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Dicritical singularities and laminar currents on Levi-flat hypersurfaces

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Abstract. We establish an effective criterion for a dicritical singularity of a real analytic Levi-flat hypersurface. The criterion is stated in terms of Segre varieties. As an application, we obtain a structure theorem for a certain class of currents in the non-dicritical case.

Keywords: Levi-flat set, dicritical singularity, foliation, current.

§ 1. Introduction

The study of Levi-flat hypersurfaces arises naturally in several areas of complex geometry. Our approach is inspired by the theory of holomorphic foliations. This aspect of Levi-flat geometry has been considered by several authors [1]–[9]. By a classical theorem of E. Cartan, a non-singular real analytic Levi-flat hypersurface is locally biholomorphic to a real hyperplane. The present paper studies local properties of Levi-flat hypersurfaces near singular points.

Our main result (Theorem 3.1) gives a complete effective characterization of dicritical singular points of a Levi-flat real analytic hypersurface in terms of the geometry of its Segre varieties. This answers a question communicated to the second and third authors by Jiri Lebl (see also [8]). As an application, we prove a structure theorem for currents supported on non-dicritical hypersurfaces (Proposition 4.2).

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§ 2. Real analytic Levi-flat hypersurfaces in $\mathbb{C}^n$

2.1. Real analytic sets and their complexifications. Let $\Omega \subset \mathbb{R}^n$ be a domain. A real analytic set $\Gamma \subset \Omega$ is a closed set locally defined as the zero locus of a finite collection of real analytic functions. In fact, we can always take just one function as locally defining any real analytic set. We say that Γ is *irreducible* in Ω if it cannot be represented as the union $\Gamma = \Gamma_1 \cup \Gamma_2$ of two real analytic sets Γ_j in Ω

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with $\Gamma_j \setminus (\Gamma_1 \cap \Gamma_2) \neq \varnothing$, $j = 1, 2$ (this is geometric irreducibility). Γ is called a *real hypersurface* if there is a point $q \in \Gamma$ such that, near q, Γ is a real analytic submanifold of dimension $n-1$. Such points q are called *regular* points of the real hypersurface Γ. The set of all regular points is called the regular locus and is denoted by Γ^*. Its complement $\Gamma_{\text{sing}} := \Gamma \setminus \Gamma^*$ is called the *singular locus* of Γ. Note that our convention is different from the usual definition of a regular point in semianalytic or subanalytic geometry, where a similar notion is less restrictive and a real analytic set is allowed to be a submanifold of *some* dimension near a regular point. By our definition, those points of Γ, where Γ is a submanifold of dimension smaller than $n-1$, belong to the singular locus. Therefore Γ^* need not be dense in Γ, and this can happen even for irreducible Γ (the so-called umbrellas). Note that Γ_{sing} is a closed semianalytic subset of Γ (possibly empty) of real dimension at most $n-2$.

In local questions, we are interested in the geometry of a real hypersurface Γ in an arbitrarily small neighbourhood of a given point $a \in \Gamma$, that is, the geometry of the germ of Γ at a. If the germ is irreducible at a, we may consider a sufficiently small neighbourhood U of a and the representative of the germ which is irreducible at a (see [10] for details). In what follows we will not distinguish between the germ of Γ at a given point a and its particular representative in a suitable neighbourhood of a.

Let $\Gamma \subset \mathbb{R}^n_x$ be the germ of a real analytic set at the origin. We denote by $\Gamma^{\mathbb{C}}$ the complexification of Γ, that is, a complex analytic germ at the origin in $\mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$, $z = x + iy$, with the property that every holomorphic function that vanishes on Γ, necessarily vanishes on $\Gamma^{\mathbb{C}}$. Equivalently, $\Gamma^{\mathbb{C}}$ is the smallest complex analytic germ in $\mathbb{C}^n$ that contains Γ. It is well known that the dimension of Γ equals the complex dimension of $\Gamma^{\mathbb{C}}$ and that the germ of $\Gamma^{\mathbb{C}}$ is irreducible at the origin whenever the germ of Γ is irreducible (see Narasimhan's book [10] for further details and proofs). Also, given a real analytic germ $\sum_{|j|\geqslant 0} a_j\, x^j$, $a_j \in \mathbb{R}$, $x \in \mathbb{R}^n$, we define its complexification to be the complex analytic germ $\sum a_j z^j$.

While the complexification of a germ of a real analytic set is canonical and is independent of the choice of the defining function, the following lemma gives a convenient way of constructing the complexification of a real analytic hypersurface using a suitably chosen defining function. We will need the following notion of a minimal defining function for a complex hypersurface. Given a complex hypersurface $A = \{z \in \Omega \colon f(z) = 0\}$ in a domain $\Omega \subset \mathbb{C}^n$, f is said to be *minimal* if, for every open subset $U \subset \Omega$ and every holomorphic function g on U such that $g = 0$ on $A \cap U$, there is a holomorphic function h on U such that $g = hf$. If f is a minimal defining function, then the singular locus of A coincides with the set $f = df = 0$. Locally, every irreducible complex hypersurface admits a minimal defining function (see the book by Chirka [11]).

