Meromorphic Convexity on Stein Manifolds

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ABSTRACT. A compact set $K \in \mathbb{C}^n$ is said to be rationally convex if every point p outside of K admits a holomorphic polynomial whose zero locus passes through p but does not intersect K. There are two main generalizations of this to a general Stein manifold X: one where the polynomials are replaced with entire functions, and another where the zero locus of the polynomial is replaced by a complex hypersurface. We show that the latter is precisely the notion of convexity with respect to meromorphic functions, while the former is precisely the notion of convexity with respect to strong meromorphic functions. Various approximation results and a Duval–Sibony-type theorem are shown for each notion of convexity. Other generalizations of rational convexity to Stein manifolds are discussed.

Given a compact set K in $\mathbb{C}^n$, its rationally convex hull is defined as

$$\mathcal{R}\text{-hull}\, K = \Big\{ z \in \mathbb{C}^n : |R(z)| \le \|R\|_K, \quad \text{for all rational functions } R \text{ with poles off } K \Big\}. \tag{0.1}$$

A compact K is called *rationally convex* if $K = \mathcal{R}\text{-hull}\, K$. This is a standard definition of convexity with respect to a family of functions $\mathcal{F}$, when $\mathcal{F}$ is chosen to be the family of rational functions. Here, K is rationally convex if and only if for any point $p \in \mathbb{C}^n \setminus K$ there exists a complex algebraic hypersurface that passes through p but avoids K.

In this paper, we prove new results concerning generalizations of rational convexity to Stein manifolds. In this setting, the defining family $\mathcal{F}$ can be chosen to be meromorphic or strongly meromorphic functions. As it turns out, this corresponds to convexity with respect to complex hypersurfaces or principal hypersurfaces. This distinction leads to different notions of convexity, which we call *meromorphic* and *strong meromorphic* convexity. In Theorem 2.1, we give general approximation results corresponding to (strong) meromorphic convexity.

While determining whether a given compact is rationally or meromorphically convex may be a difficult problem, there exists a characterization by Duval–Sibony [6] of rational convexity of a special, but important class of compacts: namely, totally real manifolds. This characterization establishes a strong connection between rational convexity and Kähler geometry. In Section 3, we give a generalization of this result for strong meromorphic convexity. In Section 4 we give a sufficient and necessary conditions for a meromorphically convex compact and totally real manifolds to be strongly meromorphically convex. Section 4 generalizes this circle of ideas to subsemigroups G of the Picard group $\mathrm{Pic}(X)$ by showing that a holomorphic function defined on a neighbourhood of a G-meromorphically convex compact (see Definition 5.1) can be approximated uniformly on K by quotients of the "strong" form s_1/s_2, where $s_1, s_2 \in \Gamma(X, L)$ for some $L \in G$.

1. MEROMORPHIC CONVEXITY.

It is easy to see that given a compact $K \subset \mathbb{C}^n$ its rationally convex hull $\mathcal{R}\text{-hull}\, K$, as defined by (0.1), consists of points $z \in \mathbb{C}^n$ with the property that if we have $f(z) = 0$ for some polynomial f, then f vanishes somewhere on K. This means that the complement of $\mathcal{R}\text{-hull}\, K$ is a union of algebraic hypersurfaces. Note that any complex hypersurface in $\mathbb{C}^n$ is principal, that is, is the zero locus of a single entire function (holomorphic on $\mathbb{C}^n$). Any holomorphic function on a rationally convex K can be approximated uniformly on K by rational functions, according to the classical Oka–Weil theorem.

Let now X be a Stein manifold. A natural generalization of rational convexity to X is to replace the family of rational functions with meromorphic functions on X or, as considered by many authors, to replace complex algebraic hypersurfaces simply with complex hypersurfaces. Let us start with meromorphic functions. A general meromorphic function can be defined as follows. Let $\mathcal{M}_p$ be the quotient field of $\mathcal{O}_p$, that is, $\mathcal{M}_p$ is the field of germs of meromorphic functions at a point p of a complex manifold X. A meromorphic function m on X is a map

$$m : X \to \bigcup_{p \in X} \mathcal{M}_p, \quad p \to \mathcal{M}_p,$$

such that $m_p \in \mathcal{M}_p$ for all $p \in X$, and for every $p \in X$ there exist a connected neighbourhood $U \subset X$ and holomorphic functions $f, g \in \mathcal{O}(U)$, $g \not\equiv 0$, such that $m_z = f_z/g_z$ for all $z \in U$. The quotient of two entire functions is clearly a meromorphic function, and on Stein manifolds one may construct global meromorphic functions from compatible local data by solving an additive Cousin problem. A point $p \in X$ is called a point of indeterminacy of a meromorphic function m if $m_p = f_p/g_p$, the germs f_p and g_p are coprime, and $f(p) = g(p) = 0$. The set of all indeterminacy points is called the indeterminacy locus of the meromorphic function m and will be denoted by $\mathcal{I}(m)$.

The Poincaré problem asks whether every global meromorphic function on X is the quotient of entire functions on X. The *strong* Poincaré problem requires that the entire functions of the quotient satisfy an additional property: their germs are relatively prime at every point of X. We will call the latter type of meromorphic function a *strong meromorphic function*. We denote by $\mathcal{M}(X)$ and $\mathcal{SM}(X)$ the spaces of meromorphic and strong meromorphic functions on X, respectively.

On a Stein manifold X the solution to the (weak) Poincaré problem is an immediate consequence of Cartan's Theorem A. The strong Poincaré problem is, however, solvable precisely when the topological condition $H^2(X,\mathbb{Z}) = 0$ is satisfied. Since $H^2(X,\mathbb{Z}) = 0$ implies the universal solvability of the multiplicative Cousin problem on Stein manifolds (they are in fact equivalent), its sufficiency for the solvability of the strong Poincaré problem is clear. On the other hand, its necessity cannot immediately be concluded from classical results, and was shown by Ephraim [7] to be a consequence of Cartan's Theorem B. (For a thorough treatment of this subject, see Fritzsche–Grauert [10].)

Consider also the situation where every (weak) meromorphic function on a Stein manifold X has zero divisor corresponding to a torsion element of $\text{Pic}(X) \cong H^2(X,\mathbb{Z})$. Then, for every $m \in \mathcal{M}(X)$ with zero divisor D there are a positive integer k and an $h \in \mathcal{O}(X)$ for which $\text{div}(h) = kD$, so $h/m^k \in \mathcal{O}(X)$, and hence one sees that $m^k \in \mathcal{SM}(X)$ via the representation $m^k = h/(h/m^k)$. Therefore, in such a situation the hulls defined below will coincide despite $H^2(X,\mathbb{Z}) \neq 0$ in general.

Definition 1.1. Let X be a Stein manifold and K be a compact subset of X. Define

$$\begin{aligned} \mathcal{M}\text{-hull}(K) = \Big\{ & p \in X : |m(p)| \le \|m\|_K \\ & \text{for all } m \in \mathcal{M}(X) \text{ with } \mathcal{I}(m) \cap (K \cup \{p\}) = \emptyset \Big\}, \\ \mathcal{SM}\text{-hull}(K) = \Big\{ & p \in X : |m(p)| \le \|m\|_K \\ & \text{for all } m \in \mathcal{SM}(X) \text{ with } \mathcal{I}(m) \cap (K \cup \{p\}) = \emptyset \Big\}. \end{aligned}$$

We call K *meromorphically convex* (respectively, *strongly meromorphically convex*) if $\mathcal{M}\text{-hull}(K) = K$ (respectively, $\mathcal{SM}\text{-hull}(K) = K$).

