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Divergent CR-equivalences and meromorphic differential equations

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Abstract. Using the analytic theory of differential equations, we construct, in any positive CR-dimension and CR-codimension, examples of formally but not holomorphically equivalent real-analytic CR-submanifolds in complex space.

Keywords. CR-manifolds, formal mappings, holomorphic classification

Contents

1. Introduction

Let M, M' be smooth real-analytic generic CR-submanifolds in $\mathbb{C}^N$, $N \geq 2$, passing through the origin (in what follows we assume that all CR-submanifolds considered are generic). The germs $(M, 0)$ and $(M', 0)$ of these hypersurfaces at the origin are called *holomorphically equivalent* if there exists a germ of an invertible holomorphic mapping $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, called a *holomorphic equivalence between* $(M, 0)$ *and* $(M', 0)$, such that $F(M) \subset M'$. Starting with the celebrated papers of Poincaré [39],

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É. Cartan [10], Tanaka [44], Chern and Moser [12], problems related to holomorphic equivalence of real submanifolds in complex spaces have been intensively studied.

In particular, the following remarkable fact, demonstrating the difference between complex analysis in one and several variables, was discovered in [12]. To describe it, we need the following definition. A *formal mapping* $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is an N-tuple of formal power series in N variables without a constant term. If $(M, 0)$ and $(M', 0)$ are the germs at the origin of smooth real-analytic CR submanifolds of CR-dimensions n and k, given by the defining equations $\theta(z, \bar{z}) = 0$ and $\theta'(z, \bar{z}) = 0$ respectively, we say that $(M, 0)$ and $(M', 0)$ are *formally equivalent* if there exists an invertible formal mapping $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, called a *formal equivalence between* $(M, 0)$ *and* $(M', 0)$, and a $k \times k$ matrix-valued formal power series $\lambda(z, \bar{z})$ with an invertible constant term such that $\theta'(F(z), \bar{F}(\bar{z})) = \lambda(z, \bar{z}) \cdot \theta(z, \bar{z})$. Holomorphic equivalence of hypersurfaces obviously implies formal equivalence. In the other direction, the convergence of the Chern–Moser [12] normalizing transformation for real hypersurfaces implies

If real-analytic hypersurfaces $M, M' \subset \mathbb{C}^N$ are Levi-nondegenerate, then any formal equivalence between them is in fact convergent.

Recall that a real hypersurface is called *Levi-nondegenerate* (resp. *Levi-nonflat*) if the restriction of the complex Hessian of its defining function to the complex tangent is nondegenerate (resp. not identically zero). The problem of convergence of formal mappings between real submanifolds is closely related to the problem of analyticity of smooth CR-mappings (see [5]). Additional motivation comes from the fact that if the convergence phenomenon for some class of CR-submanifolds is established, then a merely *formal* normal form solves the holomorphic equivalence problem for this class of CR-submanifolds. For example, the normal form of Kolář [28] for real-analytic hypersurfaces of finite type in $\mathbb{C}^2$ is in general divergent, as shown by Kolář [29]. However, convergence results for formal CR-maps known in this setting (e.g. [6]) show that Kolář's normal form gives a *complete* set of local biholomorphic invariants of a finite type hypersurface in $\mathbb{C}^2$.

Starting with the work of Baouendi, Ebenfelt and Rothschild [4], who established the Chern–Moser convergence phenomenon for a wider class of hypersurfaces, a lot of work has been done to investigate the convergence problem for formal maps; see the recent survey [35] of Mir with a discussion of its connection with Artin's Approximation Problem [3]. For hypersurfaces the problem is best understood when $M \subset \mathbb{C}^N$ is *holomorphically nondegenerate* (i.e., not locally biholomorphically equivalent at a generic point to a product of a positive-dimensional complex space and a real submanifold of smaller dimension), and *minimal* at a reference point $p \in M$ (i.e., there is no germ at p of a complex hypersurface $X \subset M$, see [46]). In fact, holomorphic nondegeneracy, as well as minimality at a generic point $q \in M$, are necessary for convergence of all formal automorphisms of $(M, 0)$ (see [4, 5]). The only known result in the nonminimal case is due to Juhlin and Lamel [26], who proved convergence of formal equivalences for 1-nonminimal hypersurfaces in $\mathbb{C}^2$. Nevertheless, the general question of convergence of formal equivalences between merely holomorphically nondegenerate hypersurfaces has remained open; in particular, it was conjectured in [7] that the groups $\mathrm{Aut}(M, p)$ and $\mathcal{F}(M, p)$ of respectively holomorphic and formal self-equivalences of the germ (M, p) coincide for a holomorphically nondegenerate hypersurface M.

Even a more intriguing question in the context of formal CR-maps, which has remained open for any CR-dimension $n > 0$ and CR-codimension $k > 0$, is the relationship between formal and holomorphic equivalences of CR-manifolds. Namely, the following question has been a long-standing problem in CR-geometry:

Does the local holomorphic classification of real-analytic CR-manifolds of CR-dimension n and CR-codimension k (in particular, of real hypersurfaces) coincide with the formal classification?

Just recently a positive answer to the above open problem was conjectured in [35]. We should also mention the work by Moser and Webster [37], and by Gong [21], who found examples of formally but not holomorphically equivalent real surfaces in $\mathbb{C}^2$ near complex points. However, complex points constitute *CR-singularities*, and so such surfaces do not fall into the category of CR-manifolds. Similar results for Lagrangian submanifolds in $\mathbb{C}^2$ are due to Webster [48] and Gong [20].

The main results of this paper give a negative answer to the conjectures of Mir [35] and Baouendi, Mir and Rothschild [7]. To formulate the results precisely, we need the following definition. Let $M \subset \mathbb{C}^2$ be a real-analytic Levi-nonflat hypersurface which is nonminimal at the origin (note that a real-analytic hypersurface $M \subset \mathbb{C}^2$ is Levi-nonflat if and only if it is holomorphically nondegenerate [5]). Then in suitable coordinates $(z, w) \in \mathbb{C}^2$ near the origin (see, e.g., [16]) M can be represented by a defining equation

$$\operatorname{Im} w = (\operatorname{Re} w)^m \Phi(z, \bar{z}, \operatorname{Re} w), \tag{1.1}$$

where the power series $\Phi(z, \bar{z}, \operatorname{Re} w)$ contains no harmonic terms, and $\Phi(z, \bar{z}, 0) \not\equiv 0$. The integer $m \geq 1$ in (1.1), known to be a biholomorphic invariant of $(M, 0)$, is called the *nonminimality order* of M at 0. If M is given by equation (1.1), it is called *m-nonminimal.* The existence of the representation (1.1) is equivalent to M being Levi-nonflat.

Theorem A. *For any integer $m \geq 2$ there exist real-analytic hypersurfaces $M, M' \subset \mathbb{C}^2$ nonminimal at the origin of nonminimality order m such that the germs $(M, 0)$ and $(M', 0)$ are formally equivalent, but holomorphically inequivalent.*

The real hypersurfaces in Theorem A can be described explicitly, namely, using elementary functions and solutions of rational complex differential equations (see Theorem 4.7 below and also Remark 5.2).

Theorem A provides the first known examples of formally but not holomorphically equivalent CR-manifolds.

The next result shows that the answer is also negative for automorphisms. Recall that a (formal) holomorphic vector field L near the origin such that its real part $L + \bar{L}$ is (formally) tangent to M is called a (formal) *infinitesimal automorphism* of M.

Theorem B. *For any integer $m \geq 2$ there exist real-analytic hypersurfaces $M, M' \subset \mathbb{C}^2$ nonminimal at the origin of nonminimality order m with a divergent formal infinitesimal automorphism L vanishing to order m at 0. In particular, the real flow $H^t(z, w)$ generated by L consists of divergent formal automorphisms of the germ $(M, 0)$ for any $t \in \mathbb{R} \setminus \{c\mathbb{Z}\}$, where $c \in \mathbb{R}$ is some constant.*

Theorem 5.1 of Section 5 gives a refinement of Theorem B.

It is possible to give a generalization of the phenomenon in Theorems A and B to higher dimensions. For a real submanifold $M \subset \mathbb{C}^N$, $M \ni 0$, we distinguish its stability algebra $\mathfrak{aut}(M, 0)$ at the origin and the formal stability algebra $\mathfrak{f}(M, 0)$ (see Section 2 for more details).

