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Analytic continuation of holomorphic correspondences and equivalence of domains in $\mathbb{C}^n$

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Abstract. The following result is proved: Let D and D' be bounded domains in $\mathbb{C}^n$, ∂D is smooth, real-analytic, simply connected, and $\partial D'$ is connected, smooth, real-algebraic. Then there exists a proper holomorphic correspondence $f : D \to D'$ if and only if there exist points $p \in \partial D$ and $p' \in \partial D'$, such that ∂D and $\partial D'$ are locally CR-equivalent near p and p'. This leads to a characterization of the equivalence relationship between bounded domains in $\mathbb{C}^n$ modulo proper holomorphic correspondences in terms of local CR-equivalence of their boundaries.

1. Definitions and main results

Following Stein [St1] a holomorphic correspondence between two domains D and D' in $\mathbb{C}^n$ is defined to be a complex-analytic set $A \subset D \times D'$ which satisfies: (i) A is of pure complex dimension n, and (ii) the natural projection $\pi : A \to D$ is proper. A correspondence A is called proper, if in addition the natural projection $\pi' : A \to D'$ is proper. We may also think of A as the graph of a multiple valued mapping defined by $F := \pi' \circ \pi^{-1}$. Holomorphic correspondences were studied for instance, in [BB1], [BB2], [Be1], [BSu], [DFY], [DP1], [P3], [St1], [St2], and [V].

In this paper we address the following question: Given two domains D and D', when does there exist a proper holomorphic correspondence $F : D \to D'$? Note (see [St2]) that the existence of a correspondence F defines an equivalence relation $D \sim D'$. This equivalence relation is a natural generalization of biholomorphic equivalence of domains in $\mathbb{C}^n$.

To illustrate the concept of equivalence modulo holomorphic correspondences, consider domains of the form $\Omega_{p,q} = \{|z_1|^{2p} + |z_2|^{2q} < 1\}$, $p, q \in \mathbb{Z}^+$. Then $f(z) = ({z_1}^{p/s}, {z_2}^{q/t})$ is a proper holomorphic correspondence between $\Omega_{p,q}$ and $\Omega_{s,t}$, $s, t \in \mathbb{Z}^+$, while a proper holomorphic map from $\Omega_{p,q}$ to $\Omega_{s,t}$ exists only if $s|p$ and $t|q$, or $s|q$ and $t|p$. For details see [Bd] or [La].

The main result of this paper is the following theorem.

Theorem 1. *Let D and D' be bounded domains in $\mathbb{C}^n$, $n > 1$. Let ∂D be smooth, real-analytic, connected and simply connected, and let $\partial D'$ be connected, smooth, real-algebraic. Then there exists a proper holomorphic correspondence $F : D \to D'$ if and only if there exist points $p \in \partial D$, $p' \in \partial D'$, neighborhoods $U \ni p$ and $U' \ni p'$, and a biholomorphic map $f : U \to U'$, such that $f(p) = p'$ and $f(U \cap \partial D) = U' \cap \partial D'$.*

In other words, we show that a germ of a biholomorphic mapping between the boundaries extends analytically to a holomorphic correspondence of the domains. By the example above, the extension will not be in general single-valued. Note that we do not require pseudoconvexity of either D or D'. Also $\partial D'$ is not assumed to be simply connected. If both D and D' have real-algebraic boundary, i.e. each is globally defined by an irreducible real polynomial, then we can drop the requirement of simple connectivity of ∂D.

Theorem 2. *Let D and D' be bounded domains in $\mathbb{C}^n$, $n > 1$, with connected smooth real-algebraic boundaries. Then there exists a proper holomorphic correspondence between D and D' if and only if there exist points $p \in \partial D$, $p' \in \partial D'$, neighborhoods $U \ni p$ and $U' \ni p'$, and a biholomorphic map $f : U \to U'$, such that $f(p) = p'$ and $f(U \cap \partial D) = U' \cap \partial D'$.*

The proof of Theorem 2 is simpler than that of Theorem 1 due to the fact that when both domains are algebraic, Webster's theorem [We] can be applied. We give a separate complete proof of Theorem 2 in Sect. 3 to emphasize the ideas of the proof of Theorem 1 and the difficulties that arise in the case when one of the domains is not algebraic.

Local CR-equivalence of the boundaries, which is used in the above theorems to characterize correspondence equivalence, is a well-studied subject. Chern and Moser [CM] gave the solution to the local equivalence problem for real-analytic hypersurfaces with non-degenerate Levi form both in terms of normalization of Taylor series of the defining functions and in terms of intrinsic differential-geometric invariants of the hypersurfaces. See also [Se], [Ca] and [T]. Note that for a bounded domain in $\mathbb{C}^n$ with smooth real-analytic boundary, the set of points where the Levi form is degenerate is a closed nowhere dense set. Thus we may reformulate Theorem 1 in the following way.

Theorem 3. *Let D and D' be as in Theorem 1. Then D and D' are correspondence equivalent if and only if there are points $p \in \partial D$ and $p' \in \partial D'$ such that the Levi form of the defining functions are non-degenerate at p and p', and the corresponding Chern–Moser normal forms are equivalent.*

Theorem 1 generalizes the result in [P1], which states that a bounded, strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with connected, simply-connected, real-analytic boundary is biholomorphically equivalent to $\mathbb{B}^n = \mathbb{B}(0, 1)$, the unit ball centered at the origin, if and only if there exists a local biholomorphism of the boundaries. It was later shown in [CJ] (see also [BER1]) that simple connectivity of ∂D can be replaced by simple connectivity of the domain D and connectedness of ∂D. For related results see also [A], [Ro], [Ru], and [Wo].

In [P2] Pinchuk also established sufficient and necessary conditions of equivalence for bounded, strictly pseudoconvex domains with real-analytic, connected and simply-connected boundaries. Two such domains D and D' are biholomorphically equivalent if and only if there exist points $p \in \partial D$ and $p' \in \partial D'$ such that ∂D and $\partial D'$ are locally equivalent at p and p'.