For brevity, we also write $\mathcal{M}$-convex (respectively, $\mathcal{SM}$-convex) for meromorphic (respectively, strong meromorphic) convexity. Clearly,

$$K \subset \mathcal{M}\text{-hull}(K) \subset \mathcal{SM}\text{-hull}(K). \tag{1.1}$$

We will see that for a compact K on a Stein manifold X, the sets $\mathcal{M}\text{-hull}(K)$ and $\mathcal{SM}\text{-hull}(K)$ are also compact. Then, by the definition, we have $\mathcal{M}\text{-hull}(K) = \mathcal{M}\text{-hull}(\mathcal{M}\text{-hull}(K))$, and similarly for the strong meromorphically convex hull. If $X = \mathbb{C}^n$, then both definitions agree with rational convexity because entire functions on $\mathbb{C}^n$ can be approximated by polynomials uniformly on compact sets.

As in the case of rational convexity, $\mathcal{M}$- and $\mathcal{SM}$-convexity can be also formulated in terms of complex hypersurfaces. What differentiates the two definitions is whether one requires the hypersurfaces to be principal. More precisely, let K be a compact subset of X, and let us define the following hulls:

$$H(K) = \{x \in X : f^{-1}(0) \cap K \neq \emptyset, \ \forall f \in \mathcal{O}(X) \text{ satisfying } f(x) = 0\},$$
$$h(K) = \Big\{x \in X : \text{every complex hypersurface in } X \text{ passing through } x \text{ intersects } K\Big\}.$$

Clearly, $h(K) \subseteq H(K)$, and it was shown by Colţoiu [3] that these hulls coincide for all $K \subset X$ if and only if $\operatorname{Hom}(H_2(X,\mathbb{Z}),\mathbb{Z}) = 0$, a slightly weaker condition than $H^2(X,\mathbb{Z}) = 0$. The next result shows that the hulls with respect to the families of meromorphic functions and hypersurfaces coincide.

Proposition 1.2. *Let X be a Stein manifold and K be a compact subset of X. Then,*

$$h(K) = \mathcal{M}\text{-hull}(K), \quad H(K) = \mathcal{SM}\text{-hull}(K).$$

In view of the proposition, meromorphic convexity can also be called *convexity with respect to hypersurfaces*.

Proof. Throughout the proof, it will be convenient to view a meromorphic function m on X as a holomorphic map from $X \setminus \mathcal{I}(m)$ into $\mathbb{CP}^1$.

Suppose that $p \notin H(K)$. Then, there exists a $f \in \mathcal{O}(X)$ such that $f(p) = 0$ and whose zero locus avoids K. It follows that $1/f \in \mathcal{SM}(X)$ with $\infty = 1/|f(p)| > \|1/u\|_K$. This shows that $(X \setminus H(K)) \subset (X \setminus \mathcal{SM}\text{-hull}(K))$.

For the opposite inclusion, suppose $p \in X \setminus \mathcal{SM}\text{-hull}(K)$, so there exists a strong meromorphic function f/g with $\mathcal{I}(f/g) \cap (K \cup \{p\}) = \emptyset$ satisfying $|f(p)/g(p)| > \|f/g\|_K$. Set $z := f(p)/g(p)$. If $z = \infty$, then since f and g are coprime, g is the entire function whose zero locus passes through p but avoids K, and so $p \in X \setminus H(K)$. If $z \neq \infty$, then the holomorphic function $f - z \cdot g$ vanishes at p. If $f(q) - z \cdot g(q) = 0$ for some $q \in K$, then either $f(q) = g(q) = 0$ or $f(q)/g(q) = z$ must hold. Since f and g are relatively prime at q, the former case implies that $q \in \mathcal{I}(f/g)$, which is not possible. The latter case contradicts the inequality $|f(p)/g(p)| > \|f/g\|_K$. Therefore, such a $q \in K$ does not exist, and hence $p \notin H(K)$. This completes the proof of the first equality of the proposition.

For the proof of the second identity we use a lemma by Ephraim [7], along with a small modification of the proof, so for convenience we provide these below.

Lemma 1.3. *Let X be a Stein manifold, $\{H_j\}_{j \in J}$ be a locally finite family of irreducible complex hypersurfaces in X, and $\nu_j \geq 0$ be integers. Then, there exists an entire function f that vanishes on H_j to order ν_j for $j \in J$.*

Proof. Let $\mathcal{J}$ be the sheaf of germs of holomorphic functions vanishing on each H_j to order at least ν_j. For each $j \in J$, choose a point $p_j \in H_j$. Then,

$P = \{p_j : j \in J\}$ is a complex analytic subset of X of dimension 0. Let $\mathfrak{n}$ be the sheaf of germs of holomorphic functions vanishing on P—a coherent sheaf of ideals.

If $p \notin P$, then $\mathcal{J}_p = (\mathfrak{n} \cdot \mathcal{J})_p$. On the other hand, by Nakayama's lemma, for $p \in P$ we have $\mathcal{J}_p \neq (\mathfrak{n} \cdot \mathcal{J})_p$. It follows that $(\mathcal{J}/\mathfrak{n} \cdot \mathcal{J})_p \neq 0$ if and only if $p \in P$. Since P is discrete, we may find a section $s \in \Gamma(X, \mathcal{J}/\mathfrak{n} \cdot \mathcal{J})$ for which $s_p \neq 0$ for any $p \in P$.

By Cartan's Theorem B there exists a section $f \in \Gamma(X, \mathcal{J})$ whose image in $\Gamma(X, \mathcal{J}/\mathfrak{n} \cdot \mathcal{J})$ is precisely s. This means that f is a holomorphic function which vanishes to order at least ν_j on H_j for all $j \in J$. But since $s_p \neq 0$ for all $p \in P$, it follows that $f_p \notin (\mathfrak{n} \cdot \mathcal{J})_p$ for all $p \in P$. Taking $p = p_j$, we see that f vanishes to order at most ν_j on H_j. ◻

Note that the function f constructed in the lemma may vanish somewhere outside $\bigcup_j H_j$.

Returning to the proof of the proposition, suppose that $q \notin h(K)$. Then there exists a hypersurface Z passing through q but avoiding K. Without loss of generality, we can assume that Z is irreducible. By Lemma 1.3, there exists an $f \in \mathcal{O}(X)$ whose zero locus contains Z. Further, we can choose q as one of the members of the discrete set P in the proof of the lemma to ensure that there is a neighbourhood U of q for which $U \cap f^{-1}(0) = U \cap Z$. Write $f^{-1}(0) = Z \cup E$, where E is a hypersurface in X that necessarily avoids U. We apply Lemma 1.3 again to find another $g \in \mathcal{O}(X)$ such that $E \subset g^{-1}(0)$ and g has multiplicity zero along Z. By a similar modification of the proof of the lemma, we can ensure that $g \neq 0$ near q. Therefore, $f/g \in \mathcal{M}(X)$, $q \notin \mathcal{I}(f/g)$, and the zero divisor of f/g and Z agree (as sets). It follows that $\infty = |g(q)/f(q)| > \|g/f\|_K$ since f/g has no zeroes on K. This shows that $(X \setminus h(K)) \subset (X \setminus \mathcal{M}\text{-hull}(K))$.

