega}$ such that we can write

$$\gamma = \sum_{j=1}^{N} \left(\iota^j_*(\alpha_j)\right)^{0,1}. \tag{1.9}$$

We will call the distributions $\alpha_1, \dots, \alpha_N$ the *face distributions* associated with the current γ.
(3) The third condition, which we call *canonicality of face distributions* is somewhat technical, and will be fully explained below in Sect. 4.5. Informally, it can be understood as follows. Given a function $f \in \mathcal{A}^{-\infty}(\Omega)$, there exists an extension of f as a distribution in $\mathcal{D}'_0(\mathfrak{M})$ with the property that it vanishes outside $\overline{\Omega}$ and its values on $\partial\Omega$ are determined in a limit process from the values in Ω, see Theorem 2.4. This will be called the *canonical extension* of f. A similar canonical extension exists for the distributions $\alpha_j \in \mathcal{D}'_0(\partial\Omega_j)$ defined by (1.9). The condition now is that the canonical extensions of α_j agree with α_j, see (4.20) below for the exact statement. In particular, this condition ensures that one can talk about boundary values of the face distributions themselves along higher codimensional strata.

We note that all three conditions above are satisfied by boundary currents of holomorphic functions. In fact, we have the following characterization of the distributional boundary values of holomorphic functions on product domains:

Theorem 1.4 *Let Ω be a product domain as above. Then for each $f \in \mathcal{A}^{-\infty}(\Omega)$, we have $\mathsf{bc} f \in \mathcal{Y}^{0,1}_{\Omega}(\mathfrak{M})$, and the map*

$$\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{Y}^{0,1}_{\Omega}(\mathfrak{M})$$

is an isomorphism of topological vector spaces.

In the last section, we relate this notion of boundary value with the more usual notion of boundary value on the Šilov boundary.

1.3 Remarks

In classical results on distributional boundary values, one considers an edge (a higher-codimensional generic CR manifold) and holomorphic functions defined on a wedge attached to the edge. If the holomorphic function has polynomial growth as one approaches the edge, one obtains a distributional boundary value *defined on the edge*. This setup is inadequate for considering distributional analogs of the Bochner–Hartogs phenomenon, since to state such results we need a "CR" object on the whole boundary $\partial\Omega$. This is the motivation for the new notion of a boundary current introduced in this paper, which coincides with the classical notion of distributional boundary values on smooth boundaries. It is shown in Sect. 5.4 below that on product domains, the classical boundary value on the Šilov boundary can be recaptured starting from our notion of boundary current.

The condition of genericity of the corners is necessary in this investigation, since we can obtain boundary values of all holomorphic functions with polynomial growth in the sense considered in this paper only if the corners are generic. This is explained through an example in [7]. This should not come as a surprise, since we can think of the corners as edges of a wedge which is contained in the domain Ω. It is well-known that genericity of the edge is necessary for the existence of boundary values (see [1,2,4,22]).

2 Existence and basic properties of boundary currents

2.1 The space $\mathcal{A}^{-\infty}(\Omega, \mathsf{X})$

We consider holomorphic functions and distributions with values in a Banach space, see [25] for basic facts on Banach-valued holomorphic functions and distributions. This will be needed in Proposition 2.1, which will be later used in the proof of Proposition 4.6. Let $\mathfrak{M}$ be a complex manifold, and let $\Omega \Subset \mathfrak{M}$ be a relatively compact domain. We endow the manifold $\mathfrak{M}$ with an arbitrary Riemannian metric. All distances are measured with respect to this metric. Let X be a Banach space, and denote by $\mathcal{O}(\Omega, \mathsf{X})$ the space of all X-valued holomorphic functions on Ω. We say that a function $f \in \mathcal{O}(\Omega, \mathsf{X})$ is of *polynomial growth* if there is a non-negative integer k and a $C > 0$ such that

$$\|f(z)\|_{\mathsf{X}} \leq \frac{C}{\operatorname{dist}(z, b\Omega)^k}. \tag{2.1}$$

For a fixed k, we denote by $\mathcal{A}^{-k}(\Omega, \mathsf{X})$ the space of X-valued holomorphic functions on Ω which satisfy the estimate (2.1). Then $\mathcal{A}^{-k}(\Omega, \mathsf{X})$ is a Banach space with the norm

$$\|f\|_{\mathcal{A}^{-k}} = \sup_{z\in\Omega} \left\{ \|f(z)\|_{\mathsf{X}} \operatorname{dist}(z, b\Omega)^k \right\}.$$

We denote the space of all X-valued holomorphic functions of polynomial growth on Ω by $\mathcal{A}^{-\infty}(\Omega, \mathsf{X})$:

$$\mathcal{A}^{-\infty}(\Omega, \mathsf{X}) = \bigcup_{k=0}^{\infty} \mathcal{A}^{-k}(\Omega, \mathsf{X}), \tag{2.2}$$

and endow $\mathcal{A}^{-\infty}(\Omega, \mathsf{X})$ with the inductive limit topology.

2.2 Notation for currents

For de Rham currents we will use the following standard notation and terminology. If $\mathfrak{M}$ is a differentiable manifold of dimension N, we denote by $\mathcal{D}^q(\mathfrak{M})$ the space of smooth compactly supported q-forms on $\mathfrak{M}$, which is a topological vector space with the standard inductive limit topology. We denote by $\mathcal{D}'_q(\mathfrak{M})$ (space of currents of degree q, or q-currents) the topological dual of the space $\mathcal{D}^{N-q}(\mathfrak{M})$, endowed with the strong topology (the topology of uniform convergence on bounded subsets of $\mathcal{D}^{N-q}(\mathfrak{M})$, see [25, p. 198ff]. A *distribution* is a 0-current, and given a locally integrable function u on a manifold $\mathfrak{M}$, we identify u with the distribution (i.e., 0-current) $\phi \mapsto \int_{\mathfrak{M}} u\phi$, where $\phi \in \mathcal{D}^{\dim_{\mathbb{R}} \mathfrak{M}}(\mathfrak{M})$ is a compactly supported form of the top degree on $\mathfrak{M}$. If X is a Banach space, then an X-*valued distribution* is an element of the space $\mathcal{D}'_0(\mathfrak{M}, \mathsf{X})$ of continuous linear maps from $\mathcal{D}^N(\mathfrak{M})$ to X.

When $\mathfrak{M}$ is a complex manifold of complex dimension n, we let $\mathcal{D}^{p,q}(\mathfrak{M})$ be the space of smooth compactly supported (p, q)-forms, and $\mathcal{D}'_{p,q}(\mathfrak{M})$ the space of (p, q)-currents, i.e., the dual (with the strong topology) of $\mathcal{D}^{n-p,n-q}(\mathfrak{M})$. If $f : \mathfrak{M} \to \mathfrak{N}$ is a mapping of smooth manifolds, and γ is a current on $\mathfrak{M}$, we denote by $f_*\gamma$ the *pushforward* of the current γ by the map f. Recall that

$$\langle f_*\gamma, \phi\rangle = \left\langle \gamma, f^*\phi \right\rangle, \tag{2.3}$$

where ϕ is a smooth compactly supported form of appropriate degree and f^* denotes the pullback operator on forms.

2.3 Distributional extensions

We now consider the problem of extending a holomorphic function defined on the domain Ω to a distribution defined on $\mathfrak{M}$. A 0-current $F \in \mathcal{D}'_{0,0}(\mathfrak{M}, \mathsf{X})$ will be called an (X-*valued*) *distributional extension* of f if $F|_\Omega = f$.

Since X is a Banach space, any X-valued current γ is locally of finite order. This means that, after choosing an arbitrary Riemannian metric on $\mathfrak{M}$, for each compact $K \subset \mathfrak{M}$, there is a $C > 0$ and an integer $k \geq 0$ (the local order of γ on K) such that for any test form ϕ of appropriate degree with support in K we have

$$\|\langle \gamma, \phi\rangle\|_{\mathsf{X}} \leq C \, \|\phi\|_{\mathcal{C}^k(\mathfrak{M})},$$

where the $\mathcal{C}^k$-norm is defined with respect to the Riemannian metric on $\mathfrak{M}$. In particular, it follows that the distributional extension F, if it exists, being compactly supported, is of finite order on the whole of $\mathfrak{M}$.

Proposition 2.1 *If $f \in \mathcal{O}(\Omega, \mathsf{X})$ admits a distributional extension $F \in \mathcal{D}'_{0,0}(\mathfrak{M}, \mathsf{X})$, then $f \in \mathcal{A}^{-\infty}(\Omega, \mathsf{X})$.*

Proof After using a system of local holomorphic coordinates centered at a boundary point, it is sufficient to prove the result when $\mathfrak{M} = \mathbb{C}^n$. We use a classic argument of Bell [6,

Lemma 2]. Let $\chi \in \mathcal{D}(\mathbb{C}^n)$ be a compactly supported smooth radial function with support in the unit ball such that $\int \chi dV = 1$, and for a fixed z in Ω define the form $\phi \in \mathcal{D}^{n,n}(\mathfrak{M})$ by

$$\phi(w) = \left(\frac{\operatorname{dist}(z, \partial\Omega)}{2}\right)^{-2n} \cdot \chi\left(\frac{w - z}{\frac{1}{2}\operatorname{dist}(z, \partial\Omega)}\right) d\mathrm{vol},$$

where $d\mathrm{vol}$ is the standard volume form of $\mathbb{C}^n$. Then $\int \phi = 1$ as well, and ϕ is radially symmetric about z. A direct computation reveals that there is a constant C_0 depending only on the function χ and independent of $z \in \Omega$ such that

$$\|\phi\|_{\mathcal{C}^k(\mathbb{C}^n)} \leq \frac{C_0}{\operatorname{dist}(z, \partial\Omega)^{k+2n}}. \tag{2.4}$$

Assume that F is of order k. Then, by the mean value theorem,

$$\|f(z)\|_{\mathsf{X}} = \left\| \int_\Omega f\phi \right\|_{\mathsf{X}} = \|\langle F, \phi\rangle\|_{\mathsf{X}} \leq C\, \|\phi\|_{\mathcal{C}^k(\mathbb{C}^n)} \leq \frac{C}{\operatorname{dist}(z, \partial\Omega)^{k+2n}}.$$

□

2.4 $\mathcal{A}^{-\infty}(\Omega, \mathsf{X})$ as a DFS space

Recall that a *DFS space* is a topological vector space isomorphic to the strong dual of a Fréchet–Schwartz space. We note the following two facts:

Proposition 2.2 *If Ω is a relatively compact domain in $\mathfrak{M}$, then $\mathcal{A}^{-\infty}(\Omega, \mathsf{X})$ is a DFS space for any Banach space X.*

Proof By a result in functional analysis ([17, Proposition 25.20] or [18, Appendix A, Sections 5 and 6]), a DFS space may be characterized as the inductive limit $E = \operatorname{ind}_n E_n$ of an increasing sequence of Banach spaces $\{E_n\}_{n\in\mathbb{N}}$ with the property that for each n, there is an $m > n$ such that the embedding $E_n \to E_m$ is compact. Therefore, it suffices to show that the inclusion map $\mathcal{A}^{-k}(\Omega, \mathsf{X}) \hookrightarrow \mathcal{A}^{-(k+1)}(\Omega, \mathsf{X})$ is a compact linear map of Banach spaces. Let $\{f_\nu\}$ be a sequence in the unit ball of $\mathcal{A}^{-k}(\Omega, \mathsf{X})$. By (2.1), on each compact $K \subset \Omega$, the family $\{f_\nu\}$ is uniformly bounded and therefore, by a Banach-valued version of Montel's theorem, there exists $f \in \mathcal{O}(\Omega, \mathsf{X})$ such that a subsequence f_{ν_j} converges to f uniformly on compact subsets of Ω. Note that f_{ν_j} need not to converge to f in $\mathcal{A}^{-k}(\Omega, \mathsf{X})$. The estimate (2.1) implies that the limit f lies in the closed unit ball of $\mathcal{A}^{-k}(\Omega, \mathsf{X})$, so that $\left\|f - f_{\nu_j}\right\|_{\mathcal{A}^{-k}} \leq 2$ for each j. Let $g_j(z) = \left\|f(z) - f_{\nu_j}(z)\right\|_{\mathsf{X}} \operatorname{dist}(z, \partial\Omega)^{k+1}$, and let $\epsilon > 0$. On the open set $\{z \in \Omega\colon \operatorname{dist}(z, \partial\Omega) < \frac{\epsilon}{2}\}$, we see that $g_j(z) \leq \left\|f - f_{\nu_j}\right\|_{\mathcal{A}^{-k}} \operatorname{dist}(z, \partial\Omega) < \epsilon$. On the complementary compact set $\{z \in \Omega\colon \operatorname{dist}(z, \partial\Omega) \geq \frac{\epsilon}{2}\}$, as $j \to \infty$, we have $f_{\nu_j} \to f$ uniformly, so that we can find an N_ϵ so large that if $j > N_\epsilon$, then $\left\|f(z) - f_{\nu_j}(z)\right\|_{\mathsf{X}} < \frac{\epsilon}{\delta^{k+1}}$, where $\delta = \max_{z\in\Omega} \operatorname{dist}(z, \partial\Omega)$. Then for $j > N_\epsilon$ and for each $z \in \Omega$, we have $g_j(z) < \epsilon$. It follows that $f_{\nu_j} \to f$ in $\mathcal{A}^{-(k+1)}(\Omega, \mathsf{X})$ and therefore, the map $\mathcal{A}^{-k}(\Omega, \mathsf{X}) \hookrightarrow \mathcal{A}^{-(k+1)}(\Omega, \mathsf{X})$ is compact. □

We will also use the following fact, which is a consequence in the usual way of the closed graph theorem for DFS spaces, a proof of which can be found in [18, Appedix A, Corollary A.6.4].