If in Theorem 1 ∂D is not assumed to be simply connected, then the result is no longer true. Indeed, in a famous example Burns and Shnider [BS] constructed a domain Ω with the boundary given by

$$\partial\Omega = \{z \in \mathbb{C}^2 : \sin(\ln|z_2|) + |z_1|^2 = 0,\ e^{-\pi} \leq |z_2| \leq 1\}, \tag{1.1}$$

which is real-analytic and strictly pseudoconvex but not simply connected. There exists a mapping $f : \mathbb{B}^2 \to \Omega$ such that f does not extend even continuously to a point on $\partial\mathbb{B}^2$. The inverse to f gives a local biholomorphic map from $\partial\Omega$ to $\partial\mathbb{B}^2$, but nevertheless f^{-1} does not extend to a global proper holomorphic correspondence between Ω and $\mathbb{B}^2$. Furthermore, suppose that there exists a proper correspondence $g : \Omega \to \mathbb{B}^2$. Since $\mathbb{B}^2$ is simply connected, g^{-1} is a proper holomorphic mapping, which extends holomorphically to a neighborhood of $\overline{\mathbb{B}^2}$. Let $p \in \partial\Omega$, $q^1 \in f^{-1}(p)$, and $q^2 \in g(p)$. By the result in [BB1] g splits at every point in $\overline{\Omega}$. Therefore, with a suitable choice of the branch of g near p, $\phi := g \circ f$ defines a local biholomorphic mapping from a neighborhood $U_1 \ni q^1$ to some neighborhood $U_2 \ni q^2$. Moreover, $\phi(U_1 \cap \partial\mathbb{B}^2) \subset (U_2 \cap \partial\mathbb{B}^2)$. By the theorem of Poincaré-Alexander (see e.g. [A]), ϕ extends to a global automorphism of $\mathbb{B}^2$. Thus after a biholomorphic change of coordinates, by the uniqueness theorem f and g^{-1} must agree in $\mathbb{B}^2$. But $f : \mathbb{B}^2 \to \Omega$ is not a proper map. This contradiction shows that there are no proper holomorphic correspondences between Ω and $\mathbb{B}^2$. Thus the condition of simple connectivity of ∂D in Theorem 1 cannot be in general weakened.

One direction in the proof of Theorems 1 and 2 is essentially contained in the work of Berteloot and Sukhov [BSu]. The proof in the other direction is based on the idea of extending the mapping f along ∂D as a holomorphic correspondence.

It is not known whether Theorem 1 holds if $\partial D'$ is real-analytic. The main difficulty is to prove local analytic continuation of holomorphic mappings along real-analytic hypersurfaces in the case when the hypersurfaces are not strictly pseudoconvex. In particular, Lemma 3 cannot be directly established for real-analytic hypersurfaces.

In Sect. 2 we present background material on Segre varieties and holomorphic correspondences. In Sect. 3 we prove a technical result, important for the proof of the main theorems. Section 4 contains the proof of Theorem 2. In Sect. 5 we prove local extendability of holomorphic correspondences along hypersurfaces. Theorem 1 is proved in Sect. 6.

2. Background material

Let Γ be an arbitrary smooth real-analytic hypersurface with a defining function $\rho(z,\overline{z})$ and let $z^0 \in \Gamma$. In a suitable neighborhood $U \ni z^0$ to every point $w \in U$ we can associate to Γ its so-called Segre variety defined as

$$Q_w = \{z \in U : \rho(z,\overline{w}) = 0\}, \tag{2.1}$$

where $\rho(z,\overline{w})$ is the complexification of the defining function of Γ. From the reality condition we conclude that for $z, w \in U$,

$$z \in Q_w \Leftrightarrow w \in Q_z. \tag{2.2}$$

After a local biholomorphic change of coordinates near z^0, we can find neighborhoods $U_1 \Subset U_2$ of z^0, where

$$U_2 = {}'U_2 \times {}''U_2 \subset \mathbb{C}^{n-1}_{'z} \times \mathbb{C}_{z_n}, \tag{2.3}$$

such that for any $w \in U_1$, Q_w is a closed smooth complex-analytic hypersurface in U_2. Here $z = ('z, z_n)$. Furthermore, a Segre variety can be written as a graph of a holomorphic function,

$$Q_w = \left\{('z, z_n) \in {}'U_2 \times {}''U_2 :\ z_n = h('z,\overline{w})\right\}, \tag{2.4}$$

where $h(\cdot,\overline{w})$ is holomorphic in ${}'U_2$. Following [DP1] we call U_1 and U_2 a *standard* pair of neighborhoods of z^0. A detailed discussion of elementary properties of Segre varieties can be found in [DW], [DF2], [DP1] or [BER2].

The map $\lambda : z \to Q_z$ is called the Segre map. We define $I_w = \lambda^{-1} \circ \lambda(w)$. This is equivalent to

$$I_w = \{z \in U_1 : Q_z = Q_w\}. \tag{2.5}$$

If Γ is of finite type in the sense of D'Angelo, or more generally, if Γ is essentially finite, then there exists a neighborhood U of Γ such that for any $w \in U$ the set I_w is finite. Due to the result in [DF1], this is the case for compact smooth real-analytic hypersurfaces, in particular for the boundaries of D and D' in Theorem 1. The last observation is crucial for the construction of proper holomorphic correspondences used throughout this paper. We remark that our choice of a standard pair of neighborhoods of any point $z \in \Gamma$ is always such that for any $w \in U_2$, the set I_w is finite.

If Γ is a smooth real-algebraic hypersurface, then (2.1) defines an algebraic variety in $\mathbb{C}^n$ of dimension $n-1$ for all $z \in \mathbb{C}^n$ outside some exceptional variety E. From the above considerations, $E \cap \Gamma = \varnothing$, and we can find a neighborhood U of Γ such that Q_z is well defined for all $z \in U$. In this case we treat I_z as a set globally defined in U, i.e.

$$I_z = \{w \in U : Q_z = Q_w\}. \tag{2.6}$$

For the proof of Theorem 1 we will need a simple lemma.

Lemma 1. *Let Γ be a compact, smooth, real-algebraic hypersurface. Then there exist a neighborhood U of Γ and an integer $m \geq 1$ such that for almost any $z \in U$, $\#I_z = m$.*

The proof follows from [DF2]. The following statement describes the invariance property of Segre varieties under holomorphic mappings. It is analogous to the classical Schwarz reflection principle in dimension one.

Suppose that Γ and Γ' are real-analytic hypersurfaces in $\mathbb{C}^n$, (U_1, U_2) and (U_1', U_2') are standard pairs of neighborhoods for $z_0 \in \Gamma$ and $z_0' \in \Gamma'$ respectively. Let $f : U_2 \to U_2'$ be a holomorphic map, $f(U_1) \subset U_1'$ and $f(\Gamma \cap U_2) \subset (\Gamma' \cap U_2')$. Then

$$f(Q_w \cap U_2) \subset Q'_{f(w)} \cap U_2', \text{ for all } w \in U_1. \tag{2.7}$$

Moreover, a similar invariance property also holds for a proper holomorphic correspondence $f : U_2 \to U_2'$, $f(\Gamma \cap U_2) \subset (\Gamma' \cap U_2')$. In this case (2.7) indicates that any branch of f maps any point from Q_w to $Q_{w'}$ for any $w' \in f(w)$. For details, see [DP1] or [V].