The reverse inclusion is a modification of a standard argument. Suppose that $q \in X$ is a point for which there exists a function

$$\frac{f}{g} \in \mathcal{M}(X) \quad \text{with } \mathcal{I}\left(\frac{f}{g}\right) \cap (K \cup \{q\}) = \emptyset$$

and satisfying $|f(q)/g(q)| > \|f/g\|_K$. Set $z := f(q)/g(q)$. If $z = \infty$, then the zero divisor of g/f is the desired hypersurface. If $z \neq \infty$, then

$$\left\|\frac{f}{g} - z\right\|_K \geq |z| - \left\|\frac{f}{g}\right\|_K > |z| - \left|\frac{f(q)}{g(q)}\right| = 0.$$

Since this inequality is strict, the zero divisor of the meromorphic function $f/g - z$ is the required hypersurface. ◻

Proposition 1.4. *Let X be a Stein manifold and $\Phi : X \to \mathbb{C}^N$ be a proper holomorphic embedding. A compact $K \subset X$ is $\mathcal{SM}$-convex if and only if $\Phi(K)$ is rationally convex in $\mathbb{C}^N$.*

Proof. Suppose K is $\mathcal{SM}$-convex. Applying Cartan's Theorem A to the sheaf of germs of holomorphic functions on $\mathbb{C}^N$ vanishing on the complex analytic set $\Phi(X)$ yields global generators $h_1, \ldots, h_M \in \mathcal{O}(\mathbb{C}^N)$ with the property that $\bigcap_{j=1}^M h_j^{-1}(0) = \Phi(X)$. Accordingly, fix $a \in \mathbb{C}^N \setminus \Phi(K)$. If $a \notin \Phi(X)$, then $h_j(a) \neq 0$ for one of the generators $h_1, \ldots, h_M$, in which case $z \mapsto h_j(z) - h_j(a)$ is a holomorphic function on $\mathbb{C}^N$ which passes through a but avoids $\Phi(K)$. Approximation by Taylor polynomials provides a polynomial with the same property. If $a \in \Phi(X) \setminus \Phi(K)$, then the strong meromorphic convexity of K (and Proposition 1.2) yields a $f \in \mathcal{O}(X)$ with $f(\Phi^{-1}(a)) = 0$ but $f^{-1}(0) \cap K = \emptyset$. Applying the Oka-Cartan extension theorem to $f \circ \Phi^{-1} \in \mathcal{O}(\Phi(X))$ provides an entire function F with the property that $F(a) = 0$ but $F^{-1}(0) \cap \Phi(K) = \emptyset$. This shows that $\Phi(K)$ is rationally convex.

Conversely, if $\Phi(K)$ is rationally convex, then for any $a \in \Phi(X) \setminus \Phi(K)$ there exists a holomorphic polynomial on $\mathbb{C}^N$ that vanishes at a but does not vanish on $\Phi(K)$. Its restriction to $\Phi(X)$ is an entire function that defines a principal hypersurface through a that avoids $\Phi(K)$. It follows from Proposition 1.2 that K is $\mathcal{SM}$-convex. □

The rationally convex hull of any compact in $\mathbb{C}^N$ is compact, and so it follows from Proposition 1.4 that $\mathcal{SM}$-hull(K) is compact for any compact $K \subset X$. If $p \notin \mathcal{M}$-hull(K) and h is a hypersurface on X passing through p and avoiding K, then h can be perturbed (see Example 2.3 for details) so that it passes through any point in a small neighbourhood of p in X still avoiding K. This shows that $\mathcal{M}$-hull(K) is a closed set, and so in view of (1.1), it is compact. A different proof of this is given in [14].

2. MEROMORPHIC APPROXIMATION

We now discuss approximation on (strongly) meromorphically convex compacts. Rossi [16, Theorem 3.4] already proved a version of the Oka–Weil theorem for compacts on a Stein manifold that are meromorphically convex: *any holomorphic function defined on a neighbourhood of a compact subset K of a Stein manifold with $\mathcal{M}$-hull$(K) = K$ is the uniform limit on K of a sequence of meromorphic functions without poles on K.* On the other hand, Hirschowitz [14, Theorem 2] showed the same conclusion holds if one replaces the meromorphically convex compact with a hypersurface convex compact. These two notions of convexity are the same (Proposition 1.2), so it may appear that meromorphic/hypersurface convexity is the proper analogue of rational convexity to Stein manifolds. However, our next result shows that the desired notion of convexity depends on the class of meromorphic functions by which one wishes to approximate. More precisely, the following holds.

Theorem 2.1. *Let X be a Stein manifold and K be a strongly meromorphically convex subset of X. For any $\varphi \in \mathcal{O}(K)$ and $\varepsilon > 0$ there exist $f, g \in \mathcal{O}(X)$ that are pointwise relatively prime at each point of X and satisfy $\|\varphi - f/g\| < \varepsilon$.*

The conclusion $\|\varphi - f/g\|_K < \varepsilon$ implies in particular that f/g has no poles on K.

Proof of Theorem 2.1. Our proof uses the methods of Hirschowitz [14]. Let K be a meromorphically compact subset of X, and let f be a holomorphic function defined on a neighbourhood U of K. Define the compact set $L := \hat{K}_X \setminus U$, where $\hat{K}_X$ denotes the holomorphically convex hull of K in X. If H is a principal hypersurface in X, then $X \setminus H$ is Stein, and if H avoids K, then $\hat{K}_{X\setminus H}$ is compact in $X \setminus H$. Therefore, the set $L_H = L \cap \hat{K}_{X\setminus H}$ is compact, and $\bigcap_{K\cap H=\emptyset} L_H = \emptyset$. Since L is compact, we can find finitely many principal hypersurfaces $H_1, \dots, H_k$ avoiding K and

$$L \cap \hat{K}_{X\setminus H_1} \cap \cdots \cap \hat{K}_{X\setminus H_k} = \emptyset.$$

Writing $H := \bigcup_{j=1}^k H_j$ we see that $\hat{K}_{X\setminus H} \subset U$, and hence by the classical Oka–Weil theorem [8, Theorem 18] we may approximate f uniformly on $\hat{K}_{X\setminus H}$ by members of $\mathcal{O}(X \setminus H)$.