Theorem C. (a) *For any integers $n, k > 0$ there exist real-analytic holomorphically nondegenerate CR-submanifolds $M, M' \subset \mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k through the origin such that the germs $(M, 0)$ and $(M', 0)$ are formally equivalent but holomorphically inequivalent. In particular, the holomorphic and formal equivalence problems for real-analytic holomorphically nondegenerate CR-submanifolds of CR-dimension n and CR-codimension k are not equivalent.*

(b) *For any integers $N, m \geq 2$ there exists a real-analytic holomorphically nondegenerate hypersurface $M \subset \mathbb{C}^N$ through the origin with a divergent formal infinitesimal automorphism L vanishing to order m. The real flow H^t of L consists of divergent formal automorphisms of the germ $(M, 0)$ for any $t \in \mathbb{R} \setminus \{c\mathbb{Z}\}$, where $c \in \mathbb{R}$ is some constant.*

For a real-analytic submanifold $M \subset \mathbb{C}^N$ through the origin one can consider its *holomorphic isotropy dimension* $\dim \mathfrak{aut}(M, 0)$ as well as its *formal isotropy dimension* $\dim \mathfrak{f}(M, 0)$.

Corollary 1.1. *The holomorphic and formal isotropy dimensions do not coincide in general for a holomorphically nondegenerate hypersurface $M \subset \mathbb{C}^N$.*

The main tool of the paper is a development, in the Levi-*degenerate* case, of the fundamental connection between CR-geometry and the geometry of completely integrable systems of complex PDEs, first observed by É. Cartan and B. Segre. In particular, the geometry of real-analytic Levi-nondegenerate hypersurfaces in $\mathbb{C}^2$ is closely related to that of second order ODEs, as discussed in Section 2. For a modern treatment of the subject in the nondegenerate case, see also Sukhov [42, 43], Gaussier and Merker [19, 34], and Nurowski and Sparling [36]. By discovering a way to connect *nonminimal* real-analytic hypersurfaces in $\mathbb{C}^2$ with *singular* complex linear second order ODEs with a reality condition, we obtain the desired counterexamples. These examples arise from local holomorphic dynamics of second order ODEs with an isolated non-Fuchsian (irregular) singularity.

We point out that the paper contains an intermediate result which is a characterization of real hypersurfaces nonminimal at the origin and spherical at a generic point and having the infinitesimal automorphism $iz\frac{\partial}{\partial z}$ ("rotations inside the complex tangent")—see Theorem 3.15 and the algorithm at the end of Section 3. Real-analytic hypersurfaces of this type were intensively studied by Ebenfelt, Lamel and Zaitsev [17], Beloshapka [8], Kolář and Lamel [30] and the present authors [31]. As the construction of each example in the cited papers is technically quite involved, the explicit description, given in Section 3 of this paper, is of independent interest. In fact, one can show that this description is *complete* (see Remark 3.19).

The paper is organized as follows. Because we use tools from a broad range of topics in complex analysis and dynamical systems, in Section 2 we provide relevant background material. In Section 3 we introduce a class of 2-parameter families of planar holomorphic curves, which can potentially be the Segre families of real hypersurfaces nonminimal at the origin and *spherical at a generic point*, and, at the same time, serve as a family of integral curves for certain second order linear ODEs with an isolated meromorphic singularity (we call these *m-admissible ODEs with a real structure*). The explicit characterization of these ODEs, given in Theorem 3.15, allows us to obtain in Section 4 nonminimal real hypersurfaces for which the associated ODE has essentially prescribed behaviour of solutions. Then, by finding a divergent formal equivalence between holomorphically inequivalent ODEs with a real structure, we obtain in Propositions 4.2 and 4.3 the potential formal equivalence, and the rest of the section is dedicated to proving that this formal mapping is the mapping between the initial real hypersurfaces, which proves Theorem A and the first statement of Theorem C. In Section 5 we apply the divergent transformation from Theorem A to infinitesimal automorphisms, which gives the proof of Theorem B and the second statement in Theorem C. We also give a description of the hypersurface M' from Theorem A by elementary functions, and a hint of a similar description for M (see Remark 5.2). Finally, we formulate some open problems and conjectures, arising from the results of this paper.

2. Preliminaries and background material

2.1. Segre varieties

Let M be a smooth connected real-analytic hypersurface in $\mathbb{C}^{n+1}$, $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}$, $0 \in M$, and let U be a neighbourhood of the origin such that $M \cap U$ admits a real-analytic defining function $\phi(Z, \bar{Z})$ holomorphic in $U \times \bar{U}$. For every point $\zeta \in U$ we can associate to M its *Segre variety* in U defined as

$$Q_\zeta = \{Z \in U : \phi(Z, \bar{\zeta}) = 0\}.$$

Segre varieties depend holomorphically on the variable $\bar{\zeta}$. One can find a pair of neighbourhoods $U_2 = U_2^z \times U_2^w \subset \mathbb{C}^n \times \mathbb{C}$ and $U_1 \Subset U_2$ such that

$$Q_\zeta = \{(z, w) \in U_2^z \times U_2^w : w = h(z, \bar{\zeta})\}, \qquad \zeta \in U_1,$$

is a closed complex analytic graph. Here h is a holomorphic function. Following [15] we call U_1, U_2 a *standard pair of neighbourhoods* of the origin. The antiholomorphic $(n+1)$-parameter family $\{Q_\zeta\}_{\zeta \in U_1}$ of complex hypersurfaces is called *the Segre family of M at the origin.* From the definition and the reality condition on the defining function the following basic properties of Segre varieties follow (we assume $Z, \zeta \in U_1$ below):

$$\begin{aligned} Z \in Q_\zeta \;&\Leftrightarrow\; \zeta \in Q_Z, \\ Z \in Q_Z \;&\Leftrightarrow\; Z \in M, \\ \zeta \in M \;&\Leftrightarrow\; \{Z \in U_1 : Q_\zeta = Q_Z\} \subset M. \end{aligned} \tag{2.1}$$

The fundamental role of Segre varieties for holomorphic mappings is illuminated by their invariance property: if $f : U \to U'$ is a holomorphic map sending a smooth real-analytic hypersurface $M \subset U$ into another such hypersurface $M' \subset U'$, and U is as above, then

$$f(Z) = Z' \ \Rightarrow \ f(Q_Z) \subset Q'_{Z'}.$$

For the proofs of these and other properties of Segre varieties, see, e.g., [49], [14], [15], [40], or [5].

In the particularly important case when M is a *real hyperquadric*, i.e.,

$$M = \{[\zeta_0, \dots, \zeta_N] \in \mathbb{CP}^N : H(\zeta, \bar{\zeta}) = 0\},$$

where $H(\zeta, \bar{\zeta})$ is a nondegenerate Hermitian form in $\mathbb{C}^{N+1}$ with $k+1$ positive and $l+1$ negative eigenvalues, $k + l = N - 1$, $0 \le l \le k \le N - 1$, the Segre variety of a point $\zeta \in \mathbb{CP}^N$ is the projective hyperplane $Q_\zeta = \{\xi \in \mathbb{CP}^N : H(\xi, \bar{\zeta}) = 0\}$. The Segre family $\{Q_\zeta : \zeta \in \mathbb{CP}^N\}$ coincides in this case with the space $(\mathbb{CP}^N)^*$ of all projective hyperplanes in $\mathbb{CP}^N$.

The space $\{Q_Z : Z \in U_1\}$ of Segre varieties can be identified with a subset of $\mathbb{C}^K$ for some $K > 0$ in such a way that the *Segre map* $\lambda : Z \to Q_Z$ is holomorphic (see [14]). For a hypersurface M Levi-nondegenerate at a point p, its Segre map is one-to-one in a neighbourhood of p. When M contains a complex hypersurface X, for any $p \in X$ we have $Q_p = X$ and $Q_p \cap X \neq \emptyset \Leftrightarrow p \in X$, so that the Segre map λ sends the entire X to a unique point in $\mathbb{C}^K$, and so λ is not even finite-to-one near each $p \in X$ (i.e., M is *not essentially finite* at points $p \in X$). For a hyperquadric $\mathcal{Q} \subset \mathbb{CP}^N$ the Segre map λ' is a global natural one-to-one correspondence between $\mathbb{CP}^N$ and $(\mathbb{CP}^N)^*$.