Let $f : D \to D'$ be a holomorphic correspondence. We say that f is *irreducible*, if the corresponding analytic set $A \subset D \times D'$ is irreducible. The condition that f is proper is equivalent to the condition that

$$\sup\{\text{dist}(f(z), \partial D')\} \to 0, \text{ as dist}(z, \partial D) \to 0. \tag{2.8}$$

Recall that if $A \subset D \times D'$ is a proper holomorphic correspondence, then $\pi : A \to D$ and $\pi' : A \to D'$ are proper. There exists a complex subvariety $S \subset D$ and a number m such that

$$f := \pi' \circ \pi^{-1} = (f^1(z), \dots, f^m(z)), \tag{2.9}$$

where f^j are distinct holomorphic functions in a neighborhood of $z \in D \setminus S$. The set S is called the *branch locus* of f. We say that the correspondence f *splits* at $z \in \overline{D}$ if there is an open subset $U \ni z$ and holomorphic maps $f^j : D \cap U \to D'$, $j = 1, 2, \dots, m$, that represent f.

Given a proper holomorphic correspondence A, one can find the system of canonical defining functions

$$\Phi_I(z, z') = \sum_{|J| \leq m} \phi_{IJ}(z) z'^J, \ |I| = m, \ (z, z') \in \mathbb{C}^n \times \mathbb{C}^n, \tag{2.10}$$

where $\phi_{IJ}(z)$ are holomorphic on D, and A is precisely the set of common zeros of the functions $\Phi_I(z, z')$. For details see e.g. [Ch].

We define analytic continuation of analytic sets as follows. Let A be a locally complex analytic set in $\mathbb{C}^n$ of pure dimension p. We say that A *extends analytically* to an open set $U \subset \mathbb{C}^n$, $A \cap U \neq \varnothing$, if there exists a (closed) complex-analytic set A^* in U such that (i) $\dim A^* = p$, (ii) $A \cap U \subset A^*$ and (iii) every irreducible component of A^* has a nonempty intersection with A of dimension p. Note that if conditions (i) and (ii) are satisfied, then the last condition can always be met by removing certain components of A^*. It follows from the uniqueness theorem that such analytic continuation of A is uniquely defined. From this we define analytic continuation of holomorphic correspondences:

Definition 1. *Let U and U' be open sets in $\mathbb{C}^n$. Let $f : U \to U'$ be a holomorphic correspondence, and let $A \subset U \times U'$ be its graph. We say that f extends as a holomorphic correspondence to an open set $V \subset \mathbb{C}^n$, $U \cap V \neq \varnothing$, if there exists an open set $V' \subset \mathbb{C}^n$ such that $A \cap (V \times V') \neq \varnothing$, A extends analytically to a set $A^* \subset V \times V'$, and $\pi : A^* \to V$ is proper.*

Note that we can always choose $V' = \mathbb{C}^n$ in the definition above. In general, the correspondence $g = \pi' \circ \pi^{-1} : V \to V'$, where $\pi' : A^* \to V'$ is the natural projection, may have more branches in $U \cap V$ than f. The following lemma gives a simple criterion for the extension to have the same number of branches.

Lemma 2. *Let $A^* \subset V \times \mathbb{C}^n$ be a holomorphic correspondence which is an analytic extension of the correspondence $A \subset U \times \mathbb{C}^n$. Suppose that for any $z \in (V \cap U)$,*

$$\#\{\pi^{-1}(z)\} = \#\{\pi^{*-1}(z)\}, \tag{2.11}$$

where $\pi : A \to U$ and $\pi^* : A^* \to V$. Then $A \cup A^*$ is a holomorphic correspondence in $(U \cup V) \times \mathbb{C}^n$.

Proof. We only need to check that $A \cup A^*$ is closed in $(U \cup V) \times \mathbb{C}^n$. If not, then there exists a sequence $\{q^j\} \subset A^*$ such that $q^j \to q^0$ as $j \to \infty$, $q^0 \in U \times \mathbb{C}^n$, and $q^0 \notin A$. Then $q^j \notin A$ for j sufficiently large. Since by the definition of analytic continuation of correspondences $A \cap (U \cap V) \times \mathbb{C}^n \subset A^*$, we have

$$\#\{\pi^{-1}(\pi^*(q^j))\} < \#\{\pi^{*-1}(\pi^*(q^j))\}.$$

But this contradicts (2.11). □

3. Extension along Segre varieties

Before we prove the main results of the paper we need to establish a technical result of local extension of holomorphic correspondence along Segre varieties. This will be used on several occasions in the proof of the main theorems.

Lemma 3. *Let $\Gamma \subset \mathbb{C}^n$ be a smooth, real-analytic, essentially finite hypersurface, and let $\Gamma' \subset \mathbb{C}^n$ be a smooth, real-algebraic, essentially finite hypersurface. Let $0 \in \Gamma$, and let U_1, U_2 be a sufficiently small standard pair of neighborhoods of the origin. Let $f : U \to \mathbb{C}^n$ be a holomorphic correspondence such that $f(U \cap \Gamma) \subset \Gamma'$, where U is some neighborhood of the origin. Then there exists a neighborhood V of $Q_0 \cap U_1$ and an analytic set $\Lambda \subset V$, $\dim_{\mathbb{C}} \Lambda \le n-1$, such that f extends to $V \setminus \Lambda$ as a holomorphic correspondence.*

Proof. In the case when Γ' is strictly pseudoconvex and f is a germ of a biholomorphic mapping, the result was established in [S]. Here we prove the lemma in a more general situation.

Lemma 3 only makes sense if $U \subset U_1$. We shrink U and choose V in such a way that for any $w \in V$, the set $Q_w \cap U$ is non-empty and connected. Note that if $w \in Q_0$, then $0 \in Q_w$ and $Q_w \cap U \neq \varnothing$. Let $S \subset U$ be the branch locus of f, and let

$$\Sigma = \{z \in V : (Q_z \cap U) \subset S\}. \tag{3.1}$$

Since $\dim_{\mathbb{C}} S = n - 1$ and Γ is essentially finite, Σ is a finite set. Define

$$A = \left\{(w, w') \in (V \setminus \Sigma) \times \mathbb{C}^n : f\,(Q_w \cap U) \subset Q'_{w'}\right\}. \tag{3.2}$$

We establish the following facts about the set A:

(i) A is not empty
(ii) A is locally complex analytic
(iii) A is closed
(iv) $\Sigma \times \mathbb{C}^n$ is a removable singularity for A.

(i) $A \neq \varnothing$ since by the invariance property of Segre varieties, A contains the graph of f.