It suffices now to approximate functions in $\mathcal{O}(X\setminus H)$ by strongly meromorphic functions. To do this, choose a proper holomorphic embedding $\Phi : X \to \mathbb{C}^N$ for some large N, which exists, since X is Stein. The hypersurface H is principal, since the H_j are, and so $H = h^{-1}(0)$ for some $h \in \mathcal{O}(X)$. Then, $\Psi := (\Phi, h)$ embeds $X \setminus H$ into $\mathbb{C}^{N+1} \setminus \{z_{N+1} = 0\}$. Let $f \in \mathcal{O}(X \setminus H)$. The Oka–Cartan theorem allows us to extend the function $f \circ \Psi^{-1} : \Psi(X \setminus H) \to \mathbb{C}$ to a function $F : \mathbb{C}^{N+1} \setminus \{z_{N+1} = 0\} \to \mathbb{C}$, which in turn may be approximated uniformly on compacts by partial sums of its Laurent series expansion with respect to z_{N+1}:

$$F(z', z_{N+1}) = \sum_{k\in\mathbb{Z}} a_k(z') z_{N+1}^k,$$

where the a_k are entire functions of the first N variables—in fact by taking an appropriate Taylor polynomial, it can be assumed that the a_k are polynomials. Taking a partial sum of the above series and precomposing with Ψ yields normal approximation of f by meromorphic functions of the form

$$\begin{aligned}\sum_{k=-m}^{m} a_k(\Phi)h^k &= \frac{a_{-m}(\Phi)}{h^m} + \frac{a_{-m+1}(\Phi)}{h^{m-1}} + \cdots + a_m(\Phi)h^m \\ &= \frac{a_{-m}(\Phi) + a_{-m+1}(\Phi)h + \cdots + a_m(\Phi)h^{2m}}{h^m}.\end{aligned}$$

The meromorphic function above is strong if the polynomial a_{-m} is not identically zero on any irreducible component of the complex-analytic set $\Phi(h)$. Since the set of polynomials in $\mathbb{C}^N$ satisfying this property is dense in the space of all polynomials, taking a small perturbation of a_{-m} if necessary ensures that the approximating meromorphic functions are strong. ∎

Remark 2.2. See Theorem 5.2 for a generalization of Theorem 2.1 involving sections of holomorphic line bundles.

A natural question is whether approximation by strong meromorphic functions is possible on compacts that are only meromorphically convex. The following example shows there exist (weakly) meromorphically convex compact sets $K \subset X$ admitting holomorphic functions which cannot be approximated by *strong* meromorphic functions.

Example 2.3. Let X be a Stein manifold for which there exists a compact K satisfying

$$\mathcal{M}\text{-hull}(K) \neq \mathcal{SM}\text{-hull}(K).$$

Such a manifold is known to exist due to the work of Colţoiu [3]. Fix a point $p \in \mathcal{SM}\text{-hull}(K) \setminus \mathcal{M}\text{-hull}(K)$. We claim that $\tilde{K} := \mathcal{M}\text{-hull}(K) \cup \{p\}$ is meromorphically convex. Indeed, let $q \notin \tilde{K}$. Then, there exists a hypersurface Z which passes through q but avoids K. Suppose Z passes through p. There is a small perturbation of Z that passes through q but avoids $\tilde{K}$. Indeed, let L be the holomorphic line bundle corresponding to Z; then, there exists a holomorphic section $s : X \to L$ whose zero locus is precisely Z. It is well known that any line bundle L admits a bundle embedding into the trivial bundle $X \times \mathbb{C}^N$ for some $N > 0$ (see, e.g., [9, Corollary 7.3.2]). Treating s as a map into $\mathbb{C}^N$, we may find a map $\tilde{s} : X \to \mathbb{C}^N$, which is a small perturbation of s and $\tilde{s}(q) \neq 0$ and $\tilde{s}^{-1}(0) \cap K = \emptyset$. Then, the projection of $\tilde{s}$ to L is a holomorphic section of L whose zero locus avoids $\tilde{K}$. This shows that $\mathcal{M}\text{-hull}(\tilde{K}) = \tilde{K}$ as claimed.

Suppose that any holomorphic function defined on a neighbourhood of $\tilde{K}$ is the uniform limit on $\tilde{K}$ of a sequence of strong meromorphic functions. Consider the function

$$\psi(z) = \begin{cases} 0, & \text{when } z \in K, \\ 1, & \text{when } z = p. \end{cases}$$

By trivial extension, we may consider ψ as a holomorphic function defined in a neighbourhood of $\tilde{K}$. By our assumption, given $\varepsilon > 0$ there is a $f/g \in \mathcal{SM}(X)$ without poles on $\tilde{K}$ such that $\|\psi - (f/g)\|_{\tilde{K}} < \varepsilon$, in particular,

$$\left| \frac{1 - f(p)}{g(p)} \right| < \varepsilon.$$

On the other hand, since $p \in \mathcal{SM}\text{-hull}(\tilde{K})$, we have

$$\left| \frac{f(p)}{g(p)} \right| \leq \left\| \frac{f}{g} \right\|_K < \varepsilon.$$

This is a contradiction for small $\varepsilon > 0$.

In the case of $X = \mathbb{C}^n$, a partial converse to Oka–Weil is known (Theorem 1.2.10 in [18]): *A compact $K \subset \mathbb{C}^n$ is rationally convex whenever any continuous function on K is the uniform limit on K of rational functions with poles off K.* We will now show the analogues for strong and weak meromorphic convexity to hold. This is where the equivalence of meromorphic and hypersurface convexity plays an important role.

For a compact $K \subset X$ denote by $\mathcal{M}(K)$ (respectively, $S\mathcal{M}(K)$) the uniformly closed subalgebra of $C(K)$ that consists of all the functions that can be approximated uniformly on K by meromorphic (respectively, strongly meromorphic) functions with poles off K.

Theorem 2.4. *A compact subset K of a Stein manifold X is meromorphically convex if $\mathcal{M}(K) = C(K)$. It is strongly meromorphically convex if $S\mathcal{M}(K) = C(K)$.*

Proof. We follow Stout [18]. First, note that every $m \in \mathcal{M}(K)$ has a natural extension to the compact set $\mathcal{M}$-hull(K). Indeed, if $\{m_k\}_{k=1}^{\infty}$ is a sequence of meromorphic functions with poles off K converging to m uniformly on K, then $\{m_k\}_{k=1}^{\infty}$ is in fact Cauchy and hence convergent at any point $a \in \mathcal{M}$-hull(K), since $|m_k(a) - m_\ell(a)| \leq \|m_k - m_\ell\|_K$ for $k, \ell \in \mathbb{N}$. We denote by

$$\hat{m} \in \mathcal{M}(\mathcal{M}\text{-hull}(K))$$

this extension of m to $\mathcal{M}$-hull(K). This then yields a natural identification of $\mathcal{M}(\mathcal{M}\text{-hull}(K))$ with $\mathcal{M}(K)$.

Seeking a contradiction, suppose K is not meromorphically convex, and choose $a \in \mathcal{M}$-hull$(K) \setminus K$. Then, the $\mathbb{C}$-linear functional T on $\mathcal{M}(K)$ defined by $m \mapsto \hat{m}(a)$ satisfies $T(u \cdot v) = T(u)T(v)$ for each $u, v \in \mathcal{M}(K)$. Since $\mathcal{M}(K) = C(K)$ by assumption, T is also a $\mathbb{C}$-linear functional on $C(K)$ satisfying $T(f \cdot g) = T(f)T(g)$ for all $f, g \in C(K)$. It is well known [18, Theorem 1.2.8] that all such functionals on $C(K)$ can be realized as evaluation functionals at a unique point of K, that is, there exists a unique point $z \in K$ such that $T(f) = f(z)$ for all $f \in C(K)$. But this means that $T(m) = m(a) = m(z)$ for all $m \in \mathcal{M}(Mhull(K))$, and since the algebra $\mathcal{M}(\mathcal{M}\text{-hull}(K))$ separates points (X is Stein), this is a contradiction.