2.2. *Real hypersurfaces and second order differential equations*

Using the Segre family of a Levi-nondegenerate real hypersurface $M \subset \mathbb{C}^N$, one can associate to it a system of second order holomorphic PDEs with 1 dependent and $N - 1$ independent variables. The corresponding remarkable construction goes back to É. Cartan [11], [10] and Segre [41], and was recently revisited in [42], [43], [36], [19], [34] (see also references therein). We describe the procedure for $N = 2$, which will be relevant for our purposes. In what follows we denote the coordinates in $\mathbb{C}^2$ by (z, w), and write $z = x+iy$, $w = u + iv$. Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface passing through the origin, and let (U_1, U_2) be its standard pair of neighbourhoods. In this case one associates to M a second order holomorphic ODE, uniquely determined by the condition that it is satisfied by the Segre family $\{Q_\zeta\}_{\zeta \in U_1}$ of M in a neighbourhood of the origin where the Segre varieties are considered as graphs $w = w(z)$. More precisely, it follows from the Levi-nondegeneracy of M near the origin that the Segre map $\zeta \mapsto Q_\zeta$ is injective and also that the Segre family has the *transversality property*: if two distinct Segre varieties intersect at a point $q \in U_2$, then their intersection at q is transverse. Thus, $\{Q_\zeta\}_{\zeta \in U_1}$ is a 2-parameter family of holomorphic curves in U_2, holomorphic with respect to $\bar{\zeta}$ with the transversality property. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [24]) that there exists a unique second order holomorphic ODE $w'' = \Phi(z, w, w')$ satisfied by the graphs $\{Q_\zeta\}_{\zeta \in U_1}$.

This procedure can be made more explicit if one considers the so-called *complex defining equation* (see, e.g., [5]) $w = \rho(z, \bar{z}, \bar{w})$ of M near the origin, which can be obtained by substituting $u = \frac{1}{2}(w+\bar{w})$, $v = \frac{1}{2i}(w-\bar{w})$ into the real defining equation and applying the holomorphic implicit function theorem. The complex defining function ρ here satisfies an additional reality condition

$$w \equiv \rho(z, \bar{z}, \bar{\rho}(\bar{z}, z, w)), \tag{2.2}$$

reflecting the fact that M is a real hypersurface. The Segre variety Q_p of a point $p = (a, b)$ close to the origin is given by

$$w = \rho(z, \bar{a}, \bar{b}). \tag{2.3}$$

Differentiating (2.3) once, we obtain

$$w' = \rho_z(z, \bar{a}, \bar{b}). \tag{2.4}$$

Considering (2.3) and (2.4) as a holomorphic system of equations with the unknowns $\bar{a}, \bar{b}$, and applying the implicit function theorem near the origin, we get

$$\bar{a} = A(z, w, w'), \qquad \bar{b} = B(z, w, w').$$

The implicit function theorem is applicable here because the Jacobian of the system coincides with the Levi determinant of M for $(z, w) \in M$ (see, e.g., [34]). Differentiation of (2.3) twice and substitution of the expressions for $\bar{a}, \bar{b}$ finally yields

$$w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: \Phi(z, w, w'). \tag{2.5}$$

Now (2.5) is the desired holomorphic second order ODE $\mathcal{E}$.

The concept of a PDE system associated with a CR-manifold can be generalized to arbitrary l-nondegenerate, $l \geq 1$, CR-submanifolds (see [5] for the definition of this nondegeneracy condition). Namely, to any l-nondegenerate CR-submanifold $M \subset \mathbb{C}^{n+k}$ of CR-dimension n and codimension k one can assign a completely integrable system $\mathcal{E}(M)$ of holomorphic PDEs with n independent and k dependent variables. The correspondence $M \mapsto \mathcal{E}(M)$ has the following fundamental properties:

(1) Every local holomorphic equivalence $F : (M, 0) \to (M', 0)$ between two l-nondegenerate CR-submanifolds is an equivalence between the corresponding PDE systems $\mathcal{E}(M), \mathcal{E}(M')$.
(2) The complexification of the infinitesimal automorphism algebra $\mathfrak{hol}(M, 0)$ of M at the origin coincides with the Lie symmetry algebra of the associated PDE system $\mathcal{E}(M)$ (for the details, see, e.g., [38]).

For the proof and applications of the properties (1) and (2) we refer to [42], [43], [36], [19], and [34]. In general, for a hypersurface $M \subset \mathbb{C}^2$ nonminimal at the origin there is *no* a priori way to associate to M a second order ODE or even a more general PDE system near the origin. However, in Section 3 we provide a way to connect a special class of nonminimal real hypersurfaces in $\mathbb{C}^2$ with a class of complex linear differential equations with an isolated singularity.

2.3. Equivalences of differential equations

For simplicity, here we consider only scalar ordinary differential equations, even though all the constructions below can be applied to arbitrary systems of PDEs. We refer to Olver [38] as a general reference for this subsection. Also note that our approach is nothing but a simple interpretation of a more general concept of a *jet bundle*.

Consider two ODEs $\mathcal{E} = \{y^{(n)} = \Phi(x, y, y', \dots, y^{(n-1)})\}$ and $\mathcal{E}^* = \{y^{(n)} = \Phi^*(x, y, y', \dots, y^{(n-1)})\}$, where the functions Φ and Φ^* are holomorphic in some neighbourhood of the origin in $\mathbb{C}^n$. We say that a biholomorphism $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *transforms $\mathcal{E}$ into $\mathcal{E}^*$* if it sends (locally) graphs of solutions of $\mathcal{E}$ to graphs of solutions of $\mathcal{E}^*$. Introducing the $(n + 2)$-dimensional *n-jet space* $J^{(n)}$, which is a linear space with the coordinates $x, y, y_1, \dots, y_n$, corresponding to the independent variable x, the dependent variable y and its derivatives up to order n, one can naturally consider $\mathcal{E}$ and $\mathcal{E}^*$ as complex submanifolds in $J^{(n)}$. Moreover, for any biholomorphism F as above, sufficiently close to the origin one may consider the *n-jet prolongation* $F^{(n)} : (J^{(n)}, 0) \to (J^{(n)}, 0)$. The jet prolongation procedure can be conveniently interpreted as follows. The first two components of the mapping $F^{(n)}$ coincide with those of F. To obtain the remaining components, we denote the coordinates in the preimage by (x, y) and in the target domain by (X, Y). Then the derivative $\frac{dY}{dX}$ can be symbolically recalculated, using the chain rule, in terms of x, y, y', so that the third coordinate Y_1 in the target jet space becomes a function of x, y, y_1. In the same manner one obtains all the n missing components of the prolongation of the mapping F. It is then nothing but a tautology to say that *the mapping F transforms the ODE $\mathcal{E}$ into $\mathcal{E}^*$ if and only if the prolonged mapping $F^{(n)}$ transforms $(\mathcal{E}, 0)$ into $(\mathcal{E}^*, 0)$ as submanifolds in the jet space $J^{(n)}$*. A similar statement can be formulated for certain singular differential equations, for example, for linear ODEs (see, e.g., [24]).

For $n = 2$ the local equivalence problem for nonsingular ODEs was solved in the celebrated papers of É. Cartan [11] and A. Tresse [45]. Of particular interest to us is the special case when the ODE is equivalent to the simplest (flat) equation $y'' = 0$. We refer to the book of Arnold [1] for a modern treatment of the problem and some further developments.

2.4. Formal power series, formal equivalences and formal flows

For the set-up and basic properties of formal power series and formal mappings we refer to [24] and [5]. Below we give a list of statements that will be useful in what follows.

- The substitution of a formal mapping $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ into a formal power series is well-defined. In particular, the composition of two formal mappings $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is always well-defined (as before, for a formal mapping $(\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ we always assume the absence of the constant term).

- A formal mapping $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is called *invertible* if there exists a formal mapping $G : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with $F \circ G$ being the identity map. Note that a formal mapping $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is invertible whenever its linear part is invertible as an element of $\mathrm{GL}_n(\mathbb{C})$.

• For any formal mapping $F(z,w) : (\mathbb{C}^m \times \mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ the following formal version of the implicit function theorem holds: if the linear part $\frac{\partial F}{\partial w}(0)$ of F with respect to w is invertible, then there exists a unique formal mapping $\varphi : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ such that $F(z, \varphi(z)) = 0$ as a formal mapping.