(ii) Let $(w, w') \in A$. Consider an open simply connected set $\Omega \Subset (U \setminus S)$ such that $Q_w \cap \Omega \neq \varnothing$. Then the branches of f are correctly defined in Ω. Since $Q_w \cap U$ is connected, the inclusion $f(Q_w \cap U) \subset Q'_{w'}$ is equivalent to

$$f^j(Q_w \cap \Omega) \subset Q'_{w'}, \ \ j = 1, \dots, m, \tag{3.3}$$

where f^j denote the branches of f in Ω. Note that such a neighborhood Ω exists for any $w \in V \setminus \Sigma$. Inclusion (3.3) can be written as a system of holomorphic equations as follows. Let $\rho'(z, \overline{z})$ be the defining function of Γ'. Then

$$\rho'(f^j(z), \overline{w'}) = 0, \quad \text{for any } z \in (Q_w \cap \Omega), \ \ j = 1, 2, \dots, m. \tag{3.4}$$

We can choose Ω in the form

$$\Omega = {}'\Omega \times \Omega_n \subset \mathbb{C}^{n-1}_{'z} \times \mathbb{C}_{z_n}. \tag{3.5}$$

Combining this with (2.4) we obtain

$$\rho'(f^j('z, h('z, \overline{w})), \overline{w'}) = 0 \tag{3.6}$$

for any $'z \in {'\Omega}$. Then (3.6) is an infinite system of holomorphic equations in (w, w') thus defining A as a locally complex analytic variety in $(V \setminus \Sigma) \times \mathbb{C}^n$.

(iii) Let us show now that A is a closed set. Suppose that $(w^j, w'^j) \to (w^0, w'^0)$, as $j \to \infty$, where $(w^j, w'^j) \in A$ and $(w^0, w'^0) \in (V \setminus \Sigma) \times \mathbb{C}^n$. Then by the definition of A, $f(Q_{w^j} \cap U) \subset Q'_{w'^j}$. Since $Q_{w^j} \to Q_{w^0}$, and $Q'_{w'^j} \to Q'_{w^0}$ as $j \to \infty$, by analyticity $f(Q_{w^0} \cap U) \subset Q'_{w'^0}$, which implies that $(w^0, w'^0) \in A$ and thus A is a closed set. Since A is locally complex-analytic and closed, it is a complex variety in $(V \setminus \Sigma) \times \mathbb{C}^n$. We now may restrict considerations only to the irreducible component of A which coincides with the graph of f at the origin. Then $\dim A \equiv n$.

(iv) Let us show now that $\overline{A}$ is a complex variety in $V \times \mathbb{C}^n$. Let $q \in \Sigma$, then

$$\overline{A} \cap (\{q\} \times \mathbb{C}^n) \subset \{q\} \times \{z' : f(Q_q) \subset Q'_{z'}\}. \tag{3.7}$$

Notice that if $w' \in f(Q_q) \subset Q'_{z'}$, then by (2.2) $z' \in Q'_{w'}$. Hence the set $\{z' : f(Q_q) \subset Q'_{z'}\}$ has dimension at most $2n - 2$, and $\overline{A} \cap (\Sigma \times \mathbb{C}^n)$ has Hausdorff $2n$-measure zero. It follows from Bishop's theorem on removable singularities of analytic sets (see e.g. [Ch]) that $\overline{A}$ is an analytic set in $V \times \mathbb{C}^n$.

Thus from (i)–(iv) we conclude that (3.2) defines a complex-analytic set in $V \times \mathbb{C}^n$ which we denote again by A. Also we observe that since Γ' is algebraic, the system of holomorphic equations in (3.6) is algebraic in w' and thus we can define the closure of A in $V \times \mathbb{P}^n$. Let $\pi : \overline{A} \to V$ and $\pi' : \overline{A} \to \mathbb{P}^n$ be the natural projections. Since $\mathbb{P}^n$ is compact, $\pi^{-1}(K)$ is compact for any compact set $K \subset V$, and thus π is proper.

This, in particular, implies that $\pi(\overline{A}) = V$. We let $\Lambda_1 = \pi(\pi'^{-1}(H_0))$, where $H_0 \subset \mathbb{P}^n$ is the hypersurface at infinity. It is easy to see that Λ_1 is a complex analytic set of dimension at most $n - 1$. We also consider the set $\Lambda_2 := \pi\{(w, w') \in \overline{A} : \dim_{\mathbb{C}} \pi^{-1}(w) \geq 1\}$. It was shown in [S] Prop. 3.3, that Λ_2 is a complex-analytic set of dimension at most $n - 2$. Let $\Lambda = \Lambda_1 \cup \Lambda_2$. Then $\pi' \circ \pi^{-1}|_{V \setminus \Lambda}$ is the desired extension of f as a holomorphic correspondence. □

4. Proof of Theorem 2

For completeness let us repeat the argument of [BSu] to prove the "only if" direction of Theorems 1 and 2. Suppose that $f : D \to D'$ is a proper holomorphic correspondence. We need to show that ∂D and $\partial D'$ are locally CR-equivalent, that is there exist points $p \in \partial D$, $p' \in \partial D'$, neighborhoods $U \ni p$, $U' \ni p'$, and a biholomorphic map $f : U \to U'$ with $f(p) = p'$ and $f(U \cap \partial D) = U' \cap \partial D'$.

If D is not pseudoconvex, then for a point $q \in \widehat{D}$, there exists a neighborhood $U \ni q$ such that all the functions in the representation (2.10) of f extend holomorphically to U. Here $\widehat{D}$ refers to the envelope of holomorphy of D. Moreover, we can replace q with a nearby point $p \in U \cap \partial D$ so that f splits at p and at least one of the holomorphic mappings of the splitting, say f^j, is biholomorphic at p. Then ∂D and $\partial D'$ are locally CR-equivalent near p and $p' = f^j(p) \in \partial D'$.

If D is pseudoconvex, then D' is also pseudoconvex. By [BSu] f extends continuously to ∂D and we can choose $q \in \partial D$ such that f splits in some neighborhood $U \ni q$ to holomorphic mappings $f^j : D \cap U \to D'$, $j = 1, \ldots, m$. Since $f^{-1} : D' \to D$ also extends continuously to $\partial D'$, the set $\{f^{-1} \circ f(q)\}$ is finite. Therefore, by [Be2], f^j extend smoothly to $\partial D \cap U$. It follows that f^j extend holomorphically to a neighborhood of q by [Be3] and [DF2]. Finally, choose $p \in U \cap \partial D$ such that f^j is biholomorphic at p for some j. Again p and $p' = f^j(p)$ is the desired pair of points.

To prove Theorem 2 in the other direction, consider a neighborhood U of $p \in \partial D$ and a biholomorphic map $f : U \to \mathbb{C}^n$ such that $f(U \cap \partial D) \subset \partial D'$. Let us show that f extends to a proper holomorphic correspondence $F : D \to D'$.

Let $\Gamma = \partial D$ and $\Gamma' = \partial D'$. Since the set of Levi non-degenerate points is dense in Γ, by Webster's theorem [We], f extends to an algebraic mapping, i.e. the graph of f is contained in an algebraic variety $X \subset \mathbb{C}^n \times \mathbb{C}^n$ of dimension n. Without loss of generality assume that X is irreducible, as otherwise consider only the irreducible component of X containing Γ_f, the graph of the mapping f.