The case of $C(K) = S\mathcal{M}(K)$ is an identical argument. □

3. Duval–Sibony for Strong Meromorphic Convexity

Recall that a submanifold S of a complex manifold X is *totally real* if for every $x \in S$ the tangent space T_xS contains no complex line. Duval and Sibony [6, Theorem 3.1] proved the following striking result: *a smooth compact totally real submanifold $S \subset \mathbb{C}^n$ is rationally convex if and only if there exists a smooth strictly plurisubharmonic function φ on $\mathbb{C}^n$ such that $\iota_S^* dd^c \varphi = 0$, where $\iota_S : S \to \mathbb{C}^n$ is the inclusion map.* Note that $\omega := dd^c \varphi$ is a Kähler form on $\mathbb{C}^n$.

We say that a Kähler form ω on a complex manifold X is a *Hodge form* if $[\omega] \in H^2(X, \mathbb{Z})$, that is, $[\omega] \in H^2_{\mathrm{dR}}(X, \mathbb{R})$ lies in the image of the morphism

$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}) \cong H^2_{\mathrm{dR}}(X,\mathbb{R})$ induced by the containment $\mathbb{Z} \hookrightarrow \mathbb{R}$. Guedj [13] further generalized the theorem of Duval–Sibony to the context of complex projective manifolds and Stein manifolds. The generalization to Stein manifolds can be stated using the terminology of the present work as follows.

Theorem 3.1 (Guedj [13, Theorem 5.8]***).*** *Let S be a smooth compact totally real submanifold of a Stein manifold X. The following are equivalent:*

(i) *S is meromorphically convex.*
(ii) *There exists a smooth Hodge form ω for X such that $\iota_S^*\omega = 0$.*

A natural question in the context of this note is whether such a characterization of *strongly* meromorphic compact totally real manifolds exists. We have the following result.

Theorem 3.2. *Let S be a smooth compact totally real submanifold of a Stein manifold X. The following are equivalent:*

(i) *S is strongly meromorphically convex.*
(ii) *There exists a smooth strictly plurisubharmonic function φ on X such that $\iota_S^*\mathrm{dd}^c\varphi = 0$.*

Observe that a Kähler form ω has a dd^c-potential—as in condition (ii) of Theorem 3.2—if and only if $[\omega] = 0 \in H^2(X,\mathbb{Z})$. Indeed, it is clear that $[\mathrm{dd}^c\varphi] = 0 \in H^2(X,\mathbb{Z})$ whenever $\varphi \in \mathrm{PSH}(X)$. Conversely, if ω is a Hodge form, then there exist a holomorphic line bundle $L \to X$ and a metric ψ on L such that $\mathrm{dd}^c\psi = \omega$ [5, Theorem 13.9 (b)]. If we know further that $[\omega] = 0 \in H^2(X,\mathbb{Z})$, then L has Chern class zero and hence is isomorphic to the trivial bundle on X. It follows that ψ can be realized as a global strongly plurisubharmonic function on X. This is the case, for example, when $X = \mathbb{C}^n$: in this setting Theorems 3.1 and 3.2 are both reduced precisely to the statement of Duval–Sibony.

Our proof of Theorem 3.2 is done through the embedding of the Stein manifold X into a complex Euclidean space, which allows us to circumvent the more sophisticated methods of Guedj [13] and use the theorem of Duval–Sibony [6] directly. We require a variation of some known results on compact Kähler manifolds; these are formulated in the lemma below, the proof of which will be given at the end of this section.

Lemma 3.3 (***cf.*** [4, Proposition 2.1; 15, Theorem 4.1; 17, Theorem 4]). *Let X be a Stein manifold and $Z \subset X$ be a closed complex submanifold equipped with a Hodge form ω. If there exists a Kähler form η on X with $[\iota_Z^*\eta] = [\omega] \in H^2(Z,\mathbb{Z})$, then ω admits an extension off of any prescribed relatively compact subset of Z to a Kähler form on X; that is, for any compact $B \Subset Z$, there exists a Kähler form $\tilde{\omega}$ on X such that $\iota_B^*\tilde{\omega} = \omega$.*

Proof of Theorem 3.2. First, we properly embed X into some Euclidean space $\mathbb{C}^N$ via the holomorphic mapping $\Phi : X \to \mathbb{C}^N$.

Suppose S is $S\mathcal{M}$-convex. Then, by Proposition 1.4, $\Phi(S)$ is rationally convex in $\mathbb{C}^N$. In view of the result of Duval–Sibony cited at the beginning of the section, there is a $\psi \in \mathrm{PSH}(\mathbb{C}^N)$ with $\iota^*_{\Phi(S)} dd^c \psi = 0$, and hence $\varphi := \psi \circ \Phi \in \mathrm{PSH}(X)$ satisfies $\iota^*_S dd^c \varphi = 0$.

Conversely, suppose there exists a smooth strongly plurisubharmonic function φ on X such that $\iota^*_S dd^c \varphi = 0$. Since

$$[dd^c(\varphi \circ \Phi^{-1})] = [\iota^*_X dd^c(|\cdot|^2)] = 0 \in H^2(X, \mathbb{Z}),$$

by Lemma 3.3, $dd^c(\varphi \circ \Phi^{-1})$ admits an extension ω off of some large ball containing $\Phi(S)$ to all of $\mathbb{C}^N$ as a Kähler form. Because $\mathbb{C}^N$ is topologically trivial, there exists a strictly plurisubharmonic ψ on $\mathbb{C}^N$ with $dd^c \psi = \omega$. Since

$$\iota^*_{\Phi(S)} dd^c \psi = \iota^*_{\Phi(S)} dd^c \omega = \iota^*_S dd^c \varphi = 0,$$

applying Duval–Sibony in the other direction shows that $\Phi(S)$ is rationally convex. It follows from Proposition 1.4 that S is strongly meromorphically convex. ◻

Proof of Lemma 3.3. Let $\psi \in C^\infty(X)$ be a strictly plurisubharmonic exhaustion function for X. Without loss of generality we can assume $B = B_1 \cap Z$, where $B_1 = \{z \in X : \psi(z) < c_1\}$ for some $c_1 > 0$. Choose $c_3 > c_2 > c_1$ so that their respective sublevel sets $B_3 := \{\psi < c_3\}$ and $B_2 := \{\psi < c_2\}$ satisfy $B_1 \Subset B_2 \Subset B_3 \Subset X$.