Lemma 2.1. *Let $\Gamma \subset \mathbb{R}^n$ be an irreducible germ of a real analytic hypersurface at the origin. Then there is a defining function $\rho(x)$ of the germ of Γ at the origin such that its complexification $\widehat{\rho}(z)$ is a minimal defining function of the complexification $\Gamma^{\mathbb{C}}$.*

Proof. Since the germ of Γ is irreducible, the complexification $\Gamma^{\mathbb{C}}$ is an irreducible germ of a complex hypersurface in $\mathbb{C}^n$. It admits a minimal defining function at the origin, $F(z) = \sum_{|j|>0} c_j z^j$. Let $c_j = a_j + ib_j,\ a_j, b_j \in \mathbb{R}$. Let $\widehat{f}(z) = \sum a_j z^j$, $\widehat{g}(z) = \sum b_j z^j$, so that $F = \widehat{f} + i\widehat{g}$. Then $\widehat{f}$ and $\widehat{g}$ are the complexifications of the real analytic germs $f(x) = \sum a_j x^j$ and $g(x) = \sum b_j x_j$ respectively. Moreover, since $F(z)\big|_{\mathbb{R}^n_x} = f + ig$ and $F(x)$ vanishes on Γ, we conclude that both f and g vanish on Γ and, therefore, $\widehat{f}$ and $\widehat{g}$ vanish on $\Gamma^{\mathbb{C}}$. Since F is a minimal defining function for $\Gamma^{\mathbb{C}}$, there are unique holomorphic germs h_1 and h_2 such that $\widehat{f} = h_1 F$ and $\widehat{g} = h_2 F$. But then $F = (h_1 + ih_2)F$, that is, $h_1 + ih_2 = 1$ identically. Hence at least one of these functions, say h_1, does not vanish at the origin. It follows that $F = {h_1}^{-1}\widehat{f}$, that is, $\widehat{f}$ is also a minimal defining function of $\Gamma^{\mathbb{C}}$. Thus $\rho = f$ is the desired choice of a defining function of Γ. $\Box$

2.2. Levi-flat hypersurfaces. Let $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$, be the standard coordinates on $\mathbb{C}^n$. Let Γ be an irreducible germ of a real analytic hypersurface at the origin defined by a function ρ provided by Lemma 2.1. In a (connected) sufficiently small neighbourhood $\Omega \subset \mathbb{C}^n$ of the origin, the hypersurface Γ is a closed irreducible real analytic subset of Ω of dimension $2n - 1$.

For $q \in \Gamma^*$ consider the complex tangent space $H_q(\Gamma) := T_q(\Gamma) \cap JT_q(\Gamma)$. The *Levi form* of Γ is a Hermitian quadratic form defined on $H_q(\Gamma)$ by the formula

$$L_q(v) = \sum_{k,j} \rho_{z_k \overline{z}_j}(q) v_k \overline{v}_j$$

for all $v \in H_q(\Gamma)$. A real analytic hypersurface Γ is said to be *Levi-flat* is its Levi form vanishes on $H_q(\Gamma)$ for every regular point q of Γ. By a classical result of Elie Cartan, for every point $q \in \Gamma^*$ there is a local biholomorphic change of coordinates centred at q such that, in the new coordinates, Γ has the form $\{z \in U\colon z_n + \overline{z}_n = 0\}$ in some neighbourhood U of the point $q = 0$. Hence, $\Gamma \cap U$ is locally foliated by complex hyperplanes $\{z_n = c,\, c \in i\,\mathbb{R}\}$. This foliation is called the *Levi foliation* of Γ^* and will be denoted by $\mathcal{L}$. We denote by $\mathcal{L}_q$ the leaf of the Levi foliation through q. Note that, by definition, it is a connected complex hypersurface and is closed in Γ^*.

Let $0 \in \overline{\Gamma^*}$. We choose a neighbourhood Ω of the origin in the form of a polydisc $\Delta(\varepsilon) = \{z \in \mathbb{C}^n \colon |z_j| < \varepsilon\}$ of radius $\varepsilon > 0$. Then, for ε small enough, the function ρ admits the convergent Taylor expansion in U:

$$\rho(z, \overline{z}) = \sum_{IJ} c_{IJ} z^I \overline{z}^J, \qquad c_{IJ} \in \mathbb{C}, \quad I, J \in \mathbb{N}^n. \tag{1}$$

The coefficients c_{IJ} satisfy the condition

$$\overline{c}_{IJ} = c_{JI} \tag{2}$$

because the function ρ is real-valued. Note that in local questions we may further shrink Ω if necessary.

For real analytic sets in complex manifolds, it is more convenient to define the complexification as follows. Denote by J the standard complex structure on $\mathbb{C}^n_z$ and define J' on $\mathbb{C}^n_w$ by the formula $J'w = -iw$. We equip $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_w$ with the complex structure $J \otimes J'$. Then the map $\iota\colon \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ given by $z \to (z, z)$ is a totally real embedding of $\mathbb{C}^n$ in $(\mathbb{C}^{2n}, J \otimes J')$. We define the complexification of a real analytic germ $\Gamma \subset \mathbb{C}^n$ to be the smallest complex analytic germ in $\mathbb{C}^{2n}$ that contains $\iota(\Gamma)$. This construction is equivalent to the definition given above. Hence all the properties of the standard complexification are preserved. Now, given a real analytic germ ρ as in (1), we define its complexification as

$$\rho(z, \overline{w}) = \sum_{IJ} c_{IJ} z^I \overline{w}^J, \tag{3}$$

that is, we replace the variable $\overline{z}$ by an independent variable $\overline{w}$. Let $\varepsilon > 0$ be chosen so small that the series (3) converges for all $(z, w) \in \Delta(\varepsilon) \times \Delta(\varepsilon)$. Note that $\rho(z, \overline{w})$ is a holomorphic function of (z, w) by the choice of the complex structure on $\mathbb{C}^{2n}$. If the reader prefers to work with the standard structure on $\mathbb{C}^{2n}$, then $\overline{w}$ should be replaced by w where appropriate.