Let $E = \{z \in \mathbb{C}^n : \dim \pi^{-1}(z) > 0\}$, where $\pi : X \to \mathbb{C}^n$ is the coordinate projection to the first component. Then E is an algebraic variety in $\mathbb{C}^n$. Let $f : \mathbb{C}^n \setminus E \to \mathbb{C}^n$ now denote the multiple valued map corresponding to X. Let $S \subset \mathbb{C}^n \setminus E$ be the branch locus of f, in other words, for any $z \in S$ the coordinate projection onto the first component is not locally biholomorphic near $\pi^{-1}(z)$. To prove Theorem 2 it is enough to show that $E \cap \Gamma = \varnothing$.

Lemma 4. *Let $p \in \Gamma$. If $Q_p \not\subset E$, then $p \notin E$.*

Proof. Suppose, on the contrary, that $p \in E$. Since $Q_p \not\subset E$, there exist a point $b \in Q_p$ and a small neighborhood U_b of b such that $U_b \cap E = \varnothing$. Choose neighborhoods U_b and U_p such that for any $z \in U_p$, the set $Q_z \cap U_b$ is non-empty and connected. Let

$$\Sigma = \{z \subset U_p : Q_z \cap U_b \subset S\}. \tag{4.1}$$

Similar to (3.2), consider the set

$$A = \left\{(w, w') \in (U_p \setminus \Sigma) \times \mathbb{C}^n : f\,(Q_w \cap U_b) \subset Q'_{w'}\right\}. \tag{4.2}$$

Then $A \neq \varnothing$. Indeed, since $\dim_{\mathbb{C}} E \leq n - 1$, there exists a sequence of points $\{p^j\} \subset (U_p \cap \Gamma) \setminus (E \cup \Sigma)$ such that $p^j \to p$ as $j \to \infty$.

By the invariance property of holomorphic correspondences, for every p^j there exists a neighborhood $U_j \ni p^j$ such that $f(Q_{p^j} \cap U_j) \subset Q'_{p'^j}$, where $p'^j \in f(p^j)$. But this implies that $f(Q_{p^j} \cap U_b) \subset Q'_{p'^j}$, and therefore $(p^j, p'^j) \in A$. Moreover, it follows that

$$A|_{U_j \times \mathbb{C}^n} = X|_{U_j \times \mathbb{C}^n}, \ \ j = 1, 2, \ldots, m. \tag{4.3}$$

Similar to the proof of Lemma 3, one can show that A is a complex analytic variety in $(U_p \setminus \Sigma) \times \mathbb{C}^n$, and that $\Sigma \times \mathbb{C}^n$ is a removable singularity for $A \subset U_p \times \mathbb{C}^n$. Denote the closure of A in $U_p \times \mathbb{C}^n$ again by A.

Without loss of generality we assume that A is irreducible, therefore in view of (4.3) we conclude that $A|_{U_p \times \mathbb{C}^n} = X|_{U_p \times \mathbb{C}^n}$. Let $\hat{f}$ be a multiple valued mapping corresponding to A. Then by analyticity, there exists $p' \in \Gamma' \cap \hat{f}(p)$. Moreover, by construction, $\hat{f}(p) = I'_{p'}$. By [DW],

$$I'_{z'} \subset \Gamma', \ \text{ for any } z' \in \Gamma'. \tag{4.4}$$

Now choose U_p so small that

$$\overline{A} \cap (U_p \times \partial U') = \varnothing, \tag{4.5}$$

where U' is a neighborhood of Γ'. This is always possible, since otherwise there exists a sequence of points $\{(z^j, z'^j), j = 1, 2, \ldots\} \subset A$, such that $z^j \to p$ and $z'^j \to z'^0 \in \partial U'$ as $j \to \infty$. Then $(p, z'^0) \in A$ and $z'^0 \notin \Gamma'$. But this contradicts (4.4).

It follows from (4.5) that $\hat{f} : U_p \to U'$ is a holomorphic correspondence extending f, which contradicts the assumption $p \in E$. □

Lemma 5. *Let $p \in \Gamma$. Then $Q_p \not\subset E$.*

Proof. Suppose that $Q_p \subset E$. Then we find a point $a \in (\Gamma \setminus E)$ such that $Q_a \cap Q_p \neq \varnothing$. The existence of such a follows, for example, from [S] Prop 4.1. (Note, that $\dim E \cap \Gamma \leq 2n - 3$). By Lemma 3 the germ of the correspondence f defined at a, extends holomorphically to a neighborhood V of Q_a. Let Λ_1 and Λ_2 be as in Lemma 3. Since $\dim \Lambda_2 < n - 1$, we may assume that $(Q_p \cap V) \not\subset \Lambda_2$. If $(Q_p \cap V) \subset \Lambda_1$, we can perform a linear-fractional transformation such that H_0 is mapped onto another complex hyperplane $H \subset \mathbb{P}^n$ and such that $H \cap \Gamma' = \varnothing$. Note that after such transformation Γ' remains compact in $\mathbb{C}^n$. Then we may also assume that $(Q_p \cap V) \not\subset \Lambda_1$. Thus holomorphic extension along Q_a defines f on a non-empty set in Q_p, which contradicts the assumption $Q_p \subset E$. □

Theorem 2 now follows. Indeed, from Lemmas 4 and 5 we conclude that $E \cap \Gamma = \varnothing$. Since D is bounded, $D \cap E = \varnothing$, and $X \cap (D \times D')$ defines a proper holomorphic correspondence from D to D'.

5. Local extension

To prove Theorem 1 we first establish local extension of holomorphic correspondences. For that we impose an additional technical condition on holomorphic correspondences as follows.

Definition 2. *Let $\Gamma \subset \mathbb{C}^n$ be a smooth, real-analytic, essentially finite hypersurface, and let $\Gamma' \subset \mathbb{C}^n$ be a smooth, compact, real-algebraic hypersurface. Let $f : U \to \mathbb{C}^n$ be a holomorphic correspondence such that $f(U \cap \Gamma) \subset \Gamma'$. Then f is called* complete *if (i) for any $z \in U \cap \Gamma$, $f(Q_z \cap U) \subset Q'_{z'}$, $z' \in f(z)$, and (ii) $f(z) = I'_{z'}$.*

We note that here $I'_{z'}$ should be understood globally as a set in U', where U' is a neighborhood of Γ' chosen as in Lemma 1. By the invariance property of Segre varieties, for any $z \in U \cap \Gamma$, $f({}_zQ_z) \subset Q_{z'}$, where $z' \in f(z)$ and ${}_zQ_z$ is the germ of Q_z at z. Condition *(i)* in the definition is somewhat stronger: it states that every connected component of $Q_z \cap U$ is mapped by f into the same Segre variety. Also note that in general Segre varieties are defined only locally, while the set U can be relatively large. In this case the inclusion $f(Q_z \cap U) \subset Q'_{z'}$ should be considered only in a suitable standard pair of neighborhoods of z.