Let $X = f_1(z)\frac{\partial}{\partial z_1} + \cdots + f_n(z)\frac{\partial}{\partial z_n}$ be a formal vector field with $X(0) = 0$. A *formal flow* of X is a 1-parameter family of formal mappings $F^t(z) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ holomorphic with respect to $t \in \mathbb{C}$ such that $\frac{d}{dt}F^t(z)\big|_{t=0} = X$ and the mapping $t \mapsto F^t$ is a group homomorphism from $(\mathbb{C}, +)$ to the group of formal invertible mappings $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. A 1-parameter group $F^t(z)$ as above is called *holomorphic* if all the *truncations* $j^k_z F^t(z)$, i.e., the Taylor polynomials of $F^t(z)$ of degree k in z, are holomorphic with respect to t.

• For any formal vector field X with $X(0) = 0$, its formal flow always exists and is uniquely determined.

Following Fels and Kaup (e.g. [27]), for a real submanifold $M \subset \mathbb{C}^N$ with $M \ni 0$ we define its *infinitesimal automorphism algebra at the origin* as the real Lie algebra $\mathfrak{hol}(M, 0)$ of holomorphic vector fields X near the origin such that their real parts $X + \bar{X}$ are tangent to M at each point. The *stability algebra* $\mathfrak{aut}(M, 0) \subset \mathfrak{hol}(M, 0)$ is the subalgebra of vector fields vanishing at 0. One can further define the *formal infinitesimal automorphism algebra* $\mathfrak{f}(M, 0)$, which consists of formal vector fields in $\mathbb{C}^N$, formally satisfying the tangency condition to M, and the *formal stability algebra* $\mathfrak{f}(M, 0)$, which consists of formal vectors fields $X \in \mathfrak{f}(M, 0)$ with $X(0) = 0$.

• A formal vector field X with $X(0) = 0$ is a formal infinitesimal automorphism of $(M, 0)$ if and only if the formal flow of X formally preserves the germ $(M, 0)$.

Finally, we will need the following property of formal CR-mappings. For a real-analytic submanifold $M \subset \mathbb{C}^N$ passing through the origin and given in some neighbourhood $U \ni 0$ by the defining equation $\theta(z, \bar{z}) = 0$, we define its *complexification* to be the complex submanifold

$$M^{\mathbb{C}} = \{(z, \zeta) \in U \times U : \theta(z, \zeta) = 0\} \subset \mathbb{C}^{2N}.$$

• Let $M_1, M_2 \subset \mathbb{C}^N$ be real-analytic submanifolds passing through the origin. A (formal) transformation $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ without a free term sends (formally) $(M_1, 0)$ to $(M_2, 0)$ if and only if the product map $(F(z), \bar{F}(\zeta)) : (\mathbb{C}^{2N}, 0) \to (\mathbb{C}^{2N}, 0)$ (called the *complexification* of F) sends (formally) $(M_1^{\mathbb{C}}, 0)$ to $(M_2^{\mathbb{C}}, 0)$.

2.5. *Complex linear differential equations with an isolated singularity*

Perhaps the most important and geometric class of complex differential equations is the class of complex linear ODEs. We refer to [24], [2], [9], [47], [13] and references therein for various facts and problems concerning complex linear differential equations. A first order linear system of n complex ODEs in a domain $G \subset \mathbb{C}$ (or simply a *linear system* in G) is a holomorphic ODE system $\mathcal{L}$ of the form $y'(w) = A(w)y$, where $A(w)$ is an $n \times n$ matrix-valued function holomorphic in G and $y(w) = (y_1(w), \dots, y_n(w))$ is an n-tuple of unknown functions. Solutions of $\mathcal{L}$ near a point $p \in G$ form a linear space of dimension n. Moreover, all the solutions $y(w)$ of $\mathcal{L}$ are defined globally in G as (possibly multiple-valued) analytic functions, i.e., any germ of a solution near a point $p \in G$ of $\mathcal{L}$ extends analytically along any path $\gamma \subset G$ starting at p. A *fundamental system of solutions for* $\mathcal{L}$ is a matrix whose columns form some collection of n linearly independent solutions of $\mathcal{L}$.

If G is a punctured disc centred at 0, we call $\mathcal{L}$ a *system with an isolated singularity at* $w = 0$. An important (and sometimes even complete) characterization of an isolated singularity is given by its *monodromy operator,* defined as follows. If $Y(w)$ is some fundamental system of solutions of $\mathcal{L}$ in G, and γ is a simple loop about the origin, then it is not difficult to see that the monodromy of $Y(w)$ with respect to γ is given by right multiplication by a constant nondegenerate matrix M, called the *monodromy matrix.* The matrix M, unique up to similarity, defines a linear operator $\mathbb{C}^n \to \mathbb{C}^n$, called the *monodromy operator* of the singularity.

If the matrix-valued function $A(w)$ is meromorphic at the singularity $w = 0$, we call it a *meromorphic singularity*. As the solutions of $\mathcal{L}$ are holomorphic in any proper sector $S \subset G$ of a sufficiently small radius with vertex at $w = 0$, it becomes important to study the behaviour of the solutions as $w \to 0$. If all solutions of $\mathcal{L}$ admit a bound $\|y(w)\| \leq C|w|^\alpha$ in any such sector (with some constants $C > 0$, $\alpha \in \mathbb{R}$, depending possibly on the sector), then $w = 0$ is called a *regular singularity*, otherwise it is an *irregular singularity.* In particular, in the case of trivial monodromy the singularity is regular if and only if all the solutions of $\mathcal{L}$ extend meromorphically to the singular point $w = 0$. L. Fuchs introduced the following condition: a singular point $w = 0$ is called *Fuchsian* if $A(w)$ is meromorphic at $w = 0$ and has a pole of order ≤ 1 there. The Fuchsian condition turns out to be sufficient for the regularity of a singular point. Another remarkable property of Fuchsian singularities can be described as follows. We call two complex linear systems $\mathcal{L}_1, \mathcal{L}_2$ with an isolated singularity *(formally) equivalent* if there exists a (formal) transformation $F : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ of the form $F(w, y) = (w, H(w)y)$ for some (formally) invertible matrix-valued function $H(w)$, which (formally) transforms $\mathcal{L}_1$ into $\mathcal{L}_2$. It turns out that two Fuchsian systems are formally equivalent if and only if they are holomorphically equivalent (moreover, any formal equivalence between them must be convergent). However, this is *not* the case for non-Fuchsian systems (see [9] and [47, Ch. IV] for some related constructions).

A scalar linear complex ODE of order n in a domain $G \subset \mathbb{C}$ is an ODE $\mathcal{E}$ of the form

$$z^{(n)} = a_n(w)z + a_{n-1}(w)z' + \cdots + a_1(w)z^{(n-1)},$$

where $\{a_j(w)\}_{j=1,\dots,n}$ is a given collection of holomorphic functions in G and $z(w)$ is the unknown function. By a reduction of $\mathcal{E}$ to a first order linear system (see the above references and also [22] for various approaches to doing that) one can naturally transfer most of the definitions and facts, relating to linear systems, to scalar equations of order n. The main difference here is in the definition of Fuchsian: a singular point $w = 0$ for an ODE $\mathcal{E}$ is called *Fuchsian* if the orders of poles p_j of the functions $a_j(w)$ satisfy the inequalities $p_j \leq j$, $j = 1, \dots, n$. It turns out that the condition of Fuchs also becomes necessary for the regularity of a singular point in the case of n-th order scalar ODEs.

Further information on the classification of isolated singularities (including Poincaré–Dulac normalization) can be found in [24], [47] or [13].

3. Meromorphic linear differential equations with real structure

The main purpose of this section is to establish a class of complex linear second order ODEs with a meromorphic singularity that generate, in a certain sense, real hypersurfaces nonminimal at the origin and spherical at a generic point. We start with a number of definitions. Denote by Δ_ε the disc in $\mathbb{C}$ centred at $w = 0$ and of radius ε, and by Δ_ε^* the corresponding punctured disc.