Proposition 2.3 *If E, F are DFS spaces, and $u : E \to F$ is a continuous linear map which is a set-theoretic bijection, then u is an isomorphism of topological vector spaces.*

2.5 Canonical extension

We now show that holomorphic functions of polynomial growth on domains with generic corners admit distributional extensions, by constructing one such extension. This extension will be called the *canonical extension* of the holomorphic function. Our proof will use a method of Barrett (see [5]). The notion of canonical extension is due to André Martineau, who noted its existence for smoothly bounded planar domains in [16].

By a *face* of the domain Ω in (1.1) we mean the subset $\partial\Omega_j \cap \partial\Omega$ of the boundary of Ω for some j.

Theorem 2.4 *Let $\mathfrak{M}$ be a complex manifold of complex dimension n, and let $\Omega \Subset \mathfrak{M}$ be a domain with generic corners. There is a unique continuous linear map*

$$\mathsf{ce} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{D}'_0(\mathfrak{M}),$$

such that the following conditions hold.

(1) $\mathsf{ce} f$ *is a distributional extension of the function f, i.e.,* $(\mathsf{ce} f)|_\Omega = f$.
(2) *If U is a coordinate neighborhood of $\mathfrak{M}$, and $\phi \in \mathcal{D}^{n,n}(\mathfrak{M})$ has support in U, and there is a vector $v \in \mathbb{C}^n$ such that in the coordinates on U, the vector v points outward from Ω along each $\partial\Omega_j$ meeting the support of ϕ, then we have*

$$\langle \mathsf{ce} f, \phi\rangle = \lim_{\epsilon\downarrow 0} \int_\Omega f_\epsilon \phi, \tag{2.5}$$

where $f_\epsilon(z) = f(z - \epsilon v)$.
(3) *In the special case when f is continuous on $\overline{\Omega}$, the distribution $\mathsf{ce} f$ is induced by the function which coincides with f on $\overline{\Omega}$ and vanishes outside $\overline{\Omega}$.*

Proof We may cover the boundary $b\Omega$ by a finite collection $\{U_j\}_{j=1}^M$ of coordinate charts, such that in the local coordinates of each chart U_j, there is a vector v_j of the type referred to above. Thus, without loss of generality we may assume that $\mathfrak{M} = \mathbb{C}^n$, and the existence of ce would follow provided we can show the existence of the limit (2.5), and its independence of the choice of the vector $v \in \mathbb{C}^n$. In the heart of our proof is the integration by parts argument due to Barrett (see [5]).

To illustrate the idea, consider first the simplest case when Ω has smooth boundary, and let with r be a defining function of Ω (see [24]). Let T be a vector field of type (0, 1) such that $Tr \equiv 1$ near $\partial\Omega$. Let U be a neighborhood of a point in $\partial\Omega$, and let v a vector such that U and v satisfy the conditions in (2) of the theorem. By shrinking U, we may assume that $Tr \equiv 1$ on U. For a (n, n)-form ϕ supported in U we write $\phi = \phi_0 dV$, where dV is the standard volume form of $\mathbb{C}^n$. Let $\epsilon > 0$, and let f_ϵ be as in (2.5). Then for any positive integer s, by applying the integration by parts formula s times we obtain that

$$\int_\Omega f_\epsilon \phi = \frac{1}{s!} \int_\Omega (r^s f_\epsilon)\, ((T^*)^s \phi_0) dV,$$

where $T^* = -(T + \operatorname{div} T)$ is the formal transpose of the vector field T, and we use the fact that $T(r^k f_\epsilon) = k r^{k-1} f_\epsilon$ on U for each integer $k \geq 0$. Note that the boundary terms vanish at each step since f_ϵ is smooth up to $\partial\Omega$. Using that f is of polynomial growth, we choose s such that the function $r^s f$ is continuous on $\overline{\Omega}$. Then as $\epsilon \to 0$, we have $r^s f_\epsilon \to r^s f$ *uniformly* on $\overline{\Omega}$, so that we obtain by letting $\epsilon \to 0$ that

$$\lim_{\epsilon\downarrow 0} \int_\Omega f_\epsilon \phi = \frac{1}{s!} \int_\Omega (r^s f)\, ((T^*)^s \phi_0) dV. \tag{2.6}$$

Therefore, the limit on the left hand side exists and is given by the expression on the right hand side, which does not involve v. Hence, the limit is independent of v, and also of s (as long as $r^s f$ is continuous on $\overline{\Omega}$), since the left hand side is independent of s. We conclude that $\langle \mathsf{ce} f, \phi\rangle$ exists for ϕ supported in U and is given by the expression on the right hand side of (2.6), and the global existence of $\mathsf{ce} f$ follows from a partition of unity argument.

In fact, for a general domain with generic corners in $\mathbb{C}^n$ it is possible to give a formula analogous to the right-hand side of (2.6) for computing $\mathsf{ce} f$ directly. To state this formula [equation (2.1a) of [5]], we use the following notation. Let r_k denote a defining function of the domain Ω_k in the representation (1.1). Note that the condition that the corners of a domain with generic corners are generic CR manifolds is equivalent to the following: at each point in the intersection $\bigcap_{j\in S} \partial\Omega_j$, we have $\bigwedge_{j\in S} \overline{\partial} r_j \neq 0$. Therefore, in a neighborhood of each point $p \in \mathbb{C}^n$, we can find N vector fields $T_1^{(p)}, \dots, T_N^{(p)}$ of type $(0, 1)$ such that $T_j^{(p)} r_k = \delta_{jk}$ in a neighborhood of p whenever $r_j(p) = r_k(p) = 0$. By a partition of unity argument, we obtain vector fields T_j, $j = 1, \dots, N$, on $\mathbb{C}^n$ of type $(0, 1)$ such that $T_j r_k = \delta_{jk}$ on a neighborhood U_{jk} of $\partial\Omega_j \cap \partial\Omega_k$. Let T_j^* denote the first order differential operator on $\mathfrak{M}$ which is the formal transpose of the vector field T_j with respect to the standard bilinear pairing $(u, v) \mapsto \int_{\mathfrak{M}} uv dV$, i.e., for smooth compactly supported u, v, we have $\int_{\mathfrak{M}} (T_j u) v dV = \int_{\mathfrak{M}} u (T_j^* v) dV$.

For a subset $S \subset \{1, 2, \dots, n\}$, let $U_S = \bigcap_{j,k\in S} U_{jk} \setminus \bigcup_{\ell\notin S} b\Omega_\ell$. In particular, $U_\emptyset = \mathbb{C}^n \setminus \bigcup_{\ell=1}^N b\Omega_\ell$. Then the family $\{U_S\}$, as S runs over all possible subsets of $\{1, 2, \dots, N\}$ including the empty set, is an open cover of $\mathbb{C}^n$. Let $\{\chi_S\}$ be a partition of unity subordinate to this cover. We now define, for an $f \in \mathcal{A}^{-\infty}(\Omega)$ and a $\phi = \phi_0 dV$, the distribution $\mathsf{ce} f$ by the prescription

$$\langle \mathsf{ce} f, \phi\rangle = \sum_{S\subset\{1,\dots,N\}} \int_\Omega \left(\prod_{j\in S} \frac{r_j^{s_j}}{s_j!} f \right) \left(\prod_{j\in S} (T_j^*)^{s_j} \right) (\chi_S \phi_0)\, dV, \tag{2.7}$$

where $(s_1, \dots, s_N) \in \mathbb{N}^N$ is such that $r_1^{s_1} \dots r_N^{s_N} f$ is continuous on $\overline{\Omega}$.

Repeated integrations by parts, using the relations $T_j r_k = \delta_{jk}$ and $T_j f = 0$ (since f is holomorphic) shows that when f bounded, we have $\langle \mathsf{ce} f, \phi\rangle = \int_\Omega f\phi$, i.e., $\mathsf{ce} f = f[\Omega]$, which shows, in particular, that at least in this case, $\mathsf{ce} f$ is defined independently of the choice of the tuple $(s_1, \dots, s_N)$ (see [5,8] for details). The general argument is similar to that in the smooth case above. Assuming ϕ and f_ϵ are as in (2.5), the same integration by parts argument shows that limit in (2.5) is given by (2.7) , which shows that $\mathsf{ce} f$ is defined by (2.7) independently of $(s_1, \dots, s_N)$, and that limit on the right-hand side of (2.5) is independent of the particular vector v used to define f_ϵ.

To show the uniqueness, assume that there exists another distribution, say, $h \in \mathcal{D}_0'(\mathfrak{M})$ that satisfies conditions (1) and (2) of the theorem for a given function f. Then it follows from (2.5) that in a coordinate neighborhood U of any point $\partial\Omega$ on which v exists, we have $h|_U = \mathsf{ce} f|_U$. This implies the uniqueness.

The continuity of $\mathsf{ce} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{D}_0'(\mathfrak{M})$ follows from the expression (2.7).

Finally, the property (3) also follows from the representation (2.7). In this case we have $s_1 = s_2 = \dots = s_N = 0$, and $\langle \mathsf{ce} f, \phi\rangle = \int_\Omega f\phi_0 dV$. □

Corollary 2.5 *The canonical extension as constructed in the previous theorem is purely local. More precisely, if p is a point on $\partial\Omega$, $\mathbb{B}$ is a coordinate ball centered at p, and f is a holomorphic function defined on $\Omega \cap \mathbb{B}$ which has polynomial growth as one approaches $\partial\Omega \cap \mathbb{B}$, then there is a distribution $\mathsf{ce} f$ on $\mathbb{B}$ which extends f and is given by (2.5) (when*

$\phi \in \mathcal{D}^{n,n}(\mathbb{B})$). If f extends continuously to $\partial\Omega$, then $\mathsf{ce}\, f$ coincides with the distribution on $\mathbb{B}$ induced by the function on $\mathbb{B}$ which coincides with f on $\overline{\Omega} \cap \mathbb{B}$ and vanishes outside $\overline{\Omega}$.

Proof This follows immediately from the local nature of the construction of $\mathsf{ce}\, f$, in particular formula (2.7). □

2.6 Proof of Theorem 1.1

We define an operator $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{D}'_{0,1}(\mathfrak{M})$ by setting

$$\mathsf{bc} = -\overline{\partial} \circ \mathsf{ce}. \tag{2.8}$$

Since ce and $\overline{\partial}$ are continuous, so is bc. Let $f \in \mathcal{A}^{-\infty}(\Omega)$. Then we have, with ψ and v as in the statement of the theorem,

$$\langle \mathsf{bc} f, \psi \rangle = \left\langle \mathsf{ce} f, \overline{\partial}\psi \right\rangle = \lim_{\epsilon \downarrow 0} \int_\Omega f_\epsilon \overline{\partial}\psi = \lim_{\epsilon \downarrow 0} \int_\Omega \overline{\partial}(f_\epsilon \psi) = \lim_{\epsilon \downarrow 0} \int_\Omega d(f_\epsilon \psi) = \lim_{\epsilon \downarrow 0} \int_{\partial\Omega} f_\epsilon \psi,$$

which completes the proof of the global statement in the first half of Theorem 1.1. For the second half, we use Corollary 2.5 and repeat the argument.

2.7 Some properties of the boundary current

The necessity of the Weinstock condition (1.3) follows essentially from Theorem 1.1.