Condition *(ii)* in the definition above indicates that f has the maximal possible number of branches. It may happen that initially a given holomorphic correspondence f misses certain branches, i.e. for $z' \in f(z)$, $f(z) \subset I'_{z'}$, but the equality does not hold. The missing branches may reappear, for example, after an analytic continuation of f. It is convenient to establish analytic continuation of complete correspondences, as such continuation does not introduce additional branches (see Lemma 1).

Lemma 6. *Let $f : U \to \mathbb{C}^n$ be a complete holomorphic correspondence, $f(\Gamma \cap U) \subset \Gamma'$, where $\Gamma \subset \mathbb{C}^n$ is a smooth, real-analytic, essentially finite hypersurface, and $\Gamma' \subset \mathbb{C}^n$ is a smooth, compact, real-algebraic hypersurface. Suppose $p \in \partial U \cap \Gamma$ is such that $Q_p \cap U \neq \varnothing$. Then there exists a neighborhood U_p of p such that f extends to a holomorphic correspondence $\hat{f} : U_p \to \mathbb{C}^n$.*

Proof. The proof of this lemma repeats that of Lemma 4. Let $b \in Q_p \cap U$. Consider a small neighborhood U_b of b, $U_b \subset U$, and a neighborhood U_p of p such that for any $z \in U_b$, the set $Q_z \cap U_p$ is non-empty and connected. As before, let $S \subset U$ be the branch locus of f, and $\Sigma = \{z \subset U_p : Q_z \cap U_b \subset S\}$. Define

$$A = \left\{(w, w') \in (U_p \setminus \Sigma) \times \mathbb{C}^n : f\left(Q_w \cap U_b\right) \subset Q'_{w'}\right\}. \qquad (5.1)$$

Observe that since f is complete, for any $w \in U \cap \Gamma$, the inclusion $f(Q_w \cap U) \subset Q'_{w'}$ is equivalent to $f(Q_w \cap U_b) \subset Q'_{w'}$. In particular, this holds for any w arbitrarily close to p. Therefore A is not empty as it contains the graph of f in $U \cap U_p$, if the neighborhood U_p is chosen sufficiently small.

Analogously to Lemma 4, one can show that A is a non-empty closed complex analytic set in $(U_p \setminus \Sigma) \times \mathbb{C}^n$. A similar argument also shows that $\Sigma \times \mathbb{C}^n$ is a removable singularity for A, and thus $\overline{A}$ defines a closed-complex analytic set in $U_p \times \mathbb{C}^n$ of pure dimension n. For simplicity denote this set again by A.

Let us show now that A defines a holomorphic correspondence $\hat{f} : U_p \to U'$, where U' is a suitable neighborhood of Γ'.

Consider the closure of A in $U_p \times \mathbb{P}^n$. Recall, that since $\mathbb{P}^n$ is compact, the projection $\pi : \overline{A} \to U_p$ is proper. In particular, $\pi(\overline{A}) = U_p$. Let U' be a neighborhood of Γ' as in Lemma 1. To simplify the notation, denote the restriction of $\overline{A}$ to $U_p \times U'$ again by A. Let $\pi : A \to U_p$ and $\pi' : A \to U'$ be the natural projections, and let $\hat{f} = \pi' \circ \pi$.

Let $Z = \pi'^{-1}(\Gamma')$. Then since $f(\Gamma \cap U) \subset \Gamma'$, $\pi^{-1}(\Gamma \cap U \cap U_p) \subset Z$. Therefore there exists at least one irreducible component of $\pi^{-1}(\Gamma \cap U_p)$ which is contained in Z. Thus for any $z \in \Gamma \cap U_p$, there exists $z' \in \Gamma'$ such that $z' \in \hat{f}(z)$. By construction, if $z \in U_p$ and $z' \in \hat{f}(z)$, then $\hat{f}(z) = I'_{z'}$. In view of (4.4) we conclude that $\hat{f}(\Gamma \cap U_p) \subset \Gamma'$. Now the same argument as in Lemma 4 shows that U_p can be chosen so small that $\hat{f}$ is a holomorphic correspondence. □

Note that without the assumption of completeness of f, the set A constructed in the proof of Lemma 6 would still be a holomorphic correspondence. However, it is not apriori clear why this correspondence would coincide with f near p.

Theorem 4 (Local extension). *Let D and D' be as in Theorem 1, $\Gamma = \partial D$ and $\Gamma' = \partial D'$. Let $f : U \to \mathbb{C}^n$ be a complete holomorphic correspondence, such that $f(\Gamma \cap U) \subset \Gamma'$, and $\Gamma \cap U$ is connected. Suppose that $p \in \partial U \cap \Gamma$, and $\Gamma \cap \partial U$ is a smooth submanifold near p. Then there exists a neighborhood U_p of the point p such that f extends to a holomorphic correspondence $\hat{f} : U_p \to \mathbb{C}^n$. Moreover, $\hat{f}|_{U \cap U_p} = f|_{U \cap U_p}$, and the resulting correspondence $F : U \cup U_p \to \mathbb{C}^n$ is complete.*

Proof. We call a point $p \in \partial U \cap \Gamma$ *regular*, if near p, $\partial U \cap \Gamma$ is a generic submanifold of Γ, i.e. $T^c_p(\partial U \cap \Gamma) = n - 2$. We prove the theorem in three steps. First we prove the result under the assumption $Q_p \cap U \neq \varnothing$, then for regular points in $\partial U \cap \Gamma$, and finally for arbitrary $p \in \partial U \cap \Gamma$.

Step 1. Suppose that $Q_p \cap U \neq \varnothing$. Then by Lemma 6 f extends as a holomorphic correspondence $\hat{f}$ to some neighborhood U_p of p. It follows from the construction that for any $z \in U \cap U_p$ the number of preimages of $f(z)$ and $\hat{f}(z)$ is the same. Thus by Lemma 2, f and $\hat{f}$ define a holomorphic correspondence in $U \cup U_p$. Denote this correspondence by F.