By Lemma 2.1, the choice of the defining function ρ guarantees that the complexification of (the germ of) Γ is given by

$$\Gamma^{\mathbb{C}} = \big\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \colon \rho(z, \overline{w}) = 0\big\}. \tag{4}$$

The hypersurface Γ lifts canonically to $\Gamma^{\mathbb{C}}$ by the formula

$$\widehat{\Gamma} = \Gamma^{\mathbb{C}} \cap \{w = z\}.$$

In what follows we write $\Gamma^{\mathbb{C}}_{\mathrm{sing}}$ for the singular locus of $\Gamma^{\mathbb{C}}$.

2.3. Segre varieties. Our key tool is the family of *Segre varieties* associated with a real analytic hypersurface Γ. For $w \in \Delta(\varepsilon)$ consider the complex analytic hypersurface given by

$$Q_w = \big\{z \in \Delta(\varepsilon) \colon \rho(z, \overline{w}) = 0\big\}. \tag{5}$$

It is called the *Segre variety* of the point w. This definition uses the defining function ρ of Γ in a neighbourhood of the origin which appears in (4). We always consider the case when the germ of Γ at the origin is irreducible. Throughout the paper, we use the defining function provided by Lemma 2.1 in a neighbourhood of the origin (the same convention is used in [9]). In general, the Segre varieties Q_w also depend on the choice of ε (some irreducible components of Q_w may disappear when we shrink ε). Throughout the paper, we consider only the Segre varieties Q_w defined by means of the complexification at the origin. The reader should keep this in mind. We also note that if 0 is a regular point of Γ, then the notion of the Segre variety Q_w is independent of the choice of a defining function ρ with non-vanishing gradient when w is close enough to the origin.

The following properties of Segre varieties are immediate.

Lemma 2.2. *Let Γ be a germ of an irreducible real analytic hypersurface in $\mathbb{C}^n$, $n > 1$. Then the following assertions hold.*

a) *$z \in Q_z$ if and only if $z \in \Gamma$.*

b) *$z \in Q_w$ if and only if $w \in Q_z$.*

We also recall the property of local biholomorphic invariance of some distinguished components of Segre varieties near regular points. Since we are working near a singularity, we state this property in detail using the notation introduced above. Consider a regular point $a \in \Gamma^* \cap \Delta(\varepsilon)$ and fix $\alpha > 0$ small enough with respect to ε. Let ρ_a be any real analytic function on the polydisc $\Delta(a,\alpha) = \{|z_j - a_j| < \alpha,\, j = 1,\dots,n\}$ such that $\Gamma \cap \Delta(a,\alpha) = \rho_a^{-1}(0)$ and the gradient of ρ_a does not vanish on $\Delta(a,\alpha)$. Then for $w \in \Delta(a,\alpha)$ we can define the Segre variety ${}^aQ_w$ ('the Segre variety with respect to the regular point a') as

$$ {}^aQ_w = \bigl\{z \in \Delta(a,\alpha) \colon \rho_a(z,\overline{w}) = 0\bigr\} $$

(we use the Taylor series of ρ_a at a to define the complexification). For α small enough, ${}^aQ_w$ is a connected non-singular complex submanifold of dimension $n-1$ in $\Delta(a,\alpha)$. This definition is independent of the choice of the local defining function ρ_a satisfying the properties above. We have the inclusion ${}^aQ_w \subset Q_w$. Note that in general Q_w can have irreducible components in $\Delta(\varepsilon)$ which do not contain ${}^aQ_w$.

Lemma 2.3 (invariance property). *Suppose that Γ, Γ' are irreducible germs of real analytic hypersurfaces, $a \in \Gamma^*$, $a' \in (\Gamma')^*$, and $\Delta(a,\alpha)$, $\Delta(a',\alpha')$ are small polydiscs. Let $f \colon \Delta(a,\alpha) \to \Delta(a',\alpha')$ be a holomorphic map such that $f(\Gamma \cap \Delta(a,\alpha)) \subset \Gamma' \cap \Delta(a',\alpha')$ and $f(a) = a'$. Then*

$$ f({}^aQ_w) \subset {}^{a'}Q'_{f(w)} $$

for all $w \in \Delta(a,\alpha)$ close enough to a. In particular, if $f \colon \Delta(a,\alpha) \to \Delta(a',\alpha')$ is biholomorphic, then $f({}^aQ_w) = {}^{a'}Q'_{f(w)}$. Here ${}^aQ_w$ and ${}^{a'}Q'_{f(w)}$ are the Segre varieties associated with Γ and Γ' at the points a and a' respectively.

For a proof see, for example, [12]. As a simple consequence of Lemma 2.2, we have the following corollary.

Corollary 2.4. *Let $\Gamma \subset \mathbb{C}^n$ be an irreducible germ of a real analytic Levi-flat hypersurface at the origin. Let $a \in \Gamma^*$. Then the following assertions hold.*

a) *There is a unique irreducible component S_a of Q_a containing the leaf $\mathcal{L}_a$. It is also the unique complex hypersurface through a which is contained in Γ.*

b) *For every $a, b \in \Gamma^*$ we have $b \in S_a \iff S_a = S_b$.*

c) *Suppose that $a \in \Gamma^*$ and $\mathcal{L}_a$ touches a point $q \in \Gamma$ such that $\dim_{\mathbb{C}} Q_q = n-1$ (the point q may be singular). Then Q_q contains S_a as an irreducible component.*

A proof is contained in [9]. We again emphasize that Corollary 2.4 concerns 'global' Segre varieties, that is, those defined by (5) using complexification at the origin.