Proposition 2.6 *If $\Omega \Subset \mathfrak{M}$ is a domain with generic corners, and $f \in \mathcal{A}^{-\infty}(\Omega)$. Then, $\mathsf{bc} f$ satisfies the Weinstock orthogonality condition with respect to the domain Ω.*

Proof of Proposition 2.6 From (2.5) it follows that if $\phi \in \mathcal{D}^{n,n}(\mathfrak{M})$ is such that $\phi \equiv 0$ on $\overline{\Omega}$, then we have $\langle \mathsf{ce} f, \phi \rangle = 0$, since for each $\epsilon > 0$, we have $\int_\Omega f_\epsilon \phi = 0$. Now let $\omega \in \mathcal{D}^{n,n-1}(\mathfrak{M})$ be such that $\overline{\partial}\omega = 0$ on $\overline{\Omega}$. Then,

$$\langle \mathsf{bc} f, \omega \rangle = \left\langle -\overline{\partial}(\mathsf{ce} f), \omega \right\rangle = \left\langle \mathsf{ce} f, \overline{\partial}\omega \right\rangle = 0.$$

□

We also have the following representation of the boundary current along the smooth part of $\partial\Omega$, which shows that on the smooth part, the boundary current is the distributional boundary value written in an invariant way:

Proposition 2.7 *Let U be an open subset of $\mathfrak{M}$ such that $\partial\Omega \cap U$ is a smooth hypersurface in U. Then there is a distribution $\alpha \in \mathcal{D}'_0(\partial\Omega \cap U)$ such that $\mathsf{bc} f|_U = \iota_*(\alpha)^{0,1}$, where $\iota : \partial\Omega \cap U \to U$ is the inclusion map.*

Proof Without loss of generality, U meets only one face $\partial\Omega_j$ of Ω. If $\phi \in \mathcal{D}^{n,n-1}(\mathfrak{M})$ has support in U, then, choosing v to be outward from Ω along $\partial\Omega_j$ and setting $f_\epsilon = f(\cdot - \epsilon v)$, we have

$$\int_{\partial\Omega} f_\epsilon \phi = \int_{\partial\Omega_j} f_\epsilon \phi = \left\langle \iota_* \left(f_\epsilon|_{\partial\Omega_j} \right)^{0,1}, \phi \right\rangle,$$

where we identify the function $f_\epsilon|_{\partial\Omega_j}$ with the distribution generated by it. Letting $\epsilon \to 0$, we note that for each ϕ, the left-hand side has a limit, therefore, $\lim_{\epsilon \downarrow 0} f_\epsilon|_{\partial\Omega_j}$ also exists and can be taken to be α. □

Proposition 2.8 *For a domain Ω with generic corners, the map* $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$ *is injective.*

Proof Let $f \in \mathcal{A}^{-\infty}(\Omega)$ be in the kernel of the linear mapping bc, i.e., $\mathsf{bc} f = 0$. Recall that this means that $\overline{\partial}(\mathsf{ce} f) = 0$, so that $\mathsf{ce} f$ is a holomorphic function on $\mathfrak{M}$. But $\mathsf{ce} f$ vanishes outside $\overline{\Omega}$, so that $\mathsf{ce} f \equiv 0$ by analytic continuation. Since $f = \mathsf{ce} f|_{\Omega}$, the result follows. □

3 Smoothly bounded domains

3.1 Proof of Theorem 1.2

First, note that if $f \in \mathcal{A}^{-\infty}(\Omega)$, then $\mathsf{bc} f \in \mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$. Indeed, by Proposition 2.6, the current $\mathsf{bc} f$ satisfies the Weinstock condition, and the existence of a face distribution α such that $\mathsf{bc} f = \iota_*(\alpha)^{0,1}$ follows from Proposition 2.7.

The injectivity of the map $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$ follows from Proposition 2.8. Now we show that $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{X}^{0,1}_{\Omega}(\mathfrak{M})$ is surjective. Without loss of generality, assume $\mathfrak{M}$ is connected. Let $\gamma \in \mathcal{X}^{0,1}(\mathfrak{M})$ be arbitrary. By definition, for each $\omega \in \mathcal{D}^{n,n-1}(\mathfrak{M})$ such that $\overline{\partial}\omega = 0$ on $\overline{\Omega}$, we have $\langle \gamma, \omega \rangle = 0$, and there is $\alpha \in \mathcal{D}'_0(\partial\Omega)$, such that $\gamma = \iota_*(\alpha)^{0,1}$, where $\iota : \partial\Omega \hookrightarrow \mathfrak{M}$ is the inclusion map. We need to show that there is an $f \in \mathcal{A}^{-\infty}(D)$ such that $\gamma = \mathsf{bc} f$.

We claim that without loss of generality, we may assume that $\mathfrak{M}$ is noncompact, and that no component of $\mathfrak{M}\backslash\Omega$ is compact. To see this, let $\mathfrak{N}$ be a noncompact open submanifold of $\mathfrak{M}$ defined in the following way. Let $\{D_i\}_{i\in I}$ be the collection of relatively compact connected components of $\mathfrak{M}\backslash\overline{\Omega}$. (The collection $\{D_i\}_{i\in I}$ may of course be empty.) For each $i \in I$, fix a point $z_i \in D_i$, and let $\mathfrak{N} = \mathfrak{M}\backslash\{z_i\}_{i\in I}$. Then $\mathfrak{N}$ is a connected noncompact complex manifold, $\Omega \Subset \mathfrak{N}$, and the complement $\mathfrak{N}\backslash\Omega$ does not have any compact components. Clearly, $\gamma|_{\mathfrak{N}} \in \mathcal{X}^{0,1}_{\Omega}(\mathfrak{N})$. The claim follows if we replace $\mathfrak{M}$ by $\mathfrak{N}$, and γ by $\gamma|_{\mathfrak{N}}$.

By a classical result of Malgrange (see [15, page 236, comments following Problème 1]), the Dolbeault cohomology group $H^{n,n}(\mathfrak{M})$ vanishes, since $\mathfrak{M}$ is noncompact and connected. Now, the transpose of the surjective linear continuous map of Fréchet spaces $\overline{\partial} : \mathcal{E}^{n,n-1}(\mathfrak{M}) \to \mathcal{E}^{n,n}(\mathfrak{M})$ (where $\mathcal{E}^{p,q}(\mathfrak{M})$ is the space of smooth (p,q)-forms on $\mathfrak{M}$) can be identified with the map $\overline{\partial} : \mathcal{E}'_{0,0}(\mathfrak{M}) \to \mathcal{E}'_{0,1}(\mathfrak{M})$, where $\mathcal{E}'_{p,q}(\mathfrak{M})$ is the space of compactly supported (p,q)-currents on $\mathfrak{M}$ (see [23, Proposition 5]). It now follows from a well-known result of functional analysis (see [25, Theorem 37.2]) that the range of $\overline{\partial} : \mathcal{E}'_{0,0}(\mathfrak{M}) \to \mathcal{E}'_{0,1}(\mathfrak{M})$ is closed in $\mathcal{E}'_{0,1}(\mathfrak{M})$ with respect to its weak topology, and therefore with respect to the usual strong topology. Therefore, the range of $\overline{\partial} : \mathcal{E}'_{0,0}(\mathfrak{M}) \to \mathcal{E}'_{0,1}(\mathfrak{M})$ can be identified with the subspace of $\mathcal{E}'_{0,1}(\mathfrak{M})$ orthogonal to the kernel of $\overline{\partial} : \mathcal{E}^{n,n-1}(\mathfrak{M}) \to \mathcal{E}^{n,n}(\mathfrak{M})$, i.e., the range of $\overline{\partial} : \mathcal{E}'_{0,0}(\mathfrak{M}) \to \mathcal{E}'_{0,1}(\mathfrak{M})$ consists precisely of those $\theta \in \mathcal{E}'_{0,1}(\mathfrak{M})$ which have the property that for each $\omega \in \mathcal{E}^{n,n-1}(\mathfrak{M})$ such that $\overline{\partial}\omega = 0$ we have $\langle \theta, \omega \rangle = 0$. It now follows from the Weinstock orthogonality condition (1.3) that γ lies in the range of $\overline{\partial} : \mathcal{E}'_{0,0}(\mathfrak{M}) \to \mathcal{E}'_{0,1}(\mathfrak{M})$, so that there is a compactly supported distribution u on $\mathfrak{M}$ such that $\overline{\partial} u = -\gamma$.

From the structure of γ given by (1.4), it follows that the support of γ is contained in the boundary $\partial\Omega$. Therefore, u is holomorphic on $\mathfrak{M}\backslash\partial\Omega$, in particular, it is holomorphic on $\mathfrak{M}\backslash\overline{\Omega}$. However, since u has compact support, and no component of $\mathfrak{M}\backslash\overline{\Omega}$ is relatively compact by assumption, it follows that u vanishes on an open subset of each component of

$\mathfrak{M}\backslash\overline{\Omega}$. Therefore, by analytic continuation, u vanishes on each component of $\mathfrak{M}\backslash\overline{\Omega}$, and so the support of u is contained in $\overline{\Omega}$.

We set $f = u|_\Omega$. Since u is holomorphic on $\mathfrak{M}\backslash\partial\Omega$, it follows that $f \in \mathcal{O}(\Omega)$. Since the holomorphic function f on Ω can be extended to the distribution u on $\mathfrak{M}$, it follows by Proposition 2.1 (with $\mathsf{X} = \mathbb{C}$) that $f \in \mathcal{A}^{-\infty}(\Omega)$. The surjectivity of bc will be established if we show that $\gamma = \mathsf{bc} f$. This is clearly a local question, so we pick a point $p \in \partial\Omega$, and a system of holomorphic coordinates around p. Let B be a ball in these coordinates centered at p, so small that $\partial\Omega$ divides B into two pieces $B^- = B \cap \Omega$ and $B^+ = B\backslash\overline{\Omega}$. Now since $H^{0,1}(B) = 0$, we can solve the $\overline{\partial}$-problem $\overline{\partial} h = -\gamma|_B$ on B, and the solution h may be represented by a Bochner–Martinelli type integral (see [3, Chapter 6]). Then h has the property that for each $\phi \in \mathcal{D}^{n,n-1}(B)$, we have (see [3, Theorem 6.1, part 5]):

$$-\gamma(\phi) = \lim_{\epsilon\downarrow 0} \int_{B\cap\partial\Omega} (h(\zeta + \epsilon\nu(\zeta)) - h(\zeta - \epsilon\nu(\zeta)))\,\phi(\zeta),$$

where ν is the outward unit normal vector field on $\partial\Omega\cap B$. Further, since $h-u$ is holomorphic on B, after subtracting a holomorphic function on B from h, we can assume that $h = u$. Therefore, $h(\zeta + \epsilon\nu(\zeta)) = 0$ and $h(\zeta - \epsilon\nu(\zeta)) = f(\zeta - \epsilon\nu(\zeta))$, and we have for each $\phi \in \mathcal{D}^{n,n-1}(B)$,

$$\gamma(\phi) = \lim_{\epsilon\downarrow 0} \int_{\partial\Omega} f(\zeta - \epsilon\nu(\zeta))\phi(\zeta),$$

so that we have $\gamma = \mathsf{bc} f$, which shows that bc is surjective.

Therefore $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{X}^{0,1}_\Omega(\mathfrak{M})$ is a continuous bijection of topological vector spaces. We know from Proposition 2.2 that $\mathcal{A}^{-\infty}(\Omega)$ is a DFS space, and $\mathcal{X}^{0,1}_\Omega(\mathfrak{M})$ is also a DFS space, since it is a closed subspace of the DFS space $\mathcal{D}'_0(\partial\Omega)$. The fact that bc is an isomorphism of topological vector spaces now follows from Proposition 2.3.

Proof of Corollary 1.3 In view of Theorem 1.2 above it suffices to show that $\gamma \in \mathcal{X}^{0,1}_\Omega(\mathbb{C}^n)$. For this we will show that γ satisfies the Weinstock criterion with respect to Ω if and only if $\overline{\partial}\gamma = 0$.

It is well-known that the compactly supported cohomology $H^{0,1}_c(\mathbb{C}^n) = 0$. Therefore, there is a compactly supported current $u \in \mathcal{E}'_0(\mathbb{C}^n)$ such that $\overline{\partial} u = \gamma$. We claim that the support of u is contained in $\overline{\Omega}$. Indeed, since u is compactly supported, u vanishes on some open subset of $U = \mathbb{C}^n\backslash\overline{\Omega}$. But by hypothesis U is connected, and u is holomorphic on U, so by analytic continuation $u \equiv 0$ on U, so the claim follows.

Now let $\omega \in \mathcal{D}^{n,n-1}(\mathbb{C}^n)$ be such that $\overline{\partial}\omega = 0$ on $\overline{\Omega}$. Then we have:

$$\langle \gamma, \omega\rangle = \left\langle \overline{\partial} u, \omega\right\rangle = -\left\langle u, \overline{\partial}\omega\right\rangle = 0,$$

since u is supported in $\overline{\Omega}$ and $\overline{\partial}\omega \equiv 0$ on $\overline{\Omega}$. □

We remark that the same type of extension result holds on any noncompact manifold $\mathfrak{M}$ such that $H^{0,1}_c(\mathfrak{M}) = 0$.