We now show that F is also complete in $U \cup U_p$. We only need to prove that for all $z \in (U \cup U_p) \cap \Gamma$, the set $Q_z \cap (U \cup U_p)$ is mapped by F

to the same Segre variety. First, observe that since f is complete, for any $z \in U \cap U_p \cap \Gamma$ arbitrarily close to p, $f(Q_z \cap U) \subset Q'_{f(z)}$. Thus if U_p is chosen sufficiently small, then for any $z \in U_p \cap \Gamma$,

$$F(Q_z \cap (U \cup U_p)) \subset Q'_{F(z)}. \tag{5.2}$$

Suppose now that there exists some point z in $(U \cup U_p) \cap \Gamma$ such that not all components of $Q_z \cap (U \cup U_p)$ are mapped by F into the same Segre variety. From the argument above, $z \notin U_p$. Since $U \cap \Gamma$ is connected, there exists a simple smooth curve $\gamma \subset \Gamma \cap U$ connecting z and p. By Lemma 3 for every point $\zeta \in \gamma$, the germ of a correspondence F at ζ extends as a holomorphic correspondence along the Segre variety Q_ζ. Moreover, for $\zeta \in \gamma$ which are close to p, the extension of F along Q_ζ coincides with the correspondence f in U (even if $Q_\zeta \cap U$ is disconnected). Since $\cup_{\zeta\in\gamma} Q_\zeta$ is connected, this property holds for all $\zeta \in \gamma$. The extension of F along Q_ζ clearly maps Q_ζ into $Q'_{F(\zeta)}$, and therefore $Q_\zeta \cap U$ is mapped by f into the same Segre variety. But this contradicts the assumption that the components of $Q_z \cap U$ are mapped into different Segre varieties. This shows that F is also a complete correspondence.

Step 2. Suppose now that $Q_p \cap U = \varnothing$, but p is a regular point. Then by [S] Prop. 4.1, there exists a point $a \in \Gamma \cap U$ such that $Q_a \cap Q_p \neq \varnothing$. We now apply Lemma 3 to extend the germ of the correspondence at a along Q_a. We note that such extension along Q_a may not in general define a complete correspondence, since apriori $Q_a \cap \Gamma$ may be disconnected from $U \cap \Gamma$. Let Λ be as in Lemma 3. Then after performing, if necessary, a linear-fractional transformation in the target space, we can find a point $b \in Q_p \cap Q_a$, such that $b \notin \Lambda$. Let U_b be a small neighborhood of b such that $U_b \cap \Lambda = \varnothing$ and f extends to U_b as a holomorphic correspondence f_b. Moreover, without loss of generality we may also assume that $U_b \cap \Gamma = \varnothing$. Then for any $z \in U \cap \Gamma$ such that $Q_z \cap U_b \neq \varnothing$, the sets $f(Q_z \cap U)$ and $f_b(Q_z \cap U_b)$ are contained in the same Segre variety. Indeed, if not, then we can connect a and z by a smooth path $\gamma \subset \Gamma \cap U$ and apply the argument that we used to prove completeness of F in Step 1.

Now we can apply the argument in Step 1 to show that f extends as a holomorphic correspondence to some neighborhood of p, and that the resulting extension is also complete.

Step 3. Suppose now that $p \in \partial U \cap \Gamma$ is not a regular point. Let

$$M = \left\{ z \in \partial U \cap \Gamma : T_z(\partial U \cap \Gamma) = T_z^c(\Gamma) \right\}. \tag{5.3}$$

It is easy to see that M is a locally real-analytic subset of Γ. Moreover, since Γ is essentially finite, $\dim M < 2n - 2$. Choose the coordinate system such that $p = 0$ and the defining function of Γ is given in the so-called normal form (see [CM]):

$$\rho(z, \overline{z}) = 2x_n + \sum_{|k|,|l|\geq 1} \rho_{k,l}(y_n)('z)^k(\overline{'z})^l, \tag{5.4}$$

where $'z = (z_1, \dots, z_{n-1})$. Since the extendability of f through regular points is already established, after possibly an additional change of variables, we may assume that f extends as a holomorphic correspondence to the points $\{z \in \partial U \cap \Gamma : x_1 > 0\}$. Let L_c denote the family of real hyperplanes in the form $\{z \in \mathbb{C}^n : x_1 = c\}$. Then there exists $\epsilon > 0$ such that for any $c \in [-\epsilon, \epsilon]$,

$$T_z^c(\Gamma) \neq T_z(L_c \cap \Gamma), \text{ for any } z \in L_c \cap \Gamma \cap \mathbb{B}(0, \epsilon). \tag{5.5}$$

Let $\Omega_{c,\delta}$ be the intersection of Γ, the δ-neighborhood of x_1-axis and the set bounded by L_c and $L_{c+\delta}$, that is

$$\Omega_{c,\delta} = \left\{ z \in \Gamma \cap \mathbb{B}(0, \epsilon) : c < x_1 < c + \delta,\ y_1^2 + \sum_{j=2}^{n} |z_j|^2 < \delta \right\}. \tag{5.6}$$

Then there exist $\delta > 0$ and $c > 0$, such that f extends as a holomorphic correspondence to a neighborhood of the set $\Omega_{c,\delta}$. Since $L_c \cap \Gamma$ consists only of regular points, from Steps 1 and 2 we conclude that f extends to a neighborhood of any point in $L_c \cap \Gamma$ that belongs to the boundary of $\Omega_{c,\delta}$. Let c_0 be the smallest number such that f extends past $L_c \cap \Gamma$. Then from (5.5) and previous steps, $c_0 < 0$, and therefore, f extends to a neighborhood of the origin. □

We remark that Theorem 4 also holds in the case when Γ is an arbitrary smooth, real-analytic, essentially finite hypersurface. The proof is the same.

6. Proof of Theorem 1

The proof of Theorem 1 in one direction, namely, that existence of a proper holomorphic correspondence $f : D \to D'$ implies the local equivalence of boundaries ∂D and $\partial D'$ near some points $p \in \partial D$ and $p' \in \partial D'$, is the same as in Theorem 2.

To prove the theorem in the other direction let us first show that a germ of a biholomorphic map $f : U \to \mathbb{C}^n$, $f(U \cap \Gamma) \subset \Gamma'$ can be replaced by a complete correspondence. Without loss of generality we assume that $0 \in U \cap \Gamma$. We choose a small neighborhood U_0 of the origin and shrink U in such a way, that $Q_w \cap U$ is non-empty and connected for any $w \in U_0$. Let U' be a neighborhood of Γ' as in Lemma 1. Define

$$A = \left\{ (w, w') \in U_0 \times U' : f(Q_w \cap U) \subset Q'_{w'} \right\}. \tag{6.1}$$

Then (6.1) defines a holomorphic correspondence, which in particular contains the germ of the graph of f at the origin. Let $\hat{f}$ be the multiple valued mapping corresponding to A.

Let us show that $\hat{f}$ is a complete correspondence. It follows from the construction of the set A and from (4.4) that

$$\hat{f}(w) = I'_{w'},\ w' \in \hat{f}(w). \tag{6.2}$$

Thus it only remains to show that for any $z \in \Gamma \cap U_0$ the correspondence $\hat{f}$ maps all connected components of $Q_z \cap U_0$ to the same Segre variety. Note that for any $z \in U_0$ the set $Q_z \cap U$ is connected, and therefore all components of $Q_z \cap U_0$ are mapped by f to the same Segre variety. We claim that

$$f(Q_z \cap U_0) \subset Q'_{z'} \Rightarrow \hat{f}(Q_z \cap U_0) \subset Q'_{z'}. \tag{6.3}$$

Indeed, the first inclusion in (6.3) implies that $z' \in I'_{f(z)}$. If $w \in Q_z \cap U_0$ and $w' \in \hat{f}(w)$, then by construction $f(Q_w \cap U_0) \subset Q'_{w'}$. From (2.2) $z \in Q_w$, and therefore, $f(z) \in Q'_{w'}$. The last inclusion implies that $w' \in Q'_{f(z)}$. Since w' was arbitrary, (6.3) follows. Finally, (6.2) and (6.3) imply that $\hat{f}$ is a complete correspondence.