§ 3. Characterization of dicritical singularities of Levi-flat hypersurfaces

Let Γ be an irreducible germ of a real analytic Levi-flat hypersurface in $\mathbb{C}^n$ at $0 \in \overline{\Gamma^*}$. Fix a local defining function ρ chosen in accordance with Lemma 2.1, so that the complexification $\Gamma^{\mathbb{C}}$ is an irreducible germ of the complex hypersurface in $\mathbb{C}^{2n}$ given as the zero locus of the complexification of ρ. As already mentioned, all Segre varieties considered are defined by means of this complexification at the origin.

We also fix a sufficiently small $\varepsilon > 0$. All considerations are in the polydisc $\Delta(\varepsilon)$ centred at the origin. A point $q \in \overline{\Gamma^*} \cap \Delta(\varepsilon)$ is called a *dicritical* singularity if q belongs to the closures of infinitely many geometrically distinct leaves $\mathcal{L}_a$. Singular points in $\overline{\Gamma^*}$ which are not dicritical are said to be *non-dicritical*.

A singular point q is said to be *Segre degenerate* if $\dim Q_q = n$. We recall that the Segre degenerate points form a complex analytic subset of $\Delta(\varepsilon)$ of complex dimension at most $n-2$. In particular, it is a discrete set if $n = 2$. For a proof, see [7], [9]. The main result of this paper is the following theorem.

Theorem 3.1. *Let $\Gamma = \rho^{-1}(0)$ be an irreducible germ of a real analytic Levi-flat hypersurface at the origin of $\mathbb{C}^n$ and $0 \in \overline{\Gamma^*}$. Then 0 is a dicritical point if and only if it is Segre degenerate.*

Proof. A dicritical point is Segre degenerate; this follows from Corollary 2.4, c). We now prove that if the origin is a Segre degenerate point, then it is dicritical. The proof is divided into four steps.

Step 1. Canonical Segre varieties. Consider the canonical projection

$$\pi\colon \Gamma^{\mathbb{C}} \to \mathbb{C}^n, \qquad \pi\colon (z,w) \mapsto w.$$

Then $Q_w = \pi^{-1}(w)$ for every w. Denote by Q_w^c the union of all irreducible components of Q_w containing the origin. We call this set *the canonical Segre variety* of w. Note that for all w in a neighbourhood of the origin in $\mathbb{C}^n$, the canonical Segre variety Q_w^c is a non-empty complex analytic hypersurface. Indeed, since 0 is a Segre degenerate singularity, it follows that $w \in Q_0 = \mathbb{C}^n$ and we obtain by Lemma 2.2, b) that $0 \in Q_w$.

Consider the set

$$\Sigma = \bigl\{(z,w) \in \Gamma^{\mathbb{C}}\colon z \notin Q_w^c\bigr\}.$$

If Σ is empty, then for every point w in a neighbourhood of the origin, the Segre variety Q_w coincides with the canonical Segre variety Q_w^c, that is, all the components of Q_w contain the origin. But for a regular point w of Γ, the closure of its Levi leaf is a component of Q_w. Hence the origin is contained in the closure of every Levi leaf and, therefore, is a dicritical point. Our goal is to prove that Σ is empty. Arguing by contradiction, we assume that Σ is non-empty. Observe that Σ is open in $\Gamma^{\mathbb{C}}$. This follows immediately from the fact that the defining function of the complex hypersurface Q_w depends continuously on the parameter w.

To prove the theorem, we are going to show that the boundary of Σ is contained in a proper complex analytic subset of $\Gamma^{\mathbb{C}}$. To do this we define the set

$$X = \{(z, w) \in \Gamma^{\mathbb{C}} \colon \dim Q_w = n\}.$$

As shown in [7], [9], the set X is contained in a complex analytic subset of $\Gamma^{\mathbb{C}}$ of dimension at most $2n-2$.

Let (z^k, w^k) be a sequence of points in Σ converging to some point (z^0, w^0). Without loss of generality we may assume that $(z^0, w^0) \in \Gamma^{\mathbb{C}} \setminus (\Gamma^{\mathbb{C}}_{\text{sing}} \cup X)$ and that (z^0, w^0) does not belong to Σ. Since (z^0, w^0) does not belong to X, we conclude that w^0 is not a dicritical singularity. Then Q_{w^0} is a complex hypersurface (in general reducible) passing through the origin, and $z^0 \in Q^c_{w_0}$.

Step 2. Analytic representation of Segre varieties. We use the notation $z = (z', z_n) = (z_1, \dots, z_{n-1}, z_n)$. Performing a complex linear change of coordinates in $\mathbb{C}^n_z$ if necessary, we can assume that the intersection of Q_{w^0} with the z_n-coordinate complex line $(0', \mathbb{C})$ is a discrete set. Then the intersection of $\Gamma^{\mathbb{C}}$ with the complex line $\{(0', \mathbb{C}, w_0)\}$ is also discrete. Let

$$\widetilde{\pi}(z', z_n, w) = (z', w)$$

be the coordinate projection. Choose a neighbourhood U of the origin in $\mathbb{C}^n_z$ and a neighbourhood V of w^0 in $\mathbb{C}^n_w$ with the following properties.

i) $U = U' \times \delta\mathbb{D}$, where U' is a neighbourhood of the origin in $\mathbb{C}^{n-1}_{z'}$, and $\mathbb{D}$ is the unit disc in $\mathbb{C}$. Choose $\delta > 0$ so small that

$$\{|z_n| < \delta\} \cap \Gamma^{\mathbb{C}} \cap \widetilde{\pi}^{-1}(0', w_0) = \{(0, w_0)\}.$$

ii) The projection $\widetilde{\pi} \colon \Gamma^{\mathbb{C}} \cap (U \times V) \mapsto U' \times V$ is proper.