Since $[\iota^*_Z] = [\omega]$ and ω is in particular Kähler, there exist a $\varphi \in C^\infty(Z)$ and an $\varepsilon > 0$ such that

$$\iota^*_Z \eta + dd^c \varphi = \omega \geq \varepsilon \cdot \iota^*_Z \eta.$$

We now proceed as in Coman–Guedj–Zeriahi [4]: choose φ_1 to be any extension of φ to X, and define

$$\varphi_2 = \varphi_1 + A\chi \, \mathrm{dist}(\cdot, Z)^2,$$

where $\chi \in C^\infty(X)$ is a cutoff function supported in a small neighbourhood of Z that is identically one near Z, and $A > 0$. Here, the distance function can be any Riemannian distance on X (e.g., the distance associated with the Kähler metric η). Now, φ_2 is another smooth extension of φ to X, and by choosing A large enough we can ensure that

$$\eta + dd^c \varphi_2 \geq \frac{\varepsilon}{2} \eta \quad \text{on } B_3.$$

Define $u = \chi \log(\mathrm{dist}(\cdot, X)^2)$; by shrinking the support of χ (and consequently increasing $A > 0$ if necessary), we can ensure that the function $\log(\mathrm{dist}(\cdot, X))^2$ is well defined and quasi-plurisubharmonic on $\mathrm{supp}(\chi)$. Hence, there is a small $\delta > 0$ such that $\delta \cdot dd^c u \geq -\eta$ on B_3.

We define one more smooth extension of φ:

$$\varphi_3 = \frac{1}{2} \log(e^{2\varphi_2} + e^{\delta u + C}).$$

A standard calculation at points of B_3 yields

$$\begin{aligned}\eta + dd^c\varphi_3 &\geq \eta + \frac{2e^{2\varphi_2}dd^c\varphi_2 + \delta e^{\delta u+C}dd^c u}{2(e^{2\varphi_2} + e^{\delta u+C})} \\ &= \frac{2e^{2\varphi_2}(\eta + dd^c\varphi_2) + e^{\delta u+C}(\eta + \delta dd^c u)}{2(e^{2\varphi_2} + e^{\delta u+C})} \\ &\geq \frac{\varepsilon e^{2\varphi_2}}{2(e^{2\varphi_2} + e^{\delta u+C})}\eta \geq \frac{\varepsilon}{4}\eta.\end{aligned}$$

It follows that $\eta + dd^c\varphi_3$ is a Kähler form on B_3 which extends ω off B.

To complete the proof we will modify this form off of B_1 so that it is Kähler on all of X, through a standard procedure. Choose a smooth function $h : \mathbb{R} \to [0, \infty)$ that is constant for $t \leq c_1$, is strictly convex and increasing for $t \in (c_1, c_2)$, and $h(t) = t$ for $t \geq c_3$. Then, $h \circ \psi$ is plurisubharmonic on X, strictly plurisubharmonic outside $\bar{B}_1$, and vanishes on $\bar{B}_1$. Next, choose a smooth function $\lambda : \mathbb{R} \to [0, 1]$ that is identically one for $t \leq c_2$ and identically zero for $t > c_3$. The form

$$\tilde{\omega} := \eta + dd^c((\lambda \circ \psi)\varphi_3) + C' \cdot dd^c(h \circ \psi)$$

extends $\iota_B^*\omega$, and is Kähler for $C' > 0$ large enough. ◻

4. CHARACTERIZATION OF STRONG MEROMORPHIC CONVEXITY

As mentioned above, Colţoiu [3] showed that if a Stein manifold X satisfies $\mathrm{Hom}(H_2(X;\mathbb{Z});\mathbb{Z}) \neq 0$, then there exist compacts $K \subset X$ for which $\mathcal{SM}\text{-hull}(K) \neq \mathcal{M}\text{-hull}(K)$. In this context, a natural question is the following: *when do* $\mathcal{M}$-hull(K) *and* $\mathcal{SM}$-hull(K) *coincide for a* given compact K? To formulate our results we first introduce some terminology.

Definition 4.1. A domain Ω on a manifold X is called meromorphically Runge if $\mathcal{M}$-hull(K) is compact in Ω for every compact subset $K \subset \Omega$, and is called strongly meromorphically Runge if $\mathcal{SM}$-hull$(K))$ is compact in Ω for every compact subset $K \subset \Omega$. Here, the respective hulls are taken with respect to the ambient manifold X.

We first consider the case when $K = S$ is a meromorphically convex totally real manifold on X. Recall that by Theorem 3.1 there exists a smooth Hodge form ω on X such that $\iota_S^*\omega = 0$. Further, there exist a holomorphic line bundle $L \to X$ and a metric φ on L such that $\iota_S^* dd^c\varphi = 0$.

Theorem 4.2. *Let S be a smooth compact totally real submanifold of a Stein manifold X that is meromorphically convex. Then, S is strongly meromorphically convex if and only if there exist a Stein neighbourhood U of S that is strongly meromorphically Runge and an integer $k > 0$ such that $L^{\otimes k}|_U$ is trivial.*

Proof. Suppose there exist a Stein neighbourhood U of S which is strongly meromorphically Runge and an integer k such that the line bundle $L^{\otimes k}|_U$ is trivial.

Then, $k\varphi|_U$ is a strictly plurisubharmonic function, so by Theorem 3.2 we see that S is convex with respect to strong meromorphic functions on U. Since U is strongly meromorphically Runge, we at least have $\mathcal{SM}$-hull$(S) \subset U$. We claim that to prove that $S = \mathcal{SM}$-hull(S) it suffices to find for every $a \in U \setminus S$ a function $g \in \mathcal{O}(X)$ whose zero locus passes through a but avoids S. Indeed, if this holds, then the compact $\mathcal{SM}$-hull(S) contains S as a connected component. A simple argument using Theorem 2.1 shows that a connected component of a $\mathcal{SM}$-convex compact is $\mathcal{SM}$-convex, and this proves the claim.

Accordingly, fix $a \in U$. Then, there exists a $f \in \mathcal{O}(U)$ such that $f(a) = 0$ and whose zero set avoids S. Since U is strongly meromorphically Runge, f can be approximated normally in U by members of $\mathcal{SM}(X)$ with poles outside a large compact of U (Theorem 2.1), so for $u/v \in \mathcal{SM}(X)$ sufficiently close to f on a neighbourhood of $S \cup \{a\}$, the function $z \mapsto u(z) - v(z)(u(a)/v(a))$ is holomorphic on X with a zero at $z = a$ and has zero locus omitting S. This proves that S is strongly meromorphically convex.

Conversely, suppose that $S = \mathcal{SM}$-hull(S). Since S is totally real, the square-distance function to S—that is, $\mathrm{dist}^2(x, S)$—is strictly plurisubharmonic in a small tubular neighbourhood U of S. Set $\rho := \mathrm{dist}^2(\cdot, S)|_U$, and note that $S = \{x \in U : \rho(x) \le 0\}$. Because $S = \mathcal{SM}$-hull(S), Boudreaux–Gupta–Shafikov [2, Theorem 1.2] shows there exists an extension of $dd^c\rho$ to a Kähler form ω on X with $[\omega] = [0] \in H^2(X, \mathbb{Z})$. Applying the converse implication of the same theorem, we see that neighbourhoods of the form $U_\varepsilon := \{x \in U : \rho(x) < \varepsilon\}$ are Stein and have closures that are convex with respect to strong meromorphic functions as well. It follows that U_ε is strongly meromorphically Runge for $\varepsilon > 0$ small enough.