4 Boundary currents on product domains

4.1 Notation and terminology

Throughout this section and the next $\mathfrak{M}_j$, $\mathfrak{M}$, Ω and Ω_j will have the same meanings as in (1.5), (1.6) and (1.7) respectively. We introduce some notation to describe the geometry of

$\mathfrak{M}$ and Ω. We set $n_j = \dim_{\mathbb{C}} \mathfrak{M}_j$. For $j = 1, \dots, N$, let $\pi_j : \mathfrak{M} \to \mathfrak{M}_j$ be the natural projection onto the j-th factor. If for each j, ϕ_j is a form on $\mathfrak{M}_j$, then we define

$$\phi_1 \otimes \cdots \otimes \phi_N = \pi_1^* \phi_1 \wedge \pi_2^* \phi_2 \wedge \cdots \wedge \pi_N^* \phi_N. \tag{4.1}$$

For $j = 1, \dots, N$, let (p_j, q_j) be ordered pairs of integers with $0 \le p_j, q_j \le n_j$, and let $P = \sum_{j=1}^N p_j$, $Q = \sum_{j=1}^N q_j$. We denote the $\mathbb{C}$-linear span of the forms $\{\phi_1 \otimes \cdots \otimes \phi_N | \phi_j \in \mathcal{D}^{p_j,q_j}(\mathfrak{M}_j)\}$ by

$$\mathcal{D}^{p_1,q_1}(\mathfrak{M}_1) \otimes \cdots \otimes \mathcal{D}^{p_N,q_N}(\mathfrak{M}_N), \tag{4.2}$$

which is the *algebraic tensor product* of the spaces $\mathcal{D}^{p_j,q_j}(\mathfrak{M}_j)$. The closure of the space (4.2) in $\mathcal{D}^{P,Q}(\mathfrak{M})$ is the *topological* tensor product of the spaces $\mathcal{D}^{p_j,q_j}(\mathfrak{M}_j)$ and is denoted by

$$\mathcal{D}^{p_1,q_1}(\mathfrak{M}_1) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{D}^{p_N,q_N}(\mathfrak{M}_N).$$

We can define tensor products of spaces of currents in the same way.

We will denote by $\widehat{\mathfrak{M}}_j$ the product of all the $\mathfrak{M}_k$ except $\mathfrak{M}_j$, i.e., $\widehat{\mathfrak{M}}_j = \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_{j-1} \times \mathfrak{M}_{j+1} \times \cdots \times \mathfrak{M}_N$, and then we will write $\mathfrak{M} = \mathfrak{M}_j \times \widehat{\mathfrak{M}}_j$. Throughout the paper we will assume that all products are reordered in the standard way, i.e., the factor $\mathfrak{M}_j$ is to be inserted into the slot between $\mathfrak{M}_{j-1}$ and $\mathfrak{M}_{j+1}$. Similarly, we can write (1.7) as $\Omega_j = D_j \times \widehat{\mathfrak{M}}_j$, and we have $\partial\Omega_j = \partial D_j \times \widehat{\mathfrak{M}}_j$, keeping in mind the reordering. We will similarly keep the same notation $\widehat{\otimes}$ for a "reordered" tensor product, for example

$$\begin{aligned} \mathcal{D}'_0(\partial\Omega_j) &= \mathcal{D}'_0(\partial D_j \times \widehat{\mathfrak{M}}_j) \\ &\cong \mathcal{D}'_0(\partial D_j) \widehat{\otimes} \mathcal{D}'_0(\widehat{\mathfrak{M}}_j), \end{aligned}$$

where in the last line we have used the Schwartz kernel theorem (see [25, Theorem 51.7]), and the tensor product is reordered. In this notation, one can write down the direct sum decomposition

$$\mathcal{D}'_{0,1}(\mathfrak{M}) = \bigoplus_{j=1}^N \mathcal{D}'_{0,1}(\mathfrak{M}_j) \widehat{\otimes} \mathcal{D}'_{0,0}(\widehat{\mathfrak{M}}_j), \tag{4.3}$$

which is easily established using the degree considerations. Given a current $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ we can therefore write uniquely

$$\gamma = \sum_{k=1}^N \gamma_k, \quad \text{with } \gamma_k \in \mathcal{D}'_{0,1}(\mathfrak{M}_k) \widehat{\otimes} \mathcal{D}'_{0,0}(\widehat{\mathfrak{M}}_k). \tag{4.4}$$

We will refer to (4.4) as the *standard decomposition* of a (0,1)-form on the product manifold $\mathfrak{M}$.

We denote by $\widehat{D}_j$ the domain in $\widehat{\mathfrak{M}}_j$ which is the product of all the D_k except D_j:

$$\widehat{D}_j = D_1 \times \cdots \times D_{j-1} \times D_{j+1} \times \cdots \times D_N,$$

and we call $\partial D_j \times \widehat{D}_j$ the j-th *open face* of Ω. Its closure is denoted by F_j, and will be called the jth *face* of Ω. Then F_j has the representations

$$\begin{aligned} F_k &= \partial D_k \times \overline{\widehat{D}_k} \\ &= (\partial D_k \times \widehat{\mathfrak{M}}_k) \cap (\overline{D_k} \times \overline{\widehat{D}_k}) \\ &= \partial\Omega_k \cap \overline{\Omega}. \end{aligned}$$

Note that the open face $\partial D_j \times \widehat{D}_j$ and the face F_j are subsets of the manifold $\partial\Omega_j$, which we will call the jth *extended face* of Ω.

We let $\jmath^k : \partial D_k \to \mathfrak{M}_k$ denote the inclusion map. If $\iota^k : \partial\Omega_k \to \mathfrak{M}$ denotes the inclusion map of the extended face $\partial\Omega_k$ in the product manifold $\mathfrak{M}$, we can write:

$$\iota^k = \jmath^k \times \mathsf{id}, \tag{4.5}$$

where $\mathsf{id} : \widehat{\mathfrak{M}}_k \to \widehat{\mathfrak{M}}_k$ denotes the identity map, and $\times$ is the reordered direct product of *maps* (with respect to the reordering $\mathfrak{M} = \mathfrak{M}_k \times \widehat{\mathfrak{M}}_k$). Then we can define pushforward maps $\jmath^k_* : \mathcal{D}'_0(\partial D_k) \to \mathcal{D}'_1(\mathfrak{M}_k)$, and $\iota^k_* : \mathcal{D}'_0(\partial\Omega_k) \to \mathcal{D}'_1(\mathfrak{M})$ as in (2.3). Thanks to (4.5), we see that these two are related by

$$\iota^k_* = \jmath^k_* \,\widehat{\otimes}\, \mathsf{id}. \tag{4.6}$$

Here id denotes the identity on $\mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$, and $\widehat{\otimes}$ has an obvious meaning as a reordered tensor product of continuous maps of topological vector spaces.

4.2 Some computations with the $\overline{\partial}$-operator on a product manifold

From the Cauchy–Riemann operators $\overline{\partial} : \mathcal{D}'_*(\mathfrak{M}_k) \to \mathcal{D}'_*(\mathfrak{M}_k)$ and $\overline{\partial} : \mathcal{D}'_*(\widehat{\mathfrak{M}}_k) \to \mathcal{D}'_*(\widehat{\mathfrak{M}}_k)$ we can construct two "partial Cauchy–Riemann operators" $\overline{\partial}_{\mathfrak{M}_k} : \mathcal{D}'_*(\mathfrak{M}) \to \mathcal{D}'_*(\mathfrak{M})$ and $\overline{\partial}_{\widehat{\mathfrak{M}}_k} : \mathcal{D}'_*(\mathfrak{M}) \to \mathcal{D}'_*(\mathfrak{M})$ using the reordered product representation $\mathfrak{M} = \mathfrak{M}_k \times \widehat{\mathfrak{M}}_k$ and setting

$$\overline{\partial}_{\mathfrak{M}_k} = \overline{\partial} \,\widehat{\otimes}\, \mathsf{id}, \tag{4.7}$$

and

$$\overline{\partial}_{\widehat{\mathfrak{M}}_k} = \mathsf{id}_\mathsf{k} \,\widehat{\otimes}\, \overline{\partial}. \tag{4.8}$$

In (4.7), $\overline{\partial}$ denotes the $\overline{\partial}$-operator on $\mathfrak{M}_k$, whereas id is the identity on $\widehat{\mathfrak{M}}_k$. On the other hand, in (4.8), $\overline{\partial}$ is the $\overline{\partial}$-operator on $\widehat{\mathfrak{M}}_k$ and id_k denotes the identity on $\mathfrak{M}_k$. Intuitively, $\overline{\partial}_{\mathfrak{M}_k}$ takes the derivative along the factor $\mathfrak{M}_k$ only and $\overline{\partial}_{\widehat{\mathfrak{M}}_k}$ takes the derivative along the factor $\widehat{\mathfrak{M}}_k$.

Proposition 4.1 *Let $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ be such that $\overline{\partial}\gamma = 0$ and let γ_k be as in (4.4). Then,*

(1) *for $k = 1, \ldots, N$, we have*

$$\overline{\partial}_{\mathfrak{M}_k}\gamma_k = 0, \tag{4.9}$$

(2) *for $j \neq k$, with $j, k = 1, \ldots, N$, we have*

$$\overline{\partial}_{\mathfrak{M}_j}\gamma_k - \overline{\partial}_{\mathfrak{M}_k}\gamma_j = 0. \tag{4.10}$$

To prove this, for $j = 1, \ldots, N$, define an operator σ_j on the space of forms on $\mathfrak{M}$ in the following way. We define σ_1 to be the identity operator and if $j \geq 2$, for forms ϕ_k on the $\mathfrak{M}_k$ of a fixed bidegree, we set

$$\sigma_j(\phi_1 \otimes \cdots \otimes \phi_N) = (-1)^{\sum_{k=1}^{j-1} \deg(\phi_k)}\, \phi_1 \otimes \cdots \otimes \phi_N,$$

and then extend by linearity and continuity to $\mathcal{D}'_*(\mathfrak{M})$.

Lemma 4.2 *On $\mathfrak{M}$, we have*

$$\overline{\partial} = \sum_{j=1}^{N} \sigma_j \overline{\partial}_{\mathfrak{M}_j}. \tag{4.11}$$

Proof By linearity and density, it is sufficient to show this for tensor products of forms of a fixed bidegree. The statement is obvious for $N = 1$, so we assume it for $N - 1$, and prove it for N, which gives the proof by induction. Let $\phi = \phi_1 \otimes \cdots \otimes \phi_N$, where ϕ_j is of fixed bidegree on $\mathfrak{M}_j$. Also let $\phi' = \phi_1 \otimes \cdots \otimes \phi_{N-1}$. Using the formula for the exterior derivative of a wedge product, we have

$$\begin{aligned}\overline{\partial}\phi &= \overline{\partial}\phi' \otimes \phi_N + (-1)^{\deg \phi'} \phi' \otimes \overline{\partial}\phi_N \\ &= \left(\sum_{j=1}^{N-1} \sigma_j \overline{\partial}_{\mathfrak{M}_j} \phi' \right) \otimes \phi_N + \sigma_N (\phi' \otimes \overline{\partial}\phi_N) \\ &= \left(\sum_{j=1}^{N} \sigma_j \overline{\partial}_{\mathfrak{M}_j} \right) \phi.\end{aligned}$$

Proof of Proposition 4.1 We write $\gamma = \sum_{k=1}^{N} \gamma_k$. Then we claim that

$$\sigma_j \overline{\partial}_{\mathfrak{M}_j} \gamma_k = \begin{cases} \overline{\partial}_{\mathfrak{M}_j} \gamma_j & \text{if } j \le k \\ -\overline{\partial}_{\mathfrak{M}_j} \gamma_j & \text{if } j > k. \end{cases}$$

Indeed, it is sufficient to consider the case when $\gamma_k = \alpha_1 \otimes \cdots \otimes \alpha_N$, where each α_ℓ is of degree 0 except α_k which is of degree $(0, 1)$. Taking the $\overline{\partial}$ with respect to $\mathfrak{M}_j$, we see that the first $j - 1$ factors in the representation of $\overline{\partial}\gamma_k$ as a tensor product are all of degree 0 if $j \le k$, but contains the single factor α_k of degree 1 if $j > k$. The claim follows from the definition of σ_j. Therefore,

$$\begin{aligned}\overline{\partial}\gamma &= \left(\sum_{j=1}^{N} \sigma_j \overline{\partial}_{\mathfrak{M}_j} \right) \left(\sum_{k=1}^{N} \gamma_k \right) \\ &= \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \overline{\partial}_{\mathfrak{M}_j} \gamma_k \\ &= \sum_{k=1}^{N} \overline{\partial}_{\mathfrak{M}_k} \gamma_k + \sum_{j<k} \left(\overline{\partial}_{\mathfrak{M}_j} \gamma_k - \overline{\partial}_{\mathfrak{M}_k} \gamma_j \right). \end{aligned} \tag{4.12}$$