We apply the Weierstrass preparation theorem to the equation (4) on the neighbourhood $U \times V$ of the point $(0, w_0) \in \Gamma^{\mathbb{C}}$ to obtain

$$\begin{aligned} \Gamma^{\mathbb{C}} = \{&(z, w) \in U \times V \colon \\ &P(z', \overline{w})(z_n) := z_n^d + a_{d-1}(z', \overline{w}) z_n^{d-1} + \dots + a_0(z', \overline{w}) = 0\}, \end{aligned} \tag{6}$$

where the coefficients $a_j(z', \overline{w})$ are holomorphic in $U' \times V$. Note that $a_0(0', \overline{w}) = 0$ for all w because every Segre variety contains the origin. The Segre varieties are then obtained by fixing w in the above equation:

$$Q_w \cap U = \{z \in U \colon P(z', \overline{w})(z_n) = 0\}, \qquad w \in V. \tag{7}$$

Step 3. Boundary points of $\boldsymbol{\Sigma}$. We noted in Step 1 that Σ is open in $\Gamma^{\mathbb{C}}$. Here we shall show that, in a neighbourhood of (z^0, w^0), the boundary of Σ is contained in a proper analytic subset of $\Gamma^{\mathbb{C}}$.

We need an analytic representation of $\Gamma^{\mathbb{C}}$ similar to (6) but in a neighbourhood of the point (z^0, w^0). Performing a linear change of coordinates (arbitrarily close to the identity map) in $\mathbb{C}^n_z$, we can assume that Step 2 holds and also the intersection of Q_{w^0} and the complex line $(z_1^0, \dots, z_{n-1}^0, \mathbb{C})$ is discrete. As in Step 2, there is a neighbourhood O' of $(z_1^0, \dots, z_{n-1}^0)$ in $\mathbb{C}^{n-1}$ and a $\delta' > 0$ such that $\Gamma^{\mathbb{C}} \cap (O \times V)$ is the zero set of some Weierstrass polynomial $\widetilde{P}(z', \overline{w})(z_n - z_n^0)$. Here $O = O' \times \delta' \mathbb{D}$ and V is the same neighbourhood of w^0 as in Step 2 (this can be achieved by shrinking V if necessary). The polynomial $\widetilde{P}$ has an expansion similar to (6) with $(z_n - z_n^0)$ instead of z_n, and its coefficients are holomorphic in $O' \times V$. For the Segre varieties Q_w, $w \in V$, we have

$$Q_w \cap O = \bigl\{ z \in O \colon \widetilde{P}(z', \overline{w})(z_n - z_n^0) = 0 \bigr\}. \tag{8}$$

We now consider the discriminant $R(z', w)$ of the polynomial $\widetilde{P}$, that is, the resultant of $\widetilde{P}$ and its derivative with respect to z_n (see, for example, [11]). The function R is holomorphic in $O' \times V$. We define the discriminant set as

$$Y = \bigl\{ (z, w) \in \Gamma^{\mathbb{C}} \cap (O \times V) \colon R(z', \overline{w}) = 0 \bigr\}. \tag{9}$$

The projection of the set Y to $\mathbb{C}^{n-1}_{z'} \times \mathbb{C}^n_w$ is formed by the points $(z', \overline{w})$ such that the polynomial $\widetilde{P}(z', \overline{w})$ has multiple roots. The set Y is a complex analytic subset of codimension 1 in $\Gamma^{\mathbb{C}} \cap (O \times V)$. We have the inclusion $\Gamma^{\mathbb{C}}_{\mathrm{sing}} \cap (O \times V) \subset Y$. In general, this inclusion is strict (see, for example, [11]).

We now use again the neighbourhood U of the origin in $\mathbb{C}^n_z$ and the neighbourhood V of w^0 defined in Step 2, so that the conditions i), ii) of Step 2 hold. In particular, $Q_w \cap U$ is given by (7) for all $w \in V$. Set $z' = 0$ in (7). This defines an algebroid d-valued function of $w \in V$, that is, an algebraic element over the commutative integral domain of functions holomorphic on V. More precisely, consider the pairs $(\zeta, w) \in \mathbb{C} \times V$ satisfying the equation

$$\zeta^d + a_{d-1}(0', \overline{w}) \zeta^{d-1} + \dots + a_0(0', \overline{w}) = 0, \tag{10}$$

where a_j are the coefficients of the polynomial P in (6). This equation defines an algebroid (d-valued) function $w \mapsto \zeta(w)$ (in other words, ζ is a holomorphic correspondence defined on V and with values in $\mathbb{C}$). The complex hypersurface determined by the equation (10) in $\mathbb{C} \times V$ is a branched analytic covering over V, and we can define branches of the algebroid function ζ in the standard way as holomorphic functions on an arbitrary simply connected subdomain in V disjoint from the branch locus; see [11].