If ∂U_0 is smooth, then by Theorem 4 we can locally extend $\hat{f}$ along Γ past the boundary of $U_0 \cap \Gamma$ to a larger open set Ω. This process of local analytic continuation of $\hat{f}$ past the boundary of Ω can be continued indefinitely. However, local extension in general does not imply that A is a closed set in $\Omega \times \mathbb{C}^n$. Indeed, there may exist a point $p \in \Gamma \cap \partial\Omega$ such that for any sufficiently small neighborhood V of p, $\Omega \cap V \cap \Gamma$ consists of two connected components, say Γ_1 and Γ_2. Then local extension from Γ_1 to a neighborhood of p may not coincide with the correspondence $\hat{f}$ defined in Γ_2. Therefore, local extension past the boundary of $\Omega \cap \Gamma$ does not lead to a correspondence defined globally in a neighborhood of $\overline{\Omega} \times \mathbb{C}^n$. Note that this cannot happen if Ω is sufficiently small.

We now show that $\hat{f}$ extends analytically along any path on Γ.

Lemma 7. *Let $\gamma : [0, 1] \to \Gamma$ be a simple curve without self-intersections, and $\gamma(0) = 0$. Then there exist a neighborhood V of γ and a holomorphic correspondence $F : V \to \mathbb{C}^n$ which extends $\hat{f}$.*

Proof. Suppose that $\hat{f}$ does not extend along γ. Then let $\zeta \in \gamma$ be the first point to which $\hat{f}$ does not extend. Let $\epsilon_0 > 0$ be so small that $\mathbb{B}(\zeta, \epsilon) \cap \Gamma$ is connected and simply connected for any $\epsilon \le \epsilon_0$. Choose a point $z \in B(\zeta, \epsilon_0/2) \cap \gamma$ to which $\hat{f}$ extends. Let δ be the largest positive number such that $\hat{f}$ extends holomorphically to $\mathbb{B}(z, \delta) \cap \Gamma$. By Theorem 4 $\hat{f}$ extends to a neighborhood of every point in $\partial\mathbb{B}(z, \delta) \cap \Gamma$. Moreover, if $\mathbb{B}(z, \delta) \subset \mathbb{B}(\zeta, \epsilon_0)$, then the extension of $\hat{f}$ is a closed complex analytic set. Thus $\delta > \epsilon/2$. This shows that $\hat{f}$ also extends to ζ, and therefore extends along γ. □

Note that analytic continuation of $\hat{f}$ along γ in Lemma 7 always yields a complete correspondence.

The Monodromy theorem cannot be directly applied for multiple valued mappings, and we need to show that analytic continuation is independent of the choice of a curve connecting two points on Γ.

Lemma 8. *Suppose that $\gamma \subset \Gamma$ is a Jordan curve $\gamma(0) = \gamma(1) = 0$. Let F be the holomorphic correspondence defined near the origin and obtained by analytic continuation of $\hat{f}$ along γ. Then $F = \hat{f}$ in some neighborhood of the origin.*

Proof. Since Γ is simply connected and compact, there exists $\epsilon_0 > 0$ such that for any $z \in \Gamma$, $\mathbb{B}(z, \epsilon) \cap \Gamma$ is connected and simply connected for any $\epsilon \le \epsilon_0$.

Let ϕ be the homotopy map, that is $\phi(t, \tau) : I \times I \to \Gamma$, $\phi(t, 0) = \gamma(t)$, $\phi(t, 1) \equiv 0$, $I = [0, 1]$. Let $\{(t_j, \tau_k) \in I \times I,\ j, k = 0, 1, 2, \dots, m\}$ be the set of points satisfying:

(i) $t_0 = \tau_0 = 0$, $t_m = \tau_m = 1$,
(ii) $\{\phi(t, \tau_k) : t_j \le t \le t_{j+1}\} \subset \mathbb{B}(\phi(t_j, \tau_k), \epsilon_0/2)$,
$\{\phi(t_j, \tau) : \tau_k \le \tau \le \tau_{k+1}\} \subset \mathbb{B}(\phi(t_j, \tau_k), \epsilon_0/2)$, for any $j, k < m$.

Suppose that f is a complete holomorphic correspondence defined in a ball B of small radius centered at $\phi(t_j, \tau_k) \in \Gamma$. By Theorem 4, f extends holomorphically past every boundary point of $\partial B \cap \Gamma$. Since $B(\phi(t_j, \tau_k), \epsilon_0)$ is connected and simply connected, f extends at least to a ball of radius $\epsilon_0/2$. Consider the closed path $\gamma_{j,k} = \{\phi(t, \tau_k) : t_j \le t \le t_{j+1}\} \cup \{\phi(t_{j+1}, \tau) : \tau_k \le \tau \le \tau_{k+1}\} \cup \{\phi(t, \tau_{k+1}) : t_j \le t \le t_{j+1}\} \cup \{\phi(t_j, \tau) : \tau_k \le \tau \le \tau_{k+1}\}$, where the second and fourth pieces are traversed in the opposite direction. Then $\gamma_{j,k}$ is entirely contained in $(B(\phi(t_j, \tau_k), \epsilon_0/2)$. Therefore, analytic continuation of f along $\gamma_{j,k}$ defines the same correspondence at $\phi(t_j, \tau_k)$.

Analytic continuation of $\hat{f}$ along γ can be reduced to continuation along paths $\gamma_{j,k}$. Since continuation along each path $\gamma_{j,k}$ does not introduce new branches of $\hat{f}$, $F = \hat{f}$. □

Thus simple connectivity of Γ implies that the process of local extension of $\hat{f}$ leads to a global extension of $\hat{f}$ to some neighborhood of Γ. Since $\hat{f}(\Gamma) \subset \Gamma'$, there exist neighborhoods U of Γ and U' of Γ' such that $\hat{f} : U \to U'$ is a proper holomorphic correspondence. Let A be the analytic set corresponding to $\hat{f}$. By (2.10) there exist functions ϕ_{IJ} holomorphic in U such that A is determined from the system $\sum_{|J| \le m} \phi_{IJ}(z) {z'}^J = 0$. By Hartog's theorem all ϕ_{IJ} extend holomorphically to a neighborhood of $\overline{D}$, (recall that $\Gamma = \partial D$). This defines a proper holomorphic correspondence $f : D \to D'$. □

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