If $j \ne k$, let $\widehat{\mathfrak{M}}_{j,k}$ denote the product of all the factors of $\mathfrak{M}$ except $\mathfrak{M}_j$ and $\mathfrak{M}_k$. We then have the direct sum decomposition for $(0, 2)$-currents:

$$\mathcal{D}'_{0,2}(\mathfrak{M}) = \bigoplus_{k=1}^{N} \mathcal{D}'_{0,2}(\mathfrak{M}_k) \widehat{\otimes} \mathcal{D}'_{0,0}(\widehat{\mathfrak{M}}_k) \oplus \left(\bigoplus_{j<k} \mathcal{D}'_{0,1}(\mathfrak{M}_j) \widehat{\otimes} \mathcal{D}'_{0,1}(\mathfrak{M}_k) \widehat{\otimes} \mathcal{D}'_{0,0}(\widehat{\mathfrak{M}}_{j,k}) \right), \tag{4.13}$$

where the tensor products are reordered so that each direct summand is a subspace of $\mathcal{D}'_{0,2}(\mathfrak{M})$. Note that the terms in (4.12) correspond to the direct summands in the decomposition (4.13), which must each vanish since $\overline{\partial}\gamma = 0$. □

We can represent the $\overline{\partial}_{\widehat{\mathfrak{M}}_k}$ operator of (4.8) in terms of the $\partial_{\mathfrak{M}_j}$. Indeed, we can show that

$$\overline{\partial}_{\widehat{\mathfrak{M}}_k} = \sum_{j \ne k} \sigma_{jk} \overline{\partial}_{\mathfrak{M}_j},$$

where $\sigma_{jk} \in \{\pm 1\}$. (The precise sign of σ_{jk}, while not difficult to find, is irrelevant for the intended application.)

Proposition 4.3 *Let γ and γ_k be as in Proposition 4.1. Then*

$$\left(\overline{\partial}_{\widehat{\mathfrak{M}}_k}\gamma_k\right)\Big|_{\partial D_k \times \widehat{D}_k} = 0. \tag{4.14}$$

Proof By conclusion (2) of Proposition 4.1 we have for $j \neq k$ that $\overline{\partial}_{\mathfrak{M}_j}\gamma_k = \overline{\partial}_{\mathfrak{M}_k}\gamma_j$. Consequently, $\overline{\partial}_{\mathfrak{M}_j}\gamma_k = 0$ outside $\operatorname{supp}(\overline{\partial}_{\mathfrak{M}_k}\gamma_j) \subset \operatorname{supp}(\gamma_j)$. Therefore, outside the set $\bigcup_{j\neq k}\operatorname{supp}(\gamma_j)$ we have

$$\overline{\partial}_{\widehat{\mathfrak{M}}_k}\gamma_k = \sum_{j\neq k}\sigma_{jk}\partial_{\mathfrak{M}_j}\gamma_k = 0.$$

The statement follows on noting that $\partial D_k \times \widehat{D}_k$ is disjoint from $\operatorname{supp}(\gamma_j) \subset \partial D_j \times \widehat{\mathfrak{M}_j}$ for each $j \neq k$. □

4.3 Existence of face distributions

We begin with the following lemma.

Lemma 4.4 *Let $\mathfrak{N}$ be a smooth manifold of dimension n, let S be a smooth hypersurface in $\mathfrak{N}$, and let $\iota : S \to \mathfrak{N}$ be the inclusion map and let $\gamma \in \mathcal{D}'_1(\mathfrak{N})$. Then there is a distribution $\alpha \in \mathcal{D}'_0(S)$ such that $\iota_*(\alpha) = \gamma$ if and only if for each $\phi \in \mathcal{D}^{n-1}(\mathfrak{N})$ such that $\iota^*\phi = 0$, we have $\langle\gamma, \phi\rangle = 0$. Consequently, the subspace of currents in $\mathcal{D}'_1(\mathfrak{N})$ of the form $\iota_*(\alpha)$ with $\alpha \in \mathcal{D}'_0(S)$ is closed.*

Proof If $\gamma = \iota_*\alpha$, then for any $\phi \in \mathcal{D}^{n-1}(\mathfrak{N})$ with $\iota^*\phi = 0$, we have $\langle\gamma, \phi\rangle = \langle\alpha, \iota^*\phi\rangle = 0$. On the other hand, there is a continuous linear extension operator $E : \mathcal{D}^{n-1}(S) \to \mathcal{D}^{n-1}(\mathfrak{N})$, such that for each $\psi \in \mathcal{D}^{n-1}(S)$, we have $\iota^*(E\psi) = \psi$. The existence of E is obvious locally using coordinates, and follows globally using a partition of unity argument. We define $\alpha \in \mathcal{D}'_0(S)$ by $\langle\alpha, \psi\rangle = \langle\gamma, E\psi\rangle$. Note that α is independent of the particular continuous extension operator E. Indeed, if $\widetilde{\psi} \in \mathcal{D}^{n-1}(\mathfrak{N})$ is any other extension of $\psi \in \mathcal{D}^{n-1}(S)$ (i.e., $\iota^*\widetilde{\psi} = \psi$), then $\langle\gamma, E\psi\rangle - \langle\gamma, \widetilde{\psi}\rangle = \langle\gamma, E\psi - \widetilde{\psi}\rangle = 0$, since $\iota^*(E\psi - \widetilde{\psi}) = 0$. For the α so defined, and any $\phi \in \mathcal{D}^{n-1}(\mathfrak{N})$, we have $\langle\iota_*\alpha, \phi\rangle = \langle\alpha, \iota^*\phi\rangle = \langle\gamma, \phi\rangle$, since ϕ is an extension of $\iota^*\phi$. □

Proposition 4.5 *Let $f \in \mathcal{A}^{-\infty}(\Omega)$. Then there are distributions $\alpha_k \in \mathcal{D}'_0(\partial\Omega_k)$ supported on $F_k = \partial\Omega_k \cap \overline{\Omega}$ such that*

$$\mathsf{bc} f = \sum_{k=1}^{N} \iota_*^k(\alpha_k)^{0,1},$$

where the notation is as in (4.5) and (4.6). Further, the k-th summand on the right-hand side is precisely the k-th component of the standard decomposition (4.4) and lies in $\mathcal{D}'_{0,1}(\mathfrak{M}_k) \widehat{\otimes} \mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$.

Proof Thanks to the decomposition (4.3) of (0,1)-forms on a product manifold, we can write $\mathsf{bc} f = \sum_{k=1}^{N}\gamma_k$, where $\gamma_k \in \mathcal{D}'_{0,1}(\mathfrak{M}_k) \widehat{\otimes} \mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$. We identify the summands γ_k. By definition of $\mathsf{bc} f$, each γ_k is supported on $\partial\Omega$. We cover Ω by open sets U of the type considered in Theorem 1.1, i.e., U is a coordinate neighborhood of $\mathfrak{M}$, and there is a vector

$v \in \mathbb{C}^n$ such that in the coordinates on U, the vector v points outward from Ω along each $\partial\Omega_j$ meeting U. Fix one such U, and let $\phi \in \mathcal{D}^{n,n-1}(U)$ be a form of degree $(n, n-1)$ with compact support in U. For f_ϵ as in Theorem 1.1, we have

$$\begin{aligned}\langle \mathsf{bc} f|_U, \phi\rangle &= \lim_{\epsilon\downarrow 0}\int_{\partial\Omega} f_\epsilon\phi \\ &= \sum_{k=1}^N \lim_{\epsilon\downarrow 0}\int_{\partial\Omega_k} f_\epsilon|_{\overline{\Omega}\cap U}\cdot\phi \\ &= \sum_{k=1}^N \lim_{\epsilon\downarrow 0}\left\langle f_\epsilon[\overline{\Omega}\cap U\cap\partial\Omega_k], (\iota^k)^*\phi\right\rangle_{\partial\Omega_k} \\ &= \sum_{k=1}^N \lim_{\epsilon\downarrow 0}\left\langle \iota^k_*\left(f_\epsilon[F_k\cap U]\right)^{0,1}, \phi\right\rangle. \end{aligned} \tag{4.15}$$

Here $F_k = \overline{\Omega}\cap\partial\Omega_k$ is the k-th face of Ω, and $[F_k\cap U]\in\mathcal{D}'_0(\partial\Omega_k)$ denotes the 0-current of integration on the set $F_k\cap U$. By Lemma 4.4 above, the subspace of $\mathcal{D}'_0(\mathfrak{M})$ consisting of currents of the type $\iota^k_*(\beta)$, (where $\beta\in\mathcal{D}'_0(\partial\Omega_k)$) is closed, so it easily follows that there is an $\alpha^U_k\in\mathcal{D}'_0(\partial\Omega_k)$ such that $\gamma_k|_U = \iota^k_*(\alpha^U_k)^{0,1}$. The existence of $\alpha_k\in\mathcal{D}'_0(\partial\Omega_k)$ such that $\gamma_k = \iota^k_*(\alpha_k)^{0,1}$ now follows by a partition of unity argument. From the representation (4.15), it follows that α_k has support in the subset F_k of $\partial\Omega_k$. □

4.4 Currents with facewise structure

In view of Proposition 4.5 above, we make the following definition: let $\mathfrak{M}$ be a product manifold as in (1.5), and let $\Omega\Subset\mathfrak{M}$ be a product domain as in (1.6). We say that a current $\gamma\in\mathcal{D}'_{0,1}(\mathfrak{M})$ has *facewise structure* with respect to Ω, if there are *face distributions* $\alpha_j\in\mathcal{D}'_0(\partial\Omega_j)$, for $j=1,\dots,N$, with support in F_j, such that

$$\gamma = \sum_{j=1}^N \iota^j_*(\alpha_j)^{0,1}, \tag{4.16}$$

where $\iota^j:\partial\Omega_j\to\mathfrak{M}$ is the inclusion map. Then, Proposition 4.5 states that the boundary current of a holomorphic function of polynomial growth on a product domain has facewise structure.

Proposition 4.6 *Let $\mathfrak{M}$ and Ω be as in (1.5), (1.6). Let $\gamma\in\mathcal{D}'_{0,1}(\mathfrak{M})$ satisfy the Weinstock condition and have facewise structure (4.16), both with respect to Ω. Then for each $k\in\{1,\dots,N\}$, we have*

$$\alpha_k|_{\partial D_k\times\widehat{D}_k}\in\mathcal{D}'_0(\partial D_k)\,\widehat{\otimes}\,\mathcal{A}^{-\infty}(\widehat{D}_k). \tag{4.17}$$

Intuitively, this says that the distributions α_k, which may be thought of as "restrictions" of γ to the faces, are each holomorphic and of polynomial growth along the complex factor, which is clearly the case for continuous boundary values on a product domain. Note also that in view of the fact that $\gamma_k = \iota^k_*(\alpha_k)$, the relation (4.17) is equivalent to

$$\gamma_k|_{\mathfrak{M}_k\times\widehat{D}_k}\in\mathcal{D}'_{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{A}^{-\infty}(\widehat{D}_k). \tag{4.18}$$

Proof From (4.6), it follows that in the decomposition of $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ into direct summands given by (4.3), the component $\gamma_k \in \mathcal{D}'_{0,1}(\mathfrak{M}_k) \widehat{\otimes} \mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$ is given by

$$\gamma_k = (j_*^k \widehat{\otimes} \mathsf{id})(\alpha_k)^{0,1}, \tag{4.19}$$

where $\alpha_k \in \mathcal{D}'_0(\partial\Omega_k)$ is the k-th face distribution associated with the current γ. Combining (4.19) and (4.8), we see that

$$\begin{aligned}\overline{\partial}_{\widehat{\mathfrak{M}}_k}\gamma_k &= (\mathsf{id} \widehat{\otimes} \overline{\partial})(j_*^k \widehat{\otimes} \mathsf{id})(\alpha_k)^{0,1} \\ &= ((j_*^k \widehat{\otimes} \overline{\partial})(\alpha_k))^{0,1} \\ &= (j_*^k \widehat{\otimes} \overline{\partial})(\alpha_k).\end{aligned}$$

Therefore, using (4.14) we see that on the manifold $\partial D_k \times \widehat{\mathfrak{M}}_k$, the current $(\mathsf{id} \widehat{\otimes} \overline{\partial})(\alpha_k)$ vanishes on the open set $\partial D_k \times \widehat{D}_k$, i.e., α_k is holomorphic in the direction of $\widehat{\mathfrak{M}}_k$ on the open set $\partial D_k \times \widehat{D}_k$ It follows that

$$\alpha_k|_{\partial D_k \times \widehat{D}_k} \in \mathcal{D}'_0(\partial D_k) \widehat{\otimes} \mathcal{O}(\widehat{D}_k).$$