Furthermore, with every point $w \in V$ the algebroid function ζ associates the set $\zeta(w) = (\zeta_1(w), \dots, \zeta_s(w))$, $s = s(w) \leqslant d$, of all (distinct) roots of the equation (10); we refer to them as the values of ζ at w. Since $a_0(0', \overline{w})$ vanishes identically in w (recall that every Segre variety Q_w contains the origin), one of the branches of ζ is identically equal to zero. In particular, the polynomial (10) is reducible. On the other hand, the function ζ has branches which are not identically equal to zero. Indeed, $(z^k, w^k) \in \Sigma$, so that the irreducible components of Q_{w^k} containing z^k do not contain the origin. Therefore the equation (10) has non-zero solutions when $w = w^k$. In particular, $a_i(0', w^k) \neq 0$ for at least one i. Let j be the smallest non-negative integer such that the coefficient $a_j(0', \overline{w})$ does not vanish identically. Dividing the equation (10) by ζ^j, we obtain

$$\zeta^{d-j} + a_{d-1}(0', \overline{w})\, \zeta^{d-j-1} + \cdots + a_j(0', \overline{w}) = 0. \tag{11}$$

Thus, all the non-zero values of the algebroid function ζ at w are solutions of this equation.

Note that 0 is one of the roots of the equation (11) for some w if and only if $a_j(0', \overline{w}) = 0$. Define the set

$$A = \big\{(z, w) \in \Gamma^{\mathbb{C}} \colon a_j(0', \overline{w}) = 0\big\}. \tag{12}$$

This is a complex analytic subset of codimension 1 in $\Gamma^{\mathbb{C}}$.

Lemma 3.2. *The boundary of Σ in a neighbourhood of (z^0, w^0) is contained in the union $A \cup X \cup Y$.*

Proof. It suffices to consider the case when the point (z^0, w^0) does not belong to $X \cup Y$. We use the neighbourhoods $O \ni z^0$ and $V \ni w^0$ defined at the beginning of Step 3. We also use the representation (8) for $Q_w \cap O$ with $w \in V$.

Since the point (z^0, w^0) is not in Y, the polynomial $\widetilde{P}((z^0)', \overline{w}^0)(z_n - z_n^0)$ in (8) has no multiple roots. It follows that this point is regular for $\Gamma^{\mathbb{C}}$ and that z^0 is a regular point of the Segre variety Q_{w^0}. The points (z^k, w^k) also do not belong to Y for sufficiently large k and are regular points for $\Gamma^{\mathbb{C}}$ and for Q_{w^k}.

Let $K_1(w), \dots, K_m(w)$ be the irreducible components of Q_w, $w \in V$. The point (z^0, w^0) belongs to exactly one of these components, say, to $K_1(w^0)$. Since Q_{w^0} has the maximal number of branches over the point $(z^0)'$, no distinct components $K_\nu(w^k)$, $\nu = 1, \dots, m$, of Q_{w^k} can glue together as w^k tends to w^0. Therefore, $K_1(w^0) \cap O$ is an irreducible component of the limit set (in the Hausdorff distance) of exactly one of these components as $w^k \to w^0$. By the uniqueness theorem for irreducible complex analytic sets, this property holds globally (not only in O). In particular, it holds in a neighbourhood of the origin in $\mathbb{C}^n$. We denote this component by $K_1(w^k)$. Note that $K_1(w^k)$ is the unique component containing z^k for k big enough.

It follows from the representations (6) and (7) that for every $w = w^k$ or $w = w^0$, the fibre $\widetilde{\pi}^{-1}(0', w) \cap K_1(w)$ is a finite set. We write it in the form $\{p^1(w), \dots, p^l(w)\}$, $l = l(k) \leqslant d$. Since $K_1(w^k)$ is a component of Q_{w^k}, each $p_n^\mu(w^k)$ is a value of the algebroid function ζ at w^k, that is, it belongs to the set $\zeta(w^k)$. We recall that $(z^k, w^k) \in \Sigma$, and the component $K_1(w^k)$ does not contain the origin. It follows that $p_n^\mu(w^k) \neq 0$ for all $\mu = 1, \dots, l$. Hence all the values $p_n^\mu(w^k)$ satisfy the equation (11) with $w = w^k$. By the choice of $K_1(w^k)$, the set $(p_n^1(w^0), \dots, p_n^l(w^0))$ is contained in the limit set of the sequence $(p_n^1(w^k), \dots, p_n^l(w^k))$ as $w^k \to w^0$. Therefore, every $p_n^\mu(w^0)$ satisfies the equation (11) with $w = w^0$. But the point (z^0, w^0) does not belong to Σ and the component $K_1(w^0)$ necessarily contains the origin. This means that $p_n^\mu(w^0) = 0$ for at least one μ. We obtain that $a_j(0', \overline{w}^0) = 0$ and $(z^0, w^0) \in A$. $\square$

By the Remmert–Stein removable singularity theorem, the closure $\overline{\Sigma}$ of Σ coincides with an irreducible component of $\Gamma^{\mathbb{C}}$. Since $\Gamma^{\mathbb{C}}$ is irreducible, we obtain that $\overline{\Sigma}$ coincides with $\Gamma^{\mathbb{C}}$.

Step 4. The complement of Σ has non-empty interior. We begin by choosing a suitable point $\widehat{w}$. First assume that $(\widehat{z}, \widehat{w})$ is a regular point of $\Gamma^{\mathbb{C}}$ and $(\widehat{z}, \widehat{w})$ is not in X. Fix a sufficiently small neighbourhood W of $\widehat{w}$. Then for all Segre varieties Q_w, $w \in W$, the number of their irreducible components is bounded above uniformly in w. Let m be the maximal number of components of Q_w for $w \in W$. Slightly perturbing $\widehat{w}$ (and $\widehat{z}$), one can assume that $Q_{\widehat{w}}$ has exactly m geometrically distinct components. Then there is a neighbourhood V of $\widehat{w}$ such that Q_w has exactly m components for all $w \in V$. Let $K_1(\widehat{w}), \dots, K_m(\widehat{w})$ be the irreducible components of $Q_{\widehat{w}}$. Note that the components $K_j(w)$ depend continuously on w in V.