We lastly must show that $L^{\otimes k}|_{U_\varepsilon}$, the line bundle $L \to X$ given to us by Theorem 3.1 (see the paragraph above the statement of Theorem 4.2), is trivial. Fix such a small $\varepsilon > 0$. By Sard's theorem, we may assume that U_ε has smooth boundary and hence has finitely generated cohomology groups. We will show that $L^{\otimes k}|_{U_\varepsilon}$ is trivial for some positive integer k. The neighbourhood U_ε is a deformation retract of S, and so in particular we have $H^2_{\mathrm{dR}}(S, \mathbb{R}) \cong H^2_{\mathrm{dR}}(U_\varepsilon, \mathbb{R})$. But

$$c_1(L|_S) = [\iota_S^* dd^c\varphi] = [0] \in H^2_{\mathrm{dR}}(S, \mathbb{R}),$$

so $c_1(L|_{U_\varepsilon}) = [0] \in H^2_{\mathrm{dR}}(U_\varepsilon, \mathbb{R})$ as well; that is, the image of the first Chern class of L in $H^2_{\mathrm{dR}}(U_\varepsilon, \mathbb{R}) \cong H^2(U_\varepsilon, \mathbb{R})$ through the morphism $H^2(U_\varepsilon, \mathbb{Z}) \to H^2(U_\varepsilon, \mathbb{R})$ induced by the containment $\mathbb{Z} \hookrightarrow \mathbb{R}$ is zero. Since the kernel of this morphism is precisely the torsion subgroup of $H^2(U_\varepsilon, \mathbb{Z})$, there exists an integer k so that $c_1(L^k|_{U_\varepsilon}) = [0] \in H^2(U_\varepsilon, \mathbb{Z})$. Furthermore, U_ε is Stein, so the operator $c_1 : \mathrm{Pic}(U_\varepsilon) \to H^2(U_\varepsilon, \mathbb{Z})$ is an isomorphism, and we can conclude that $L^k|_{U_\varepsilon}$ is trivial. □

Next, we formulate a characterization of strong meromorphic convexity for arbitrary compacts.

Theorem 4.3. *Let K be a meromorphically convex compact in a Stein manifold X. Then, K is strongly meromorphically convex if and only if it admits a strongly meromorphically Runge Stein neighbourhood U with the property that, for every $a \in U \setminus K$, there exist a line bundle L in the torsion subgroup of* $\mathrm{Pic}(U)$ *and a section $\sigma \in \Gamma(U, L)$ such that $\sigma(a) = 0$ but $\sigma^{-1}(0) \cap K = \emptyset$.*

Proof. Suppose that K admits such a neighbourhood. Since U is strongly meromorphically Stein, we at least have $\mathcal{SM}$-hull$(K) \subset U$, and so it suffices to show that for every $a \in U \setminus K$ there exists a $f \in \mathcal{O}(X)$ with $f(a) = 0$ but $f^{-1}(0) \cap K = \varnothing$. Given $a \in U \setminus K$, by assumption we know there are a line bundle L in the torsion subgroup of $\mathrm{Pic}(U)$ and a section $\sigma \in \Gamma(U, L)$ such that $\sigma(a) = 0$ and $\sigma^{-1}(0) \cap K = \emptyset$. This means there is an integer k such that $\sigma^k \in \mathcal{O}(U)$ that can then be approximated normally on U by members of $\mathcal{SM}(X)$ with poles outside of some large compact in U. We can now proceed as in the proof of the previous theorem: choosing a $u/v \in \mathcal{SM}(X)$ that approximates σ^k close enough on a neighbourhood of the compact $K \cup \{a\}$, we see that

$$z \mapsto u(z) - v(z)\frac{u(a)}{v(a)}$$

is a member of $\mathcal{O}(X)$ with the zero set that passes through a but avoids K.

The converse is trivial: if K is strongly meromorphically convex, then every neighbourhood of K has this property, since the trivial line bundle will then satisfy the hypotheses. Thus, choose U to be any (possibly large) strongly meromorphically Runge neighbourhood of K. □

5. FURTHER GENERALIZATIONS

Upon noticing that every hypersurface of X can be realized as the zero set of a global holomorphic section of some holomorphic line bundle $L \to X$, one might be drawn to consider a notion of convexity with respect to global holomorphic sections of the *fixed* line bundle L. However, note that if the zero set of a section $s \in \Gamma(X, L)$ avoids a compact $K \subset X$, then so does the zero set of the section $s^M \in \Gamma(X, L^{\otimes M})$ for any positive integer M. Therefore, a more appropriate notion of convexity of this type is to consider convexity with respect to the subgroup $\langle L \rangle \leq \mathrm{Pic}(X)$ generated by L. Here, we use the notation "$G_1 \leq G_2$" to indicate G_1 is a sub(semi)group of G_2. Recall that a semigroup G is a nonempty set equipped with an associative binary operation.

In this vein, Abe [1] considered more generally convexity with respect to subsemigroups of $\mathrm{Pic}(X)$, defined as follows.

Definition 5.1. Let X be a Stein manifold containing a compact K, and G be a subsemigroup of $\mathrm{Pic}(X)$. Define

$$G\text{-hull}(K) = \Big\{x \in X : s^{-1}(0) \cap K \neq \emptyset \quad \text{for every } L \in G \text{ and } s \in \Gamma(X, L) \text{ satisfying } s(x) = 0\Big\}.$$

We call K *meromorphically convex with respect to* G, or simply, G*-meromorphically convex*, if $G\text{-hull}(K) = K$.

It is clear that $G_2\text{-hull}(K) \subseteq G_1\text{-hull}(K)$ whenever $G_1 \leq G_2$. Furthermore, it is known that $G\text{-hull}(K)$ is compact in X for any $G \leq \mathrm{Pic}(X)$ (see Proposition 4.1 and Corollary 4.5 in [1]). It is immediate from the definitions that $\langle 1\rangle\text{-hull}(K) = H(K)$ and $\mathrm{Pic}(X)\text{-hull}(K) = h(K)$, where $1 \in \mathrm{Pic}(X)$ denotes the trivial line bundle.

Convexity with respect to a subsemigroup $G \leq \mathrm{Pic}(X)$ is of interest in view of the following generalization of the Oka–Weil theorem (see [1, Theorem 5.1]): *If K is a compact subset of a Stein manifold with G-hull$(K) = K$, then for every $f \in \mathcal{O}(K)$ and $\varepsilon > 0$ there exist $L \in G$ and $s_1, s_2 \in \Gamma(X, L)$ such that*

$$\left\| f - \frac{s_1}{s_2} \right\|_K < \varepsilon.$$

However, this statement alone undervalues Abe's work. Indeed, a compact K with $G\text{-hull}(K) = K$ is in particular meromorphically convex, so as mentioned above [14, Theorem 2] any $f \in \mathcal{O}(K)$ can be approximated uniformly on K by members of $\mathcal{M}(X)$ with poles off K. Furthermore, no control over the $L \in G$ from which the approximating meromorphic functions s_1/s_2 are built is given, so one is left to wonder why approximation by quotients of sections of L would be preferred over approximation by quotients of holomorphic functions (which are sections of the trivial bundle).