Now we note that the face distribution $\alpha_k \in \mathcal{D}'_0(\partial\Omega_k)$ (where we recall that $\partial\Omega_k = \partial D_k \times \widehat{\mathfrak{M}}_k$) is supported in $\partial D_k \times \overline{\widehat{D}_k}$. Consequently, α_k is a distribution of finite order on $\partial\Omega_k$. It follows that there is an integer K, such that

$$\alpha_k|_{\partial D_k \times \widehat{D}_k} \in \mathcal{C}^{-K}(\partial D_k) \widehat{\otimes} \mathcal{O}(\widehat{D}_k) \cong \mathcal{O}(\widehat{D}_k, \mathcal{C}^{-K}(\partial D_k)),$$

where $\mathcal{C}^{-K}(\partial D_k)$ is the Banach space of distributions of order K on ∂D_k (it is a Banach space since ∂D_k is compact), and the isomorphism of the space $\mathcal{O}(\widehat{D}_k, \mathcal{C}^{-K}(\partial D_k))$ with the topological tensor product $\mathcal{C}^{-K}(\partial D_k) \widehat{\otimes} \mathcal{O}(\widehat{D}_k)$ (which makes sense since $\mathcal{O}(\widehat{D}_k)$ is nuclear) follows as in [25, Theorem 44.1]. Using this isomorphism, interpreting $\alpha_k|_{\partial D_k \times \widehat{D}_k}$ as a Banach-valued holomorphic function on $\widehat{D}_k$, we see that it can be extended to a Banach-valued distribution on $\widehat{\mathfrak{M}}_k$. Therefore, by Proposition 2.1, we have $\alpha_k \in \mathcal{A}^{-\infty}(\widehat{D}_k, \mathcal{C}^{-K}(\partial D_k))$. Since each distribution on the compact manifold ∂D_k is of finite order, we see that

$$\alpha_k|_{\partial D_k \times \widehat{D}_k} \in \mathcal{D}'_0(\partial D_k) \widehat{\otimes} \mathcal{A}^{-\infty}(\widehat{D}_k),$$

which proves (4.17). □

4.5 The space $\mathcal{Y}^{0,1}_{\Omega}(\mathfrak{M})$

We will now state precisely the third condition in the definition of the space $\mathcal{Y}^{0,1}_{\Omega}(\mathfrak{M})$. Let $\mathfrak{M}$ and Ω be as above a product manifold and a product domain as in (1.5) and (1.6). Suppose that a current $\gamma \in \mathcal{D}'_{0,1}(\mathfrak{M})$ satisfies the Weinstock condition and has facewise structure (4.16), both with respect to Ω. Then by Proposition 4.6 the relation (4.17) holds. The third condition in the definition of $\mathcal{Y}^{0,1}_{\Omega}(\mathfrak{M})$ is the following: for $k = 1, \ldots, N$, we have

$$(\mathsf{id}_k \widehat{\otimes} \mathsf{ce}_{\widehat{k}}) \left(\alpha_k|_{\partial D_k \times \widehat{D}_k}\right) = \alpha_k, \tag{4.20}$$

where id_k is the identity map on $\mathcal{D}'_0(\partial D_k)$ and $\mathsf{ce}_{\widehat{k}} : \mathcal{A}^{-\infty}(\widehat{D}_k) \to \mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$ is the canonical extension operator. We refer to (4.20) as the *canonicality condition on face distributions.* We note that the condition (4.20) may be directly expressed in terms of the current γ as:

$$(\mathsf{id}_k \widehat{\otimes} \mathsf{ce}_{\widehat{k}}) \left(\gamma_k|_{\mathfrak{M}_k \times \widehat{D}_k}\right) = \gamma_k, \tag{4.21}$$

where γ_k is the k-th component of γ in the standard decomposition (4.4), id_k now denotes the identity map on $\mathcal{D}'_{0,1}(\mathfrak{M}_k)$ and $\mathsf{ce}_{\widehat{k}}$ is as in (4.20). Also note that (4.21) makes sense thanks to (4.18). To prove (4.21), we have

$$\begin{aligned}(\mathsf{id}_k \widehat{\otimes} \mathsf{ce}_{\widehat{k}})\left(\gamma_k|_{\mathfrak{M}_k\times\widehat{D}_k}\right) &= (\mathsf{id}_k \widehat{\otimes} \mathsf{ce}_{\widehat{k}})\left((\jmath^k \widehat{\otimes} \mathsf{id}_{\widehat{k}})(\alpha_k|_{\partial D_k\times\widehat{D}_k})\right)^{0,1}\\ &= \left((\jmath^k_* \widehat{\otimes} \mathsf{ce}_{\widehat{k}})(\mathsf{id}_k \widehat{\otimes} \mathsf{ce}_{\widehat{k}})\left(\alpha_k|_{\partial D_k\times\widehat{D}_k}\right)\right)^{0,1}\\ &= \left(\iota^k_*(\alpha_k)\right)^{0,1} \qquad \text{using (4.20)}\\ &= \gamma_k,\end{aligned}$$

where in the second line, id_k denotes the identity operator on ∂D_k. The converse implication, i.e., that (4.21) implies (4.20), can be proved by an analogous computation.

Therefore, $\mathcal{Y}^{0,1}_\Omega(\mathfrak{M})$ consists of those (0, 1)-currents in $\mathcal{D}'_{0,1}(\mathfrak{M})$ which satisfy the Weinstock criterion, have facewise structure (both with respect to Ω), and whose face distributions are canonical in the sense of (4.20) or, equivalently, (4.21). Since all three conditions are closed, it follows that $\mathcal{Y}^{0,1}_\Omega(\mathfrak{M})$ is a closed subspace of $\mathcal{D}'_{0,1}(\mathfrak{M})$. The next proposition shows that in Theorem 1.4 we have identified the correct target space.

Proposition 4.7 *Let $\Omega \Subset \mathfrak{M}$ be a product domain as in* (1.5), (1.6). *If $f \in \mathcal{A}^{-\infty}(\Omega)$, then* $\mathsf{bc} f \in \mathcal{Y}^{0,1}_\Omega(\mathfrak{M})$.

Proof In view of Propositions 2.6 and 4.5, we only need to prove (4.20), which we do in the equivalent form (4.21). We first note that the statement (4.21) is local in the following sense: to prove (4.21) it suffices to give an open cover $\mathcal{W}$ of $\mathfrak{M}_k$ and another open cover $\widehat{\mathcal{W}}$ of $\partial\widehat{D}_k$ by open sets of $\widehat{\mathfrak{M}}_k$ such that for each $W \in \mathcal{W}$ and $\widehat{W} \in \widehat{\mathcal{W}}$, we have

$$(\mathsf{id}_k \widehat{\otimes} \mathsf{ce}_{\widehat{k}})\left(\gamma_k|_{W\times(\widehat{D}_k\cap\widehat{W})}\right) = \gamma_k|_{W\times\widehat{W}}, \tag{4.22}$$

where id_k is now the identity operator on $\mathcal{D}'_{0,1}(W)$ and $\mathsf{ce}_{\widehat{k}}$ denotes the canonical extension operator from $\mathcal{A}^{-\infty}(\widehat{D}_k \cap \widehat{W}))$ to $\mathcal{D}'_0(\widehat{W})$. For any point $p \in \mathfrak{M}_k \times \partial\widehat{D}_k$, we can find a neighborhood U of p in $\mathfrak{M}$ such that there is a vector v as in Theorem 1.1 pointing outward from Ω along each $\partial\Omega_j$, so that if we define $f_\epsilon = f(\cdot - \epsilon v)$, then (1.2), (2.5) and (4.15) hold. Note further that by shrinking U around p, we may assume that $U = W \times \widehat{W}$, where W is an open set in $\mathfrak{M}_k$ and $\widehat{W}$ is an open set of $\widehat{\mathfrak{M}}_k$.

Now from (4.15) we conclude that for each $\phi \in \mathcal{D}^{n,n-1}(U)$, we have

$$\langle \gamma_k|_U, \phi\rangle = \lim_{\epsilon\downarrow 0}\left\langle \gamma_k^\epsilon, \phi\right\rangle,$$

where γ_k has the same meaning as above, and $\gamma_k^\epsilon = \iota^k_*(f_\epsilon[F_k \cap U])^{0,1}$. Let $\{\epsilon_\nu\}$ be a sequence of positive real numbers which converges to the limit 0 as $\nu \to \infty$. Recalling that in $\mathcal{D}'_{0,1}(U)$ (or more generally in the dual of a Montel space, see [25, Section 34.4]) a weak-* convergent sequence is also convergent in the usual strong topology, we have

$$\gamma_k|_U = \lim_{\nu\to\infty} \gamma_k^{\epsilon_\nu}. \tag{4.23}$$

Note now that for each ν, we have

$$\gamma_k^{\epsilon_\nu}|_{W\times(W\cap\widehat{D}_k)} \in \mathcal{D}'_{0,1}(W) \widehat{\otimes} \mathcal{A}^\infty(\widehat{W} \cap \widehat{D}_k),$$

where $\mathcal{A}^\infty(\widehat{W}\cap\widehat{D}_k)$ denotes the space of functions which are holomorphic on $\widehat{W}\cap\widehat{D}_k\subset\widehat{\mathfrak{M}}_k$ and extend smoothly to the boundary. This follows from the fact that for $\epsilon>0$, we have $\gamma_k^\epsilon=\iota_*^k(f_\epsilon[F_k\cap U])^{0,1}$, and f_ϵ is $\mathcal{C}^\infty$-smooth on $\partial\Omega\cap U$. Therefore, we have

$$(\mathsf{id}_k\,\widehat{\otimes}\,\mathsf{ce}_{\widehat{k}})\left(\gamma_k^{\epsilon_\nu}|_{W\times(\widehat{D}_k\cap\widehat{W})}\right)=\gamma_k^{\epsilon_\nu}|_{W\times\widehat{W}},$$

since for functions continuous up to the boundary, the canonical extension is precisely the extension by 0 (see the proof of Theorem 2.4). We now let $\nu\to\infty$, use (4.23) and the continuity of id_k and $\mathsf{ce}_{\widehat{k}}$ to conclude that (4.22) holds. □

5 Holomorphic extension of currents in $\mathcal{Y}_\Omega^{0,1}(\mathfrak{M})$

5.1 The structure of the direct summands

Let $\gamma\in\mathcal{Y}_\Omega^{0,1}(\mathfrak{M})$. Using the decomposition (4.3), we write $\gamma=\sum_{k=1}^N\gamma_k$, where $\gamma_k\in\mathcal{D}'_{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$. First we note the following fact.

Proposition 5.1 *For each $k=1,\dots,N$,*

$$\gamma_k|_{\mathfrak{M}_k\times\widehat{D}_k}\in\mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{A}^{-\infty}(\widehat{D}_k).\tag{5.1}$$

Proof Let $n_k=\dim_{\mathbb{C}}\mathfrak{M}_k$, and $\hat{n}_k=\dim_{\mathbb{C}}\widehat{\mathfrak{M}}_k$. We first show that $\gamma_k\in\mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$. From the definition of $\mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)$ (see (1.3) and (1.4)) combined with Lemma 4.4, we see that a current $\theta\in\mathcal{D}'_{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$ is in $\mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$ provided the following two conditions are satisfied:

(i) for each $\omega\in\mathcal{D}^{n_k,n_k-1}(\mathfrak{M}_k)$ such that $\overline{\partial}\omega=0$ on D_k, and for each $\phi\in\mathcal{D}^{\hat{n}_k,\hat{n}_k}(\widehat{\mathfrak{M}}_k)$ we have $\langle\theta,\omega\otimes\phi\rangle=0$;
(ii) denoting as usual by $\jmath^k$ the inclusion of ∂D_k in $\mathfrak{M}_k$, for each $\lambda\in\mathcal{D}^{n_k,n_k-1}(\mathfrak{M}_k)$ such that $(\jmath^k)^*(\lambda)=0$, and for each $\phi\in\mathcal{D}^{\hat{n}_k,\hat{n}_k}(\widehat{\mathfrak{M}}_k)$ we have $\langle\theta,\lambda\otimes\phi\rangle=0$.