Consider the sets $F_j = \{w \in V \colon 0 \in K_j(w)\}$. Every set F_j is closed in V. Since $0 \in Q_w$ for all w, we have $\bigcup_j F_j = V$. Therefore one of these sets, say F_1, has non-empty interior. This means that there is a small ball B centred at some point $\widetilde{w}$ such that $K_1(w)$ contains 0 for all $w \in B$. Choose a regular point $\widetilde{z}$ in $K_1(\widetilde{w})$ close to the origin. Then for every $(z, w) \in \Gamma^{\mathbb{C}}$ close to $(\widetilde{z}, \widetilde{w})$, we have $z \in K_1(w)$, that is, $(z, w) \notin \Sigma$. Hence the complement of Σ has non-empty interior. But this contradicts the conclusion of Step 3 that $\overline{\Sigma} = \Gamma^{\mathbb{C}}$, and the proof is complete. $\square$

§ 4. Uniformly laminar currents near non-dicritical singularities

We say that the Segre variety Q_w defined by (8) is *minimal* if the holomorphic function $z \mapsto \rho(z, \overline{w})$ is minimal. We have the following proposition.

Proposition 4.1. *Let Γ be a real analytic Levi-flat hypersurface in $\mathbb{C}^n$ with irreducible germ at the origin. Assume that 0 is a non-dicritical singularity for Γ. Then for every sufficiently small neighbourhood Ω of the origin there is a complex linear map $L\colon \mathbb{C} \to \mathbb{C}^n$ with the following properties.*

i) $L(\mathbb{C}) \cap Q_0 = \{0\}$.

ii) *No component of the one-dimensional real analytic set $\gamma = L(\mathbb{C}) \cap \Gamma$ is contained in Γ_{sing}.*

iii) *For every $q \in \Gamma^* \cap \Omega$ there is a point $w \in \gamma$ such that $\mathcal{L}_q$ is contained in Q_w.*

iv) *If in addition the Segre variety Q_0 is irreducible and minimal, then such a point w is unique.*

Parts i), ii), iii) are proved in [9] (Proposition 4.1) under the assumption that 0 is a Segre non-degenerate singularity. Theorem 3.1 enables us to apply this result in the non-dicritical case. Note that if Q_0 is irreducible and minimal, then the Segre varieties Q_w with w close enough to the origin enjoy the same properties. This implies iv).

A one-dimensional real analytic set γ constructed as in Proposition 4.1 is called *a transverse* for the Levi-flat hypersurface Γ at a non-dicritical singularity. In general, γ can be reducible, that is, be a finite union of real analytic curves. The existence of a transverse shows that the structure of a Levi-flat hypersurface near a non-dicritical singularity is similar to that of a non-singular foliation. In [9], Proposition 4.1 was used to extend a non-dicritical Levi foliation as a holomorphic web in a full neighbourhood of a singularity in $\mathbb{C}^n$. Here we give another application.

We use the standard terminology and notation of the theory of currents (see [11], [13]). Denote by $\mathcal{D}'_{p,q}(\Omega)$ the space of currents of bidimension (p,q) (or simply (p,q)-currents) in a domain Ω of $\mathbb{C}^n$. If A is a complex analytic subset of Ω of pure dimension p, then $[A] \in \mathcal{D}'_{p,p}(\Omega)$ stands for the current of integration over A.

The main result of this section is the following assertion.

Proposition 4.2. *Let $\Gamma = \rho^{-1}(0)$ be a real analytic Levi-flat hypersurface in $\mathbb{C}^n$ with irreducible germ at the origin. Suppose that 0 is a non-dicritical singularity, and a one-dimensional real analytic subset γ of Γ is a transverse containing the origin. Assume that the Segre variety Q_0 is irreducible and minimal. Furthermore, suppose that the sets $Q_s \setminus \Gamma_{\mathrm{sing}}$, $s \in \gamma$, are connected.*

Then there is a neighbourhood Ω of the origin in $\mathbb{C}^n$ with the following property. Every closed positive current $T \in \mathcal{D}'_{n-1,n-1}(\Omega)$ of order (*of singularity*) *0 with support in $\overline{\Gamma^*}$ can be written in the form*

$$T = \int_{s\in\gamma} [Q_s]\, d\mu(s) \tag{13}$$

with a unique positive measure μ.

In the smooth case (for C^1 Levi-flat CR-manifolds without singularities), this result is due to Demailly [14]. Proposition 4.2 shows that every current T satisfying the assumptions of the theorem is a so-called uniformly laminar current. These currents play an important role in dynamical systems and foliation theory (see [15], [16]). Note that in many cases compact Levi-flat hypersurfaces in complex manifolds necessarily have singular points. This is our motivation for Proposition 4.2.

We need some known results on currents, which we recall for the convenience of the reader. Proofs are contained in [13]. Recall that a current T is said to be normal if both T and dT are currents of order zero.