For a given $L \in \mathrm{Pic}(X)$, we say that two sections $s_1, s_2 \in \Gamma(X, L)$ are *coprime* if their zero loci share no irreducible components. With this in mind, we give the following strengthening of Abe's result.

Theorem 5.2. *Let X be a Stein manifold and G a subsemigroup of* $\mathrm{Pic}(X)$. *Let K be a compact set of X such that G-hull$(K) = K$. Then, for every $f \in \mathcal{O}(K)$ and for every $\varepsilon > 0$, there exist $L \in G$ and coprime $s_1, s_2 \in \Gamma(X, L)$ such that $\|f - s_1/s_2\|_K < \varepsilon$.*

Remarks 5.3.

(i) Meromorphic functions of the form s_1/s_2 for coprime $s_1, s_2 \in \Gamma(X, L)$ are natural generalizations of strong meromorphic functions to sections of a line bundle. Furthermore, note that every $m \in \mathcal{M}(X)$ can be written in strong form with respect to some line bundle. Indeed, the zero divisor of m can be realized as $\mathrm{div}(s)$ for some global section s of some $L \in \mathrm{Pic}(X)$. Write $s \cong \{s_i\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open cover of X by trivializations of L with associated transition functions $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. Then, we have

$$\frac{s_i}{m|_{U_i}} = g_{ij} \frac{s_j}{m|_{U_j}}$$

on $U_i \cap U_j$, and it follows that $\{s_i/m|_{U_i}\}_{i\in I}$ patches together to form a global section of L. Hence, $m = s/(s/m)$ is a strong representation of m in terms of sections of L.

(ii) Since (weak) meromorphic convexity is equivalent to convexity with respect to the entire Picard group, a consequence of Theorem 5.2 is the following: *Any holomorphic function defined in a neighbourhood of a meromorphically convex compact $K \subset X$ is the uniform limit on K of a sequence of meromorphic functions having the form s_1/s_2 for* coprime $s_1, s_2 \in \Gamma(X, L)$, *where $L \in \mathrm{Pic}(X)$ only depends on f.* This may have utility in situations where it is more desirable that the meromorphic functions by which one wishes to approximate be quotients of objects which are coprime rather than be quotients of functions themselves.

Proof of Theorem 5.2. Let U be an open subset of K on which f is defined. Define the compact set $M := \hat{K}_X \setminus U$, where $\hat{K}_X$ denotes the holomorphically convex hull of K in X. As in the proof of Theorem 2.1, for any $s \in \Gamma(X, L)$, $L \in G$, with zero locus avoiding K, the set $X \setminus s^{-1}(0)$ is Stein; in particular, $\hat{K}_{X\setminus s^{-1}(0)} \subset X \setminus s^{-1}(0)$. The set $M_s := M \cap \hat{K}_{X\setminus s^{-1}(0)}$ is compact, and so $\bigcap_s M_s = \emptyset$, where the intersection is taken over all such s, implies there are finitely many $s_j \in \Gamma(X, L_j)$, $L_j \in G$, $j = 1, \ldots, k$, with

$$M \cap \hat{K}_{X\setminus s_1^{-1}(0)} \cap \cdots \cap \hat{K}_{X\setminus s_k^{-1}(0)} = \emptyset.$$

Consequently, the section $s := \prod_{j=1}^k s_j$ is a global holomorphic section of

$$L_1 \otimes \cdots \otimes L_k \in G$$

that has $\hat{K}_{X\setminus s^{-1}(0)} \subset U$, and hence f may be approximated uniformly on $\hat{K}_{X\setminus s^{-1}(0)}$ by members of $\mathcal{O}(X \setminus s^{-1}(0))$.

To complete the proof, it suffices to show the following. Let X be a Stein manifold and $s \in \Gamma(X, L)$ for some holomorphic line bundle L. Then, every $f \in \mathcal{O}(X \setminus s^{-1}(0))$ can be approximated normally by meromorphic functions of the form σ/s^N, where the zero set of $\sigma \in \Gamma(X, L^{\otimes N})$ shares no irreducible components with $s^{-1}(0)$. Define $\mathcal{J} = (1/s) \cdot \mathcal{O}$, where $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on X. Then, $\mathcal{J}$ is a coherent subsheaf of germs of meromorphic functions on X [11, p. 119], and so by Cartan's Theorem A there exist global sections $h_1, \ldots, h_k$ of the sheaf $\mathcal{J}$ with the property that the germs $(h_1)_z, \ldots, (h_k)_z$ generate $\mathcal{J}_z$ as an $\mathcal{O}_z$-module for every $z \in X$. Given a proper holomorphic embedding $\Phi : X \to \mathbb{C}^N$, we define $\Psi = (\Phi, h_1, \ldots, h_k)$. Observe that Ψ is a proper holomorphic embedding of $X \setminus s^{-1}(0)$ into $\mathbb{C}^{N+k}$. Indeed, if $p \in s^{-1}(0)$, then $(1/s)_p = (g_1)_p(h_1)_p + \cdots + (g_k)_p(h_k)_p$ for some $(g_1)_p, \ldots, (g_k)_p \in \mathcal{O}_p$. Since $1/s \to \infty$ along any sequence in $X \setminus s^{-1}(0)$ tending towards p, the same must be true for at least one of the h_j.

Now, the Oka–Cartan theorem yields a function $F \in \mathcal{O}(\mathbb{C}^{N+k})$ which agrees, when restricted to the complex-analytic set $\Psi(X \setminus s^{-1}(0)) \subset \mathbb{C}^{N+k}$, with $f \circ \Psi^{-1}$. Let F_T be a Taylor polynomial of F and consider $F_T \circ \Psi$. Let $\{U_i\}$ be an open cover of X by trivializations of L with associated transition functions

g_{ij}. Since the $h_1, \ldots, h_k$ are in particular sections of $\mathcal{J}$, we have $h_\nu|_{U_i} = t_{i\nu}/s_i$ in U_i, $t_{i\nu} \in \mathcal{O}(U_i)$, for each ν as well; consequently, $F_T \circ \Psi|_{U_i}$ is a polynomial in $t_{i1}/s_i, t_{i2}/s_i, \ldots, t_{ik}/s_i$ and the components of Φ restricted to U_i. Putting everything under one denominator shows $F_T \circ \Psi|_{U_i}$ to be of the form

$$\frac{a_{-N}(t_i, \Phi) + a_{-N+1}(t_i, \Phi)s_i + \cdots + a_N(t_i, \Phi)s_i^{2N}}{s_i^N},$$

where $a_{-N}, \ldots, a_N$ are polynomials and t_i denotes $(t_{i1}, \ldots, t_{ik})$. Note that N is independent of the choice of i. Write σ_i for the numerator of the above expression. We have $\sigma_i = (s_i/s_j)^N \sigma_j = g_{ij}^N \sigma_j$ on $U_i \cap U_j$ for every i, j, so $\sigma = \{\sigma_i\}$ patches together to a section of $L^{\otimes N}$. Much as in Theorem 2.1, σ and s can be assumed coprime after possibly a small perturbation of a_{-N}. ◻

Remark 5.4. It should be noted that Theorem 5.2 and many of the results above can be stated more generally for Stein spaces (e.g., [1]), yet for simplicity's sake we have restricted our attention to Stein manifolds.

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