Definition 3.1. A complex linear second order ODE with an isolated singularity at the origin is called *m-admissible* if it is of the form

$$z'' = \frac{P(w)}{w^m} z' + \frac{Q(w)}{w^{2m}} z, \tag{3.1}$$

where $m \geq 1$ is an integer and $P(w), Q(w) \in \mathcal{O}(\Delta_\varepsilon)$ for some $\varepsilon > 0$.

Direct calculations show that if a germ $z(w)$ of a solution of (3.1) is invertible in some domain, then the inverse function $w(z)$ satisfies in the image domain the ODE

$$w'' = -\frac{P(w)}{w^m}(w')^2 - \frac{Q(w)}{w^{2m}}(w')^3 z. \tag{3.2}$$

We call (3.2) the *inverse ODE* for (3.1). Note that in (3.1) the independent variable is w, while z is the independent variable for the inverse ODE. Also note that without the requirement that $P(w)/w^m$ and $Q(w)/w^{2m}$ are irreducible, a complex linear ODE meromorphic at the origin is admissible for different integers $m \geq 1$.

We next introduce a class of antiholomorphic 2-parameter families of planar complex curves that potentially can be the family of solutions for an m-admissible ODE and, at the same time, the family of Segre varieties of a real hypersurface in $\mathbb{C}^2$.

Definition 3.2. *An m-admissible Segre family* is a parametrized antiholomorphic family of planar holomorphic curves in a polydisc $\Delta_\delta \times \Delta_\varepsilon$ of the form

$$w = \bar{\eta} e^{\pm i \bar{\eta}^{m-1} \psi(z\bar{\xi}, \bar{\eta})}, \tag{3.3}$$

where $m \geq 1$ is an integer, $\xi \in \Delta_\delta$ and $\eta \in \Delta_\varepsilon$ are holomorphic parameters, and the function $\psi(x, y)$ is holomorphic in the polydisc $\Delta_{\delta^2} \times \Delta_\varepsilon$ and has there an expansion

$$\psi(x, y) = x + \sum_{k \geq 2} \psi_k(y) x^k, \qquad \psi_k \in \mathcal{O}(\Delta_\varepsilon).$$

We can rewrite an m-admissible Segre family in the form

$$\mathcal{S} = \{w = \bar{\eta} e^{\pm i \bar{\eta}^{m-1}(z\bar{\xi} + \sum_{k\geq 2} \psi_k(\bar{\eta}) z^k \bar{\xi}^k)} : (\xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon\}. \tag{3.4}$$

The fact that an antiholomorphic 2-parameter family of planar complex curves is m-admissible can be easily checked: a family $w = \rho(z\bar{\xi}, \bar{\eta})$, where ρ is holomorphic in some polydisc $U \subset \mathbb{C}^2$ centred at the origin, is m-admissible if and only if the defining function ρ has the expansion $\rho(z\bar{\xi}, \bar{\eta}) = \bar{\eta} \pm i\bar{\eta}^m z\bar{\xi} + O(\bar{\eta}^m z^2\bar{\xi}^2)$.

For any real-analytic hypersurface $M \subset \mathbb{C}^2$ nonminimal at the origin with nonminimality order m, of the form

$$v = u^m\Big(\pm|z|^2 + \sum_{k\geq 2} h_k(u)|z|^{2k}\Big), \tag{3.5}$$

it is not difficult to check that its Segre family is an m-admissible Segre family. We call a real hypersurface of the form (3.5) an *m-admissible nonminimal hypersurface.* Note that in the case of m-admissible Segre families (respectively, nonminimal hypersurfaces) the integer m is uniquely determined by the Segre family (respectively, by the hypersurface). Depending on the sign in the exponent $e^{\pm i\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})}$, we call an m-admissible Segre family *positive* or *negative* respectively, and the same for real hypersurfaces. In analogy with the case of real hypersurfaces, we call the holomorphic curve in the family (3.3) corresponding to $\xi = a$, $\eta = b$ the *Segre variety of the point* $p = (a, b) \in \Delta_\delta \times \Delta_\varepsilon$, and denote it by Q_p. We point out that the *Segre map* $\lambda : p \mapsto Q_p$ *does* depend on the parametrization of the family. We call the hypersurface

$$X = \{w = 0\} \subset \Delta_\delta \times \Delta_\varepsilon$$

the *singular locus* of the m-admissible Segre family. The following proposition provides some simple properties of Segre families.

Proposition 3.3. *The following properties hold for an m-admissible family:*

(i) $Q_p \cap X \neq \emptyset \Leftrightarrow p \in X \Leftrightarrow Q_p = X$.
(ii) *The Segre mapping* $\lambda : p \mapsto Q_p$ *is injective in* $(\Delta_\delta \times \Delta_\varepsilon) \setminus X$.

Proof. The first property follows directly from (3.3). For the proof of (ii) we note that if a Segre variety Q_p is given as a graph $w = w(z)$, then, from (3.3), $w(0) = \bar{\eta}$, $w'(0) = \pm i\bar{\xi}\bar{\eta}^m$, depending on the sign of the Segre family, and that implies the global injectivity of λ in $(\Delta_\delta \times \Delta_\varepsilon) \setminus X$. □

The next definition connects admissible Segre families with second order linear ODEs with a meromorphic singularity.

Definition 3.4. We say that an m-admissible Segre family $\mathcal{S}$ *is associated with an m-admissible ODE* $\mathcal{E}$ if after an appropriate shrinking of the basic neighbourhood $\Delta_\delta \times \Delta_\varepsilon$ of the origin all the elements $Q_p \in \mathcal{S}$ with $p \notin X$, considered as graphs $w = w(z)$, satisfy the inverse ODE for $\mathcal{E}$.

Given an ODE $\mathcal{E}$, we denote the associated m-admissible Segre family by $\mathcal{S}^{\pm}_{\mathcal{E}}$, depending on the sign of the Segre family. By Proposition 3.3, $w \neq 0$ for $p \notin X$, and so we may alway substitute the Segre varieties into (3.2).

Proposition 3.5. *For any m-admissible ODE $\mathcal{E}$ (3.1) there exist a unique positive and a unique negative m-admissible (with the same m) Segre families $\mathcal{S}$ associated with $\mathcal{E}$. The ODE $\mathcal{E}$ and the associated Segre families $\mathcal{S}^{\pm}_{\mathcal{E}}$ given by (3.4) satisfy the following relations:*

$$P(w) = \pm 2i\psi_2(w) - w^{m-1}, \tag{3.6}$$

$$Q(w) = 6\psi_3(w) - 8(\psi_2(w))^2 \pm 2i(m-1)w^{m-1}\psi_2(w) \mp 2iw^m\psi_2'(w). \tag{3.7}$$

In particular, for any fixed m the correspondences $\mathcal{E} \mapsto \mathcal{S}^{+}_{\mathcal{E}}$ and $\mathcal{E} \mapsto \mathcal{S}^{-}_{\mathcal{E}}$ are injective.

Proof. Consider a positive m-admissible Segre family $\mathcal{S}$ as in (3.3), and an m-admissible ODE $\mathcal{E}$. We first express the condition that $\mathcal{S}$ is associated with $\mathcal{E}$ in the form of a differential equation. Fix $p = (\xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon$ and consider the Segre variety Q_p given by (3.3) as a graph $w = w(z)$. For the function $\psi(x, y)$ we denote by $\dot{\psi}$ and $\ddot{\psi}$ its first and second derivatives with respect to the first argument. Then one computes

$$w' = i\bar{\xi}\bar{\eta}^m e^{i\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})}\dot{\psi}(z\bar{\xi},\bar{\eta}),$$
$$w'' = i\bar{\xi}^2\bar{\eta}^m e^{i\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})}\ddot{\psi}(z\bar{\xi},\bar{\eta}) - \bar{\xi}^2\bar{\eta}^{2m-1}e^{i\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})}(\dot{\psi}(z\bar{\xi},\bar{\eta}))^2.$$

Plugging these expressions into (3.2) yields after simplifications

$$\begin{aligned}\ddot{\psi}(z\bar{\xi},\bar{\eta}) = {} & -i(\dot{\psi}(z\bar{\xi},\bar{\eta}))^2\big(\bar{\eta}^{m-1} + P(\bar{\eta}e^{i\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})})e^{i(1-m)\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})}\big)\\ & + (\dot{\psi}(z\bar{\xi},\bar{\eta}))^3 Q(\bar{\eta}e^{i\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})})e^{i(2-2m)\bar{\eta}^{m-1}\psi(z\bar{\xi},\bar{\eta})}z\bar{\xi}.\end{aligned} \tag{3.8}$$