Proposition 4.3 (first theorem on supports). *Let $T \in \mathcal{D}'_{p,p}(\Omega)$ be a normal current on a domain Ω in $\mathbb{C}^n$. If the support of T is contained in a real manifold M of* CR-*dimension $< p$, then $T = 0$.*

Let M be a Levi-flat smooth hypersurface in Ω and let I be an (open) smooth real curve. Assume that there is a submersion $\sigma\colon M \to I$ such that the set $\mathcal{L}_t = \sigma^{-1}(t)$ is a connected complex hypersurface (a Levi leaf) in M for every $t \in I$. Our second tool is the following proposition.

Proposition 4.4 (second theorem on supports). *Every closed current $T \in \mathcal{D}'_{n-1,n-1}(\Omega)$ of order zero with support contained in M can be written in the form*

$$T = \int_I [\mathcal{L}_t]\, d\mu(t)$$

for a unique complex measure μ on I. Moreover, T is positive if and only if μ is positive.

Let A be an irreducible complex p-dimensional analytic set in Ω, and let T be a closed positive current of bidimension (p,p) in Ω. The generic Lelong number of T along A is defined as

$$m(A) := \inf\{\nu(T,a) \mid a \in A\}.$$

Here $\nu(T,a)$ stands for the Lelong number of T at a, which is defined as

$$\nu(T,a) = \lim_{r\to 0^+} r^{-2p} \int_{|z-a|<r} T \wedge \left(\frac{1}{2}\, d\, d^c |z|^2\right)^p.$$

We need the following preparation result for Siu's semicontinuity theorem. Write $\mathbf{1}_A$ for the characteristic function of a set A.

Proposition 4.5. *Let T be a closed positive current of bidimension (p,p) in Ω, and let A be an irreducible p-dimensional analytic subset of Ω. Then $\mathbf{1}_A T = m(A)[A]$.*

Proof of Proposition 4.2. This is a simple consequence of the existence of a transverse γ as given by Proposition 4.1 and the above-mentioned properties of currents.

Since Q_0 is an irreducible hypersurface with a minimal defining function, every Q_s, $s \in \gamma$, is an irreducible complex hypersurface for s close enough to 0 and is contained in Γ. The set of regular points of every Q_s is connected. If a regular point of Q_s belongs to Γ^*, then Q_s coincides with some leaf of the Levi foliation near this point. However, a regular point of Q_s can in general be a singular point of Γ. For this reason, we impose the condition that the sets $Q_s \setminus \Gamma_{\text{sing}}$ are connected.

We define a set $\gamma_0 \subset \gamma$ as follows. First, it contains the singular points of γ (this is a finite set since γ is real analytic). Second, we include in γ_0 the points which are singular for Γ (this is again a finite set since γ is not contained in Γ_{sing}). Furthermore, γ_0 contains the points s such that the Segre variety Q_s is contained in Γ_{sing}. Note that γ_0 is non-empty since it contains 0. Recall that Γ_{sing} is a semianalytic set of dimension at most $2n-2$ and can be stratified into a finite union of real analytic manifolds. In particular, it contains only a finite number of Segre varieties. Considering a small enough neighbourhood Ω of the origin, we can assume that $\gamma_0 = \{0\}$. This is the reason why we treat Q_0 in a special way in the following argument. We do not assume, however, that Q_0 is contained in Γ_{sing}.

Denote by I one of the components of $\gamma \setminus \{0\}$. Consider the domains $\Omega' = \Omega \setminus Q_0$ and $\Omega'' = \Omega' \setminus \Gamma_{\text{sing}}$. The subset

$$X = \left(\bigcup_{s\in I} Q_s\right) \setminus \Gamma_{\text{sing}}$$

is a closed smooth (without singularities) Levi-flat real analytic hypersurface in Ω''. Furthermore, X coincides with a component of $\Gamma^* \cap \Omega'$.

The positive current $\mathbf{1}_X T$ is closed in Ω''. By Proposition 4.4 we conclude that

$$\mathbf{1}_X T = \int_I [Q_s]\, d\mu(s) \tag{14}$$

for a unique positive measure μ on I. Recall that $\dim \Gamma_{\mathrm{sing}} \leqslant 2n-2$. By the choice of the neighbourhood of the origin, the only complex hypersurface that Γ_{sing} may contain is Q_0. Therefore, $\Gamma_{\mathrm{sing}} \cap \Omega'$ can be stratified into a finite union of smooth real analytic CR-manifolds of CR-dimension $< n-1$. The current

$$T\big|_{\Omega'} - \int_I [Q_s]\, d\mu(s)$$

is closed in Ω', is of order 0, and its support is contained in Γ_{sing}. By Proposition 4.3, this current must vanish. Hence, (14) holds on Ω'. Repeating this argument for other components of $\gamma \setminus \{0\}$, we extend μ to $\gamma \setminus \{0\}$.

In order to extend μ to the origin, we use Proposition 4.5, which yields

$$\mathbf{1}_{Q_0} T = m(Q_0)[Q_0].$$

We set $\mu(0) = m(Q_0)$. Then μ is defined on γ and (13) holds. $\square$

The Segre varieties Q_s are defined quite explicitly as the zero sets of the functions $z \mapsto \rho(z, \overline{s})$. In combination with the Poincaré–Lelong formula [11], [13], this gives the following assertion.

Corollary 4.6. *Under the hypotheses of Proposition 4.2 we have*

$$T = \frac{i}{\pi} \int_{s \in \gamma} \partial\overline{\partial} \log |\rho(z, \overline{s})|\, d\mu(s). \tag{15}$$

One can view (15) as a 'foliated' Poincaré–Lelong formula for non-dicritical singularities. Hence, non-dicritical singularities are not 'detected' at the level of currents: the structure is the same as in the smooth case. Only dicritical singularities are essential from this point of view.

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