Note that in the algebraic tensor products $\omega\otimes\phi$ and $\lambda\otimes\phi$ above, the factors have to be reordered. To verify (i), note that if $j\neq k$, we have $\langle\gamma_j,\omega\otimes\phi\rangle=0$, since $\omega\otimes\phi\in\mathcal{D}^{n_k,n_k-1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{D}^{\hat{n}_k,\hat{n}_k}(\widehat{\mathfrak{M}}_k)$, whereas $\gamma_j\in\mathcal{D}'_{0,1}(\mathfrak{M}_j)\,\widehat{\otimes}\,\mathcal{D}'_{0,0}(\widehat{\mathfrak{M}}_j)$. If we set $\hat{\gamma}_k=\sum_{j\neq k}\gamma_j$, then we have $\langle\hat{\gamma}_k,\omega\otimes\phi\rangle=0$. Now since $\overline{\partial}\omega=0$ on D_k, we see that $\overline{\partial}(\omega\otimes\phi)$ vanishes on $D_k\times\widehat{\mathfrak{M}}_k$, and therefore vanishes a fortiori on $\Omega=D_k\times\widehat{D}_k$. Since $\gamma\in\mathcal{X}_\Omega^{0,1}(\mathfrak{M})$, we have therefore $\langle\gamma,\omega\otimes\phi\rangle=0$. We therefore have

$$\begin{aligned}\langle\gamma_k,\omega\otimes\phi\rangle&=\langle\gamma-\hat{\gamma}_k,\omega\otimes\phi\rangle\\&=\langle\gamma,\omega\otimes\phi\rangle-\langle\hat{\gamma}_k,\omega\otimes\phi\rangle\\&=0.\end{aligned}$$

For (ii), we use the representation (4.19) of γ_k. We then have

$$\begin{aligned}\langle \gamma_k, \lambda \otimes \phi\rangle &= \left\langle (\jmath_*^k \widehat{\otimes} \mathsf{id})(\alpha_k)^{0,1}, \lambda \otimes \phi\right\rangle \\ &= \left\langle \alpha_k, (\jmath^k)^* \widehat{\otimes} \mathsf{id})(\lambda \otimes \phi)\right\rangle \\ &= \left\langle \alpha_k, (\jmath^k)^* \lambda \otimes \phi)\right\rangle \\ &= \langle \alpha_k, 0\rangle \\ &= 0.\end{aligned}$$

Therefore, it follows that $\gamma_k \in \mathcal{X}^{0,1}_{D_k}(\mathfrak{M}_k) \widehat{\otimes} \mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$. Now, we can write $\gamma_k|_{\mathfrak{M}_k \times \widehat{D}_k} = \iota_*^k(\alpha_k|_{\partial D_k \times \widehat{D}_k})^{0,1} = (\jmath_*^k \widehat{\otimes} \mathsf{id})\left(\alpha_k|_{\partial D_k \times \widehat{D}_k}\right)^{0,1}$. Using (4.17), the result (5.1) now follows. □

5.2 The space $\tilde{\mathcal{A}}^{-\infty}(\Omega)$

We begin by noting some simple properties of the space of holomorphic functions of polynomial growth:

Proposition 5.2 *Let D be a domain with generic corners in the complex manifold $\mathfrak{M}$. Then*

(a) *the canonical extension map*

$$\mathsf{ce} : \mathcal{A}^{-\infty}(D) \to \mathcal{D}'_0(\mathfrak{M})$$

is an isomorphism (of TVS) onto the image $\mathsf{ce}(\mathcal{A}^{-\infty}(D))$, equipped with the subspace topology from $\mathcal{D}'_0(\mathfrak{M})$.

(b) *The space $\mathcal{A}^{-\infty}(D)$ is nuclear. Consequently, there is a naturally defined topological tensor product $\mathcal{A}^{-\infty}(D) \widehat{\otimes} \mathsf{X}$ with any locally convex topological vector space X, which can be naturally identified with a closed subspace of $\mathcal{D}'_0(\mathfrak{M}, \mathsf{X})$, the space of X-valued distributions.*

(c) *if U is a relatively compact open subset in a complex manifold $\mathfrak{N}$, then*

$$\mathcal{A}^{-\infty}(D) \widehat{\otimes} \mathcal{A}^{-\infty}(U) \subseteq \mathcal{A}^{-\infty}(D \times U).$$

Proof Since ce is obviously injective. to prove (a), it suffices to show that $\mathsf{ce}(\mathcal{A}^{-\infty}(D))$ with its subspace topology is a DFS space. Then the result would follow from Proposition 2.3. We claim that each element $\mathsf{ce} f$ of $\mathsf{ce}(\mathcal{A}^{-\infty}(D))$ induces a linear functional on $\mathcal{E}^{n,n}(\overline{D})$, the space of top degree forms smooth up to the boundary on D. If $\phi \in \mathcal{E}^{n,n}(\overline{D})$, and $\tilde{\phi}$ is any extension of ϕ to an element of $\mathcal{D}^{n,n}(\mathfrak{M})$, then we define $\langle \mathsf{ce} f, \phi\rangle = \left\langle \mathsf{ce} f, \tilde{\phi}\right\rangle$, which is well-defined, since from the definition it is clear that $\mathsf{ce} f$ vanishes on any test form vanishing on D. This embeds $\mathsf{ce}(\mathcal{A}^{-\infty}(D))$ as a closed subspace of the strong dual of $\mathcal{E}^{n,n}(\overline{D})$. Since $\mathcal{E}^{n,n}(\overline{D})$ is a Fréchet–Schwartz space, it follows that $\mathsf{ce}(\mathcal{A}^{-\infty}(D))$ is a DFS space.

Assertion (b) now follows, since ce embeds $\mathcal{A}^{-\infty}(D)$ as a closed subspace of the nuclear space $\mathcal{D}'_0(\mathfrak{M})$. This also allows us to view $\mathcal{A}^{-\infty}(D) \widehat{\otimes} \mathsf{X}$ as the subspace $\mathsf{ce}(\mathcal{A}^{-\infty}(D)) \widehat{\otimes} \mathsf{X}$ of $\mathcal{D}'_0(\mathfrak{M}) \widehat{\otimes} \mathsf{X}$.

To see (c), note that it suffices to prove that the algebraic tensor product $\mathcal{A}^{-\infty}(D) \otimes \mathcal{A}^{-\infty}(U)$ is contained in $\mathcal{A}^{-\infty}(D \times U)$. Indeed, by linearity, it suffices to show that if $f \in \mathcal{A}^{-\infty}(D)$ and $g \in \mathcal{A}^{-\infty}(U)$, then $f \otimes g \in \mathcal{A}^{-\infty}(D \times U)$. Now $\mathcal{A}^{-\infty}(D \times U)$ is closed under

multiplication (since on any domain Ω, we clearly have for $F \in \mathcal{A}^{-k}(\Omega)$, $G \in \mathcal{A}^{-\ell}(\Omega)$ that $FG \in \mathcal{A}^{-(k+\ell)}(\Omega)$). Denoting by $\mathbf{1}$ the constant function with value 1, we see that $f \otimes \mathbf{1}$ and $\mathbf{1} \otimes g$ belong to $\mathcal{A}^{-\infty}(D \times U)$, but we have $f \otimes g = (f \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes g)$. □

5.3 Proof of Theorem 1.4

Let $\Omega = D_1 \times \cdots \times D_N$ be a product domain in a product manifold as in (1.6). We define:

$$\widetilde{\mathcal{A}}^{-\infty}(\Omega) = \mathcal{A}^{-\infty}(D_1) \widehat{\otimes} \mathcal{A}^{-\infty}(D_2) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{A}^{-\infty}(D_N),$$

where the topological tensor products are well-defined thanks to Proposition 5.2. Then $\widetilde{\mathcal{A}}^{-\infty}(\Omega) \subset \mathcal{A}^{-\infty}(\Omega)$. Similarly, we denote by $\widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k)$ the topological tensor product of the $\mathcal{A}^{-\infty}(D_j)$'s with $j \neq k$. We will need the following properties of $\widetilde{\mathcal{A}}^{-\infty}(\Omega)$:

Lemma 5.3 *The map* $\mathsf{ce} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{D}'_0(\mathfrak{M})$ *restricted to the subspace* $\widetilde{\mathcal{A}}^{-\infty}(\Omega)$ *admits the representation*

$$\mathsf{ce}|_{\widetilde{\mathcal{A}}^{-\infty}(\Omega)} = \widehat{\otimes}_{k=1}^{N} \mathsf{ce}_k, \tag{5.2}$$

where $\mathsf{ce}_k : \mathcal{A}^{-\infty}(D_k) \to \mathcal{D}'_0(\mathfrak{M}_k)$ *is the canonical extension map. Similarly, the restriction of* $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{D}'_{0,1}(\mathfrak{M})$ *when admits the representation*

$$\mathsf{bc}|_{\widetilde{\mathcal{A}}^{-\infty}(\Omega)} = \sum_{k=1}^{N} \mathsf{bc}_k \widehat{\otimes} \mathsf{ce}_{\widehat{k}}, \tag{5.3}$$

where $\mathsf{bc}_k : \mathcal{A}^{-\infty}(D_k) \to \mathcal{D}'_{0,1}(\mathfrak{M}_k)$ *is the boundary current map and* $\mathsf{ce}_{\widehat{k}} : \widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k) \to \mathcal{D}'_0(\widehat{\mathfrak{M}}_k)$ *is restriction to* $\widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k)$ *of the canonical extension map.*

Proof In order to establish (5.2) it suffices to show that

$$\mathsf{ce} f = \mathsf{ce}_1 f_1 \otimes \mathsf{ce}_2 f_2 \otimes \cdots \otimes \mathsf{ce}_N f_N, \tag{5.4}$$

whenever $f_j \in \mathcal{A}^{-\infty}(\Omega_j)$ and $f = f_1 \otimes \cdots \otimes f_N$ is their tensor product, which lies in $\mathcal{A}^{-\infty}(\Omega)$ by part (c) of Proposition 5.2. Once (5.4) is established, it follows by linearity that (5.2) holds on the algebraic tensor product $\mathcal{A}^{-\infty}(D_1) \otimes \cdots \otimes \mathcal{A}^{-\infty}(D_N)$, and then (5.2) follows by density.

Note that (5.2) is a local property, in the sense that to prove it, it suffices to show that each point in the product manifold $\mathfrak{M} = \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n$ has a neighborhood W of the form $W_1 \times \cdots \times W_N$, with $W_j \subset \mathfrak{M}_j$, such that we have

$$\mathsf{ce}(f|_{\Omega \cap W}) = \mathsf{ce}_1(f_1|_{D_1 \cap W_1}) \otimes \cdots \otimes \mathsf{ce}_N(f_N|_{D_N \cap W_N}), \tag{5.5}$$

where ce_j is now the canonical extension operator from $\mathcal{A}^{-\infty}(W_j \cap D_j)$ into $\mathcal{D}'_0(W_j)$, and ce is the canonical extension operator from $\mathcal{A}^{-\infty}(W \cap D)$ into $\mathcal{D}'_0(W)$. To prove (5.5), we use the method of proof as in Proposition 4.7, i.e., represent f and f_j locally as a limit of functions smooth up to the boundary. Note that in (5.5), the only interesting case is when W is a neighborhood of a point on the boundary of Ω. After shrinking W, we may assume that that there is a vector v as in Theorem 2.4 which is transverse to each $\partial\Omega_j$ in W, and let $f_\epsilon = f(\cdot - \epsilon v)$ be also as in Theorem 2.4. Choosing a sequence of positive numbers ϵ_ν such that $\epsilon_\nu \to 0$ as $\nu \to \infty$, we see that f_{ϵ_ν} is smooth up to the boundary on $\Omega \cap W$, and we have

$$f_{\epsilon_\nu} = f_1^\nu \otimes \cdots \otimes f_N^\nu,$$

where $f_j^\nu \in \mathcal{O}(W_j \cap D_j)$ is smooth up to the boundary as well, and as in the proof of Proposition 4.7, as $\nu \to \infty$, we have $f_j^\nu \to f_j$ and $f_{\epsilon_\nu} \to f$. Since (5.5) holds when f is replaced by $f_{\epsilon_n u}$ and f_j is replaced by f_j^ν, and since the maps ce_j and ce are continuous, taking a limit as $\nu \to \infty$, (5.5) follows, and from which follows (5.2).