Consider now the differential equation

$$y'' = -i(y')^2\big(\bar{\eta}^{m-1} + P(\bar{\eta}e^{i\bar{\eta}^{m-1}y})e^{i(1-m)\bar{\eta}^{m-1}y}\big) + (y')^3 t Q(\bar{\eta}e^{i\bar{\eta}^{m-1}y})e^{i(2-2m)\bar{\eta}^{m-1}y}, \tag{3.9}$$

holomorphic near the origin, where y is the dependent variable, t is the independent variable, and $\bar{\eta}$ is a holomorphic parameter near the origin. The Cauchy problem for (3.9) with the initial data $y(0) = 0,\ y'(0) = 1$ is well-posed, as the right-hand side is polynomial with respect to y'. By the analytic dependence of solutions of a holomorphic ODE on a holomorphic parameter (see, e.g., [24]), its solution $y = y_0(t, \bar{\eta})$ is unique and holomorphic in some polydisc $U \subset \mathbb{C}^2$ centred at the origin. The comparison of (3.8) and (3.9) shows that the functions $y_0(z\bar{\xi}, \bar{\eta})$ and $\psi(z\bar{\xi}, \bar{\eta})$ coincide. Observe that the above arguments are reversible.

For the proof of the proposition, given an m-admissible ODE $\mathcal{E}$, we solve the corresponding equation (3.9) with the initial data $y(0) = 0,\ y'(0) = 1$, and obtain a solution $y_0(t, \bar{\eta}) = t + \sum_{k\geq 2,\, l\geq 0} c_{kl}t^k\bar{\eta}^l$. Then

$$w = \bar{\eta}e^{i\bar{\eta}^{m-1}y_0(z\bar{\xi},\bar{\eta})}$$

is the desired positive m-admissible Segre family $\mathcal{S} = \mathcal{S}_{\mathcal{E}}$ associated with $\mathcal{E}$. The uniqueness of $\mathcal{S}_{\mathcal{E}}$ also follows from the uniqueness of the solution of the Cauchy problem.

To prove the relations (3.6), (3.7), we substitute (3.3) into (3.2). As $(\bar{\xi},\bar{\eta}) \in \Delta_\delta \times \Delta_\varepsilon$ is arbitrary, we compare the $z^0\bar{\xi}^2\bar{\eta}^l$-terms in the resulting identity, which gives $2i\bar{\eta}^m\psi_2(\bar{\eta}) - \bar{\eta}^{2m-1} = \bar{\eta}^m P(\bar{\eta})$. This is equivalent to (3.6). Comparing then the $z^1\bar{\xi}^3\bar{\eta}^l$-terms, we get

$$6i\bar{\eta}^m\psi_3(\bar{\eta}) - 6\bar{\eta}^{2m-1}\psi_2(\bar{\eta}) - i\bar{\eta}^{3m-2} = i\bar{\eta}^m Q(\bar{\eta}) - 2iP(\bar{\eta})(2i\bar{\eta}^m\psi_2(\bar{\eta}) - \bar{\eta}^{2m-1}) \\ - imP(\bar{\eta})\bar{\eta}^{2m-1} + i\bar{\eta}^{2m}P'(\bar{\eta}).$$

From this and (3.6), we finally obtain (3.7).

The proof for the negative Segre family is analogous. □

Proposition 3.5 gives an effective algorithm for computing the m-admissible Segre family for a given linear meromorphic second order ODE. Our goal is, however, to identify those ODEs that produce Segre families with a reality condition, that is, Segre families of nonminimal real hypersurfaces.

Definition 3.6. We say that an m-admissible Segre family *has a real structure* if it is the Segre family of an m-admissible real hypersurface $M \subset \mathbb{C}^2$. We also say that an m-admissible ODE $\mathcal{E}$ *has a positive* (respectively, *negative*) *real structure* if the associated positive (respectively, negative) m-admissible Segre family $\mathcal{S}^{\pm}_{\mathcal{E}}$ has a real structure. We say that the corresponding real hypersurface M is *associated with* $\mathcal{E}$.

We will need a development of the following construction from the theory of second order ODEs, going back to A.Tresse [45] and É. Cartan [11] (see also [1], [36], [34], [19] and references therein). Let $\rho(z,\bar{\xi},\bar{\eta})$ be a holomorphic function near the origin in $\mathbb{C}^3$ with $\rho(0,0,0) = 0$ and $d\rho(0,0,0) = \bar{\eta}$. For $z, \xi \in \Delta_\delta$, w and $\eta \in \Delta_\varepsilon$, let

$$\mathcal{S} = \{w = \rho(z,\bar{\xi},\bar{\eta})\}$$

be a parametrized antiholomorphic family of holomorphic curves near the origin, parametrized by (ξ,η). We will call such a family a (*general*) *Segre family*, and for each point $p = (\xi,\eta) \in \Delta_\delta \times \Delta_\varepsilon$ we call the corresponding holomorphic curve $Q_p = \{w = \rho(z,\bar{\xi},\bar{\eta})\} \in \mathcal{S}$ its *Segre variety*. Clearly, an m-admissible Segre family is a particular example of a general Segre family.

We say that two (general) Segre families *coincide* if there exists a nonempty open neighbourhood G of the origin such that for any point $p \in G$ the Segre varieties of p in both families coincide. Further, given a (general) Segre family $\mathcal{S}$, from the implicit function theorem we conclude that the parametrized antiholomorphic family of planar holomorphic curves

$$\mathcal{S}^* = \{\bar{\eta} = \rho(\bar{\xi}, z, w)\}$$

is also a (general) Segre family for some, possibly smaller, polydisc $\Delta_{\tilde{\delta}} \times \Delta_{\tilde{\varepsilon}}$.

Definition 3.7. The Segre family $\mathcal{S}^*$ is called the *dual Segre family* for $\mathcal{S}$.

Importantly, the dual Segre family *does* depend on the parametrization of the initial family. The dual Segre family has a simple interpretation: in the defining equation of the family $\mathcal{S}$ one should consider the parameters $\bar{\xi},\bar{\eta}$ as new coordinates, and the variables z, w as new parameters. We denote the Segre variety of a point p with respect to the family $\mathcal{S}^*$ by Q^*_p.

Lemma 3.8. *Suppose that $\mathcal{S}$ is a positive (respectively, negative) m-admissible Segre family. Then $\mathcal{S}^*$ is a negative (respectively, positive) m-admissible Segre family.*

Proof. To obtain the defining function $\rho^*(z,\bar\xi,\bar\eta)$ of the general Segre family $\mathcal{S}^*$ we solve for w its defining equation

$$\bar\eta = we^{\pm i w^{m-1}(z\bar\xi + \sum_{k\geq 2}\psi_k(w)z^k\bar\xi^k)}. \tag{3.10}$$

Note that (3.10) implies

$$w = \bar\eta e^{\mp i w^{m-1}(z\bar\xi + O(z^2\bar\xi^2))}. \tag{3.11}$$

We then obtain from (3.11) $w = \rho^*(z,\bar\xi,\bar\eta) = \bar\eta(1 + O(z\bar\xi))$. Substituting this back into (3.11) gives $w = \rho^*(z,\bar\xi,\bar\eta) = \bar\eta e^{\mp i\bar\eta^{m-1}(z\bar\xi + O(z^2\bar\xi^2))}$, which proves the lemma. □

We also consider the following Segre family, connected with $\mathcal{S}$:

$$\bar{\mathcal{S}} = \{w = \bar\rho(z,\bar\xi,\bar\eta)\}.$$

Definition 3.9. The Segre family $\bar{\mathcal{S}}$ is called the *conjugate family* of $\mathcal{S}$.

If $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ is the antiholomorphic involution $(z,w) \mapsto (\bar z,\bar w)$, then one simply has $\sigma(Q_p) = \overline{Q_{\sigma(p)}}$. We will denote by $\bar Q_p$ the Segre variety of the point p with respect to the family $\bar{\mathcal{S}}$. It follows from the definition that if $\mathcal{S}$ is a positive (respectively, negative) m-admissible Segre family, then $\bar{\mathcal{S}}$ is a negative (respectively, positive) m-admissible Segre family.