We now prove (5.3) by a direct computation using the definition (2.8). We obtain

$$\begin{aligned}
\mathsf{bc}|_{\widetilde{\mathcal{A}}^{-\infty}(\Omega)} &= -\overline{\partial}\left(\mathsf{ce}|_{\widetilde{\mathcal{A}}^{-\infty}(\Omega)}\right) \\
&= -\overline{\partial}\left(\widehat{\otimes}_{k=1}^N \mathsf{ce}_k,\right) \\
&= \sum_{k=1}^N -(\overline{\partial} \circ \mathsf{ce}_k)\,\widehat{\otimes}\,\mathsf{ce}_{\widehat{k}} \\
&= \sum_{k=1}^N \mathsf{bc}_k\,\widehat{\otimes}\,\mathsf{ce}_{\widehat{k}},
\end{aligned}$$

where in the last-but-one line we have used the Leibniz product rule, and the fact that $\mathsf{ce}_{\widehat{k}}$ on $\widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k)$ is by (5.2) the tensor product $\widehat{\otimes}_{j\neq k}\mathsf{ce}_j$. □

We now prove the following result, which contains Theorem 1.4:

Proposition 5.4 *Let $\mathfrak{M}$, Ω be as in (1.5) and (1.6). Then we have the following:*

(1) $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{Y}_\Omega^{0,1}(\mathfrak{M})$ *is an isomorphism of topological vector spaces, and*
(2) $\widetilde{\mathcal{A}}^{-\infty}(\Omega) = \mathcal{A}^{-\infty}(\Omega)$

Proof We proceed by induction on N, the number of smooth factors of the product $\mathfrak{M}$. For $N = 1$, conclusion (1) is Theorem 1.2 and conclusion (2) is obvious. Therefore we assume the result when Ω has $N - 1$ smooth factors. Note that each $k = 1, \ldots, N$, we have

$$\widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k) = \mathcal{A}^{-\infty}(\widehat{D}_k),$$

since the domain $\widehat{D}_k \subset \widehat{\mathfrak{M}}_k$ is the product of $(N - 1)$ smooth factors.

We now show that $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{Y}_\Omega^{0,1}(\mathfrak{M})$ is an isomorphism of topological vector spaces. The injectivity follows from Proposition 2.8, so we need only to show that bc is surjective. By Theorem 1.2, the map $\mathsf{bc} : \mathcal{A}^{-\infty}(D_k) \to \mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)$ is an isomorphism of topological vector spaces. Denote by bc_k^{-1} its inverse, which is an isomorphism from $\mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)$ to $\mathcal{A}^{-\infty}(D_k)$. Tensoring with the identity map on $\widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k)$, we obtain an isomorphism

$$(\mathsf{bc}_k^{-1}\,\widehat{\otimes}\,\mathsf{id}) : \mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{A}^{-\infty}(\widehat{D}_k) \to \widetilde{\mathcal{A}}^{-\infty}(\Omega),$$

where we use the fact that $\widetilde{\mathcal{A}}^{-\infty}(\Omega) = \mathcal{A}^{-\infty}(D_k)\,\widehat{\otimes}\,\widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k)$.

Now let $\gamma \in \mathcal{Y}_\Omega^{0,1}(\mathfrak{M})$, and let $\gamma_k \in \mathcal{D}'_{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\mathcal{D}'_{0,0}(\widehat{\mathfrak{M}}_k)$ be as in the beginning of Sect. 5.1. Then by (5.1) and the induction hypothesis, we have that $\gamma_k|_{\mathfrak{M}_k \times \widehat{D}_k} \in \mathcal{X}_{D_k}^{0,1}(\mathfrak{M}_k)\,\widehat{\otimes}\,\widetilde{\mathcal{A}}^{-\infty}(\widehat{D}_k)$. We define $f_k \in \widetilde{\mathcal{A}}^{-\infty}(\Omega)$ as

$$f_k = (\mathsf{bc}_k^{-1}\,\widehat{\otimes}\,\mathsf{id})\left(\gamma_k|_{\mathfrak{M}_k \times \widehat{D}_k}\right),$$

which makes sense thanks to (5.1). The proof of the surjectivity of bc will be completed by showing that $\mathsf{bc}\,f_k = \gamma$.

Set $\lambda = \mathsf{bc} f_k$, and let $\lambda = \sum_{j=1}^N \lambda_j$ be the standard decomposition of λ as in (4.4). We claim that $\lambda_k = \gamma_k$. Indeed, in the representation (5.3), the summands correspond to the terms of the standard decomposition (4.4). Therefore, we have

$$\begin{aligned}\lambda_k &= (\mathsf{bc}_k \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{k}}) f_k \\ &= (\mathsf{bc}_k \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{k}})(\mathsf{bc}_k^{-1} \,\widehat{\otimes}\, \mathsf{id}) \left(\gamma_k|_{\mathfrak{M}_k \times \widehat{D}_k}\right) \\ &= (\mathsf{id}_k \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{k}}) \left(\gamma_k|_{\mathfrak{M}_k \times \widehat{D}_k}\right) \\ &= \gamma_k, \end{aligned} \tag{5.6}$$

thanks to the canonicality of the face distributions, as expressed in (4.21). To complete the proof we will show that for each pair $k, \ell \in \{1, \dots, N\}$, we have $f_k = f_\ell$, which will ensure that $\mathsf{bc} f_k = \gamma$.

Consider the continuous linear map from $\widetilde{\mathcal{A}}^{-\infty}(\Omega)$ into $\mathcal{D}'_{0,2}(\mathfrak{M})$ given by

$$\mathsf{b}_{k,\ell} = \mathsf{ce} \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathsf{bc} \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathsf{bc} \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathsf{ce},$$

where each tensor factor is ce except the k-th and ℓ-th ones, which are bc. Then $\mathsf{b}_{k,\ell} : \widetilde{\mathcal{A}}^{-\infty}(\Omega) \to \mathcal{D}'_{0,2}(\mathfrak{M})$ is injective, since each of the factor maps ce of bc of the tensor product defining $\mathsf{b}_{k,\ell}$ is injective. Using the fact that $\mathsf{bc} = -\overline{\partial} \circ \mathsf{ce}$, we obtain the representation

$$\mathsf{b}_{k,\ell} = -\overline{\partial}_{\mathfrak{M}_\ell} \circ (\mathsf{bc}_k \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{k}}), \tag{5.7}$$

where $\overline{\partial}_{\mathfrak{M}_\ell}$ is the differential operator as in (4.7), the tensor product of the $\overline{\partial}$ operator on the factor $\mathfrak{M}_\ell$ and the identity operator on $\widehat{\mathfrak{M}}_\ell$, and $\mathsf{bc}_k \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{k}}$ is as in (5.3). Of course, by symmetry we may also write

$$\mathsf{b}_{k,\ell} = -\overline{\partial}_{\mathfrak{M}_k} \circ (\mathsf{bc}_\ell \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{\ell}}). \tag{5.8}$$

Therefore, we have

$$\begin{aligned}\mathsf{b}_{k,\ell} f_k &= -\overline{\partial}_{\mathfrak{M}_\ell} \circ (\mathsf{bc}_k \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{k}}) f_k \\ &= -\overline{\partial}_{\mathfrak{M}_\ell} \gamma_k, \end{aligned}$$

and also

$$\begin{aligned}\mathsf{b}_{k,\ell} f_\ell &= -\overline{\partial}_{\mathfrak{M}_k} \circ (\mathsf{bc}_\ell \,\widehat{\otimes}\, \mathsf{ce}_{\widehat{\ell}}) f_\ell \\ &= -\overline{\partial}_{\mathfrak{M}_k} \gamma_\ell. \end{aligned}$$

Using (4.10) of Proposition 4.1 we see that $\overline{\partial}_{\mathfrak{M}_\ell} \gamma_k = \overline{\partial}_{\mathfrak{M}_k} \gamma_\ell$, therefore $\mathsf{b}_{k,\ell} f_k = \mathsf{b}_{k,\ell} f_\ell$, so that by injectivity of $\mathsf{b}_{k,\ell}$ it follows that f_k and f_ℓ are the same function in $\widetilde{\mathcal{A}}^{-\infty}(\Omega)$. This completes the proof of the surjectivity of $\mathsf{bc} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{Y}^{0,1}_\Omega(\mathfrak{M})$, and therefore it is a bijection, since we already know that bc is injective.

Note further that the inverse mapping to bc constructed during the above argument actually maps into $\widetilde{\mathcal{A}}^{-\infty}(\Omega)$. It follows that $\widetilde{\mathcal{A}}^{-\infty}(\Omega) = \mathcal{A}^{-\infty}(\Omega)$.

Finally, both $\mathcal{A}^{-\infty}(\Omega)$ and $\mathcal{Y}^{0,1}_\Omega(\mathfrak{M})$ are DFS space, the latter being a closed subspace of $\mathcal{D}'_{0,1}(\mathfrak{M})$. It follows therefore from Proposition 2.3 that bc is, in fact, an isomorphism of topological vector spaces. This completes the induction, and the proposition (and Theorem 1.4) is proved. □

5.4 Boundary value on the distinguished boundary

For the product domain $\Omega \Subset \mathfrak{M}$ of (1.6), one can also consider boundary values on the *distinguished* or *Šilov* boundary

$$\partial_{\mathrm{III}}\Omega = \partial D_1 \times \cdots \times \partial D_N,$$

which is a smooth submanifold of $\mathfrak{M}$ of codimension N. In this section we give a brief account of such boundary values, omitting the routine proofs. We consider the operator $\mathsf{bc}_{\mathrm{III}}$ from $\mathcal{A}^{-\infty}(\Omega) = \widetilde{\mathcal{A}}^{-\infty}(\Omega)$ to $\mathcal{D}'_{0,N}(\mathfrak{M})$ (currents of bidegree $(0, N)$ on $\mathfrak{M}$) given by

$$\mathsf{bc}_{\mathrm{III}} = \widehat{\bigotimes}_{k=1}^{N} \mathsf{bc}_k = \mathsf{bc}_1 \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathsf{bc}_N,$$

where on the right hand side, $\mathsf{bc}_k : \mathcal{A}^{-\infty}(D_k) \to \mathcal{X}^{0,1}_{D_k}(\mathfrak{M}) \subset \mathcal{D}'_{0,1}(\mathfrak{M})$ is the boundary current operator on the smooth domain D_k. Since for each k, the map $\mathsf{bc}_k : \mathcal{A}^{-\infty}(D_k) \to \mathcal{X}^{0,1}_{D_k}(\mathfrak{M})$ is an isomorphism of topological vector spaces, we conclude that we have an isomorphism

$$\mathsf{bc}_{\mathrm{III}} : \mathcal{A}^{-\infty}(\Omega) \to \mathcal{X}^{0,N}_{\mathrm{III}}(\mathfrak{M}),$$

where

$$\mathcal{X}^{0,N}_{\mathrm{III}}(\mathfrak{M}) = \mathcal{X}^{0,1}_{D_1}(\mathfrak{M}_1) \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathcal{X}^{0,1}_{D_N}(\mathfrak{M}_N) \subset \mathcal{D}'_{0,N}(\mathfrak{M}).$$

Using the definition of the spaces $\mathcal{X}^{0,1}_{D_k}(\mathfrak{M}_k)$ in terms of the Weinstock condition (1.3) and the existence of a face distribution (1.4), we have the following easy consequence:

Proposition 5.5 *Let $\Gamma \in \mathcal{D}'_{0,N}(\mathfrak{M})$. Then there is a holomorphic $f \in \mathcal{A}^{-\infty}(\Omega)$ such that $\Gamma = \mathsf{bc}_{\mathrm{III}} f$ if and only if the following two conditions hold*

(1) *Let $\omega \in \mathcal{D}^{n,n-N}(\mathfrak{M})$ be such that for each $k = 1, \ldots, N$, we have $\overline{\partial}_{\mathfrak{M}_k}\omega = 0$ on $\overline{\Omega}$, where $\overline{\partial}_{\mathfrak{M}_k}$ is as in (4.7). Then we have $\langle \Gamma, \omega \rangle = 0$.*
(2) *There is a distribution $A \in \mathcal{D}'_0(\Gamma)$ such that $\Gamma = I_*(A)^{(0,N)}$, where $I : \partial_{\mathrm{III}}\Omega \to \mathfrak{M}$ is the inclusion map, and the superscript $(0, N)$ denotes taking the part of bidegree $(0, N)$ of the N-current $I_*(A)$.*

Further, from (1.2), we can obtain the following local representation of the boundary value on the Šilov boundary.

Proposition 5.6 *Let $p \in \partial_{\mathrm{III}}\Omega$, let U be a coordinate chart of $\mathfrak{M}$ around p, $U = U_1 \times \cdots \times U_N$, where $U_k \subset \mathfrak{M}_k$, and let v be a vector such that its projection on each $\mathfrak{M}_k$ is transverse to ∂D_k inside U_k. Then, for each $\psi \in \mathcal{D}^{n,n-N}(U)$, we have*

$$\langle \mathsf{bc}_{\mathrm{III}} f, \psi \rangle = \lim_{\epsilon \downarrow 0} \int_{\partial_{\mathrm{III}}\Omega} f_\epsilon \psi,$$

where $f_\epsilon(z) = f(z - \epsilon v)$.

Acknowledgements We thank Christine Laurent-Thiébaut and David Barrett for sharing with us many of the ideas on which this paper is based. We also thank Sagun Chanillo, Evgeny Poletsky, Jean-Pierre Rosay, Mei-Chi Shaw and Emil Straube for interesting discussions on the topic of this paper. Finally, we would like to thank the referee for valuable remarks concerning the paper.

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