Just as for the case of an m-admissible Segre family, we say that a (general) Segre family $\mathcal{S} = \{w = \rho(z,\bar\xi,\bar\eta)\}$ *has a real structure* if there exists a smooth real-analytic hypersurface $M \subset \mathbb{C}^2$ passing through the origin such that $\mathcal{S}$ coincides with the Segre family of M (as a parametrized family!).

The use of the dual and conjugate Segre families stems from the following

Proposition 3.10. *A (general) Segre family $\mathcal{S}$ has a real structure if and only if the dual Segre family $\mathcal{S}^*$ coincides with the conjugate one: $\mathcal{S}^* = \bar{\mathcal{S}}$.*

Proof. Suppose that $\mathcal{S}$ is the Segre family at the origin of a real hypersurface $M \subset \mathbb{C}^2$ with the complex defining equation $w = \rho(z,\bar z,\bar w)$. Then $\mathcal{S}$ is given by $\{w = \rho(z,\bar\xi,\bar\eta)\}$, and if $(z,w) \in Q^*_{(\xi,\eta)}$, then $\bar\eta = \rho(\bar\xi, z, w)$, so that $(\bar\xi,\bar\eta) \in Q_{(\bar z,\bar w)}$. Then (2.1) gives $(\bar z,\bar w) \in Q_{(\bar\xi,\bar\eta)}$, and so $(z,w) \in \sigma(Q_{(\bar\xi,\bar\eta)}) = \bar Q_{(\xi,\eta)}$. In the same way one shows that $(z,w) \in \bar Q_{(\xi,\eta)}$ implies $(z,w) \in Q^*_{(\xi,\eta)}$, so that $\mathcal{S}^* = \bar{\mathcal{S}}$.

Conversely, if $\mathcal{S}^* = \bar{\mathcal{S}}$, then $[\bar\eta = \rho(\bar\xi, z, w)] \Leftrightarrow [w = \bar\rho(z,\bar\xi,\bar\eta)]$, which is possible only if

$$\bar\eta \equiv \rho(\bar\xi, z, \bar\rho(z,\bar\xi,\bar\eta)).$$

Changing notation and replacing in the latter identity the variables $\bar\eta, \bar\xi, z$ by $w, z, \bar\xi$ respectively, we obtain the complexification of the reality condition (2.2). Hence, the equation $w = \rho(z,\bar z,\bar w)$ determines the germ at the origin of a smooth real-analytic hypersurface M. This proves the proposition. □

We next transfer the above real structure criterion from m-admissible families to the associated ODEs.

Definition 3.11. Let $\mathcal{E}$ be an m-admissible ODE. We say that an m-admissible ODE $\mathcal{E}^*$ is *dual to* $\mathcal{E}$ if the negative m-admissible Segre family dual to the family $\mathcal{S}^+_{\mathcal{E}}$ is associated with $\mathcal{E}^*$, i.e.,

$$\mathcal{E}^* \text{ is dual to } \mathcal{E} \;\Leftrightarrow\; (\mathcal{S}^+_{\mathcal{E}})^* = \mathcal{S}^-_{\mathcal{E}^*}.$$

In the same manner, we say that an m-admissible ODE $\bar{\mathcal{E}}$ is *conjugate* to $\mathcal{E}$ if the negative m-admissible Segre family conjugate to $\mathcal{S}^+_{\mathcal{E}}$ is associated with $\bar{\mathcal{E}}$, i.e.,

$$\bar{\mathcal{E}} \text{ is conjugate to } \mathcal{E} \;\Leftrightarrow\; \bar{\mathcal{S}}^+_{\mathcal{E}} = \mathcal{S}^-_{\bar{\mathcal{E}}}.$$

From Proposition 3.5 we conclude that both the conjugate and dual ODEs are unique (if they exist). The existence of the conjugate ODE is obvious: if $\mathcal{E}$ is given by $z'' = \frac{P(w)}{w^m}z' + \frac{Q(w)}{w^{2m}}z$, then clearly the desired ODE $\bar{\mathcal{E}}$ is given explicitly by

$$z'' = \frac{\bar{P}(w)}{w^m}z' + \frac{\bar{Q}(w)}{w^{2m}}z. \tag{3.12}$$

However, the existence of the dual ODE is a more subtle issue. To prove it, we first need

Proposition 3.12 (Transversality Lemma). *Let $\mathcal{S}$ be an m-admissible Segre family in a polydisc $\Delta_\delta \times \Delta_\varepsilon$, and X be its singular locus. After possibly shrinking the polydisc $\Delta_\delta \times \Delta_\varepsilon$, the following property holds: if $p, q \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X$, $p \neq q$, and Q_p and Q_q intersect at a point r, then the intersection is transverse.*

Proof. Suppose first that $\mathcal{S}$ is positive. Take $p = (\xi, \eta) \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X$ and consider Q_p as a graph $w = w(z) = \bar{\eta}e^{i\bar{\eta}^{m-1}(z\bar{\xi}+O(z^2\bar{\xi}^2))}$. Then

$$w = \bar{\eta} + O(z\bar{\xi}\bar{\eta}), \qquad \frac{w'}{w^m} = i\bar{\xi} + O(z\bar{\xi}). \tag{3.13}$$

The latter implies that by shrinking the polydisc $\Delta_\delta \times \Delta_\varepsilon$, one can make the map

$$(\xi, \eta) \mapsto \left(w(z), \frac{w'(z)}{w^m(z)}\right),$$

which is defined for each z, injective in $(\Delta_\delta \times \Delta_\varepsilon) \setminus X$ (for all z). Then the same property holds for the map

$$(\xi, \eta) \mapsto (w(z), w'(z)),$$

which shows that the graphs Q_p and Q_q cannot have the same slope at a point of intersection. The proof for the negative case is analogous. $\square$

Proposition 3.13. *Let $\mathcal{E}$ be an m-admissible ODE. Then the dual ODE $\mathcal{E}^*$ always exists.*

Proof. Let $\Delta_\delta \times \Delta_\varepsilon$ be the polydisc where $\mathcal{S}^+_{\mathcal{E}}$ is defined, and X be the singular locus. For simplicity, we will assume that the dual family is defined in the same polydisc. Consider two (possibly multiple-valued) linearly independent solutions $h_1(w), h_2(w)$ of $\mathcal{E}$ in the punctured disc Δ_ε^*. Then, by the definition of the associated Segre family, for any $p = (\xi, \eta) \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X$, $\xi, \eta \neq 0$, there exist unique complex numbers $\lambda_1(\bar\xi, \bar\eta)$, $\lambda_2(\bar\xi, \bar\eta)$ such that Q_p is contained in the graph

$$z = \lambda_1(\bar\xi, \bar\eta)h_1(w) + \lambda_2(\bar\xi, \bar\eta)h_2(w).$$

As the family $\mathcal{S}$ depends on the parameters holomorphically, $\lambda_1(\bar\xi, \bar\eta)$, $\lambda_2(\bar\xi, \bar\eta)$ are two (possibly multiple-valued) analytic functions in $\Delta_\delta^* \times \Delta_\varepsilon^*$.

We claim that $\bar\xi\lambda_1(\bar\xi, \bar\eta)$ and $\bar\xi\lambda_2(\bar\xi, \bar\eta)$ are independent of $\bar\xi$. Indeed, we note from the defining equation (3.4) that the expression $z\bar\xi$ for the family $\mathcal{S}$ depends only on w and $\bar\eta$, so that in some polydisc U in $\Delta_\delta \times \Delta_\varepsilon^* \times \Delta_\delta \times \Delta_\varepsilon^*$ we have $\lambda_1(\bar\xi, \bar\eta)\bar\xi h_1(w) + \lambda_2(\bar\xi, \bar\eta)\bar\xi h_2(w) = \Psi(w, \bar\eta)$ for an appropriate holomorphic function Ψ. Differentiating this equality with respect to w and solving the resulting system of linear equations with respect to $\lambda_1\bar\xi$, $\lambda_2\bar\xi$, we get

$$(\lambda_1(\bar\xi, \bar\eta)\bar\xi, \lambda_2(\bar\xi, \bar\eta)\bar\xi) = (\Psi(w, \bar\eta), \Psi_w(w, \bar\eta)) \cdot H^{-1}(w), \tag{3.14}$$

where $H(w)$ is the Wronski matrix for the linearly independent functions $h_1(w)$, $h_2(w)$ (we consider the single-valued branches of these functions, defined in the polydisc U). As the right-hand side of (3.14) depends on $\bar\eta$ only, we conclude that $\lambda_1(\bar\xi, \bar\eta)\bar\xi$, $\lambda_2(\bar\xi, \bar\eta)\bar\xi$ are independent of $\bar\xi$, which proves the claim.

It follows from the claim that each Q_p as above is contained in the graph

$$z\bar\xi = \tau_1(\bar\eta)h_1(w) + \tau_2(\bar\eta)h_2(w)$$

for some (possibly multiple-valued) functions $\tau_1(\bar\eta)$, $\tau_2(\bar\eta)$ analytic in Δ_ε^*. It follows that for any $p = (\xi, \eta) \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X$, $\xi \neq 0$, the dual Segre variety Q_p^* is contained in the graph

$$z\bar\xi = \tau_1(w)h_1(\bar\eta) + \tau_2(w)h_2(\bar\eta). \tag{3.15}$$

We now claim that the Wronskian $d(w) = \left|\begin{smallmatrix} \tau_1(w) & \tau_2(w) \\ \tau_1'(w) & \tau_2'(w) \end{smallmatrix}\right|$ does not vanish in Δ_ε^*. Indeed, suppose $d(w_0) = 0$ for some w_0, and let $(0, w_0) \in Q^*_{(\xi_0,\eta_0)}$ for some (ξ_0, η_0), $\xi_0 \neq 0$ (one can take $(\xi_0, \eta_0) = (\xi, \bar w_0)$ for some $\xi \in \Delta_\delta^*$). We seek all (ξ, η) such that $Q^*_{(\xi,\eta)}$ passes through $(0, w_0)$ and has 1-jet there the same as $Q^*_{(\xi_0,\eta_0)}$. Clearly, such (ξ, η) are given by

$$\begin{pmatrix} \tau_1(w_0) & \tau_2(w_0) \\ \tau_1'(w_0) & \tau_2'(w_0) \end{pmatrix} \cdot \frac{1}{\bar\xi}\begin{pmatrix} h_1(\bar\eta) \\ h_2(\bar\eta) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \tag{3.16}$$

where $\alpha = (h_1(\bar\eta_0)\tau_1'(w_0) + h_2(\bar\eta_0)\tau_2'(w_0))/\bar\xi_0$. If we think of $\left(\frac{1}{\bar\xi}h_1, \frac{1}{\bar\xi}h_2\right)$ as the unknown variables in the above linear system, then since $d(w_0) = 0$, its solution contains an affine line L passing through $\left(\frac{1}{\bar\xi_0}h_1(\bar\eta_0), \frac{1}{\bar\xi_0}h_2(\bar\eta_0)\right)$. The linear independence of $h_1(w)$ and $h_2(w)$ implies that the map $H : (\xi, \eta) \mapsto \left(\frac{1}{\xi}h_1(\eta), \frac{1}{\xi}h_2(\eta)\right)$ is locally biholomorphic near $(\bar\xi_0, \bar\eta_0)$, so that there exist points $(\xi, \eta) \neq (\xi_0, \eta_0)$ near (ξ_0, η_0) with $H(\bar\xi, \bar\eta) \in L$. We conclude that there exists a 1-dimensional family of dual Segre varieties Q_p^* passing through $(0, w_0)$ that have the same 1-jet at $w = w_0$. But this contradicts Proposition 3.12, and so $d(w_0) \neq 0$.

It follows that the graphs (3.15) satisfy the linear differential equation

$$W(z,\tau_1,\tau_2)=\begin{vmatrix} z & \tau_1(w) & \tau_2(w)\\ z' & \tau_1'(w) & \tau_2'(w)\\ z'' & \tau_1''(w) & \tau_2''(w)\end{vmatrix}=0,$$

which can be rewritten as

$$z''=A(w)z'+B(w)z, \tag{3.17}$$

with the inverse ODE being

$$w''=-A(w)(w')^2-B(w)(w')^3z \tag{3.18}$$

for some functions $A(w), B(w)$ holomorphic in Δ_ε^*. The relation (3.18) is satisfied by every dual Segre variety Q_p^*, $p\in(\Delta_\delta\times\Delta_\varepsilon)\setminus X$.

On the other hand, we may consider relations (3.13), applied to the dual family $\mathcal{S}^*$, and use them to obtain a second order ODE satisfied by all Q_p^*, $p\in(\Delta_\delta\times\Delta_\varepsilon)\setminus X$. To do so, we apply the implicit function theorem to (3.13) to obtain

$$\bar\xi=\Lambda(z,w,w'/w^m),\qquad \bar\eta=\Omega(z,w,w'/w^m)$$

for some functions $\Lambda(z,w,\zeta)=i\zeta+O(z\zeta)$, $\Omega(z,w,\zeta)=w+O(zw\zeta)$, holomorphic in a polydisc $V\subset\mathbb{C}^3$ centred at the origin. We next differentiate twice the relation (3.4), applied to the dual family $\mathcal{S}^*$, with respect to z and get $w''=O(\bar\xi^2\bar\eta^m)$. Plugging $\bar\xi=\Lambda(z,w,w'/w^m)$, $\bar\eta=\Omega(z,w,w'/w^m)$ into this, one gets a second order ODE

$$w''=\Phi(z,w,w'/w^m) \tag{3.19}$$

for some function $\Phi(z,w,\zeta)$ holomorphic in a polydisc $\tilde V\subset\mathbb{C}^3$ centred at the origin (compare this with the procedure in Section 2.2). The ODE (3.19) is satisfied by all Q_p^* with $p\in(\Delta_\delta\times\Delta_\varepsilon)\setminus X$. The function $\Phi(z,w,\zeta)$ also satisfies $\Phi(z,w,\zeta)=O(\zeta^2w^m)$.

We now compare (3.19) with (3.18). We set $\zeta:=w'/w^m$ and observe that in some domain $G\subset\tilde V$ we have $\Phi(z,w,\zeta)=-A(w)w^{2m}\zeta^2-B(w)w^{3m}\zeta^3z$, which shows that the function $\Phi(z,w,\zeta)$ is cubic with respect to the third argument. Since, in addition, $\Phi(z,w,\zeta)=O(\zeta^2w^m)$, we conclude that $\Phi(z,w,\zeta)$ has the form $w^m(\Phi_2(w)\zeta^2+\Phi_3(w)\zeta^3z)$ for some $\Phi_2(w)$ and $\Phi_3(w)$ holomorphic in a disc $\Delta_r\subset\mathbb{C}$, $r>0$. Then the substitution $\zeta=w'/w^m$ turns (3.19) into an m-admissible ODE, rewritten in the inverse form. This proves the proposition. □

Combining Proposition 3.13 with Propositions 3.5 and 3.10, we immediately obtain a crucial

Corollary 3.14. *An m-admissible ODE $\mathcal{E}$ has a positive real structure if and only if the conjugate ODE coincides with the dual one: $\mathcal{E}^*=\bar{\mathcal{E}}$.*

We can now prove the main result of this section.

Theorem 3.15. *Let $\mathcal{E} : z'' = \frac{P(w)}{w^m} z' + \frac{Q(w)}{w^{2m}} z$ be an m-admissible ODE, $w \in \Delta_r$, $r > 0$. Then $\mathcal{E}$ has a positive real structure if and only if the functions $P(w)$, $Q(w)$ have the form*

$$P(w) = 2ia(w) - mw^{m-1}, \qquad Q(w) = b(w) + iw^m a'(w), \tag{3.20}$$

where $a(w) = \sum_{j=0}^{\infty} a_j w^j$, $a_j \in \mathbb{R}$, and $b(w) = \sum_{j=0}^{\infty} b_j w^j$, $b_j \in \mathbb{R}$, are power series convergent in Δ_r. Moreover, if $\mathcal{E}$ has a positive real structure, then the associated real hypersurface $M \subset \mathbb{C}^2$ is Levi-nondegenerate and spherical outside the complex locus $X = \{w = 0\}$.

Proof. Let $\mathcal{E}^*$ be given as $z'' = (P^*(w)/w^m)z' + (Q^*(w)/w^{2m})z$. As previously observed, the conjugate ODE $\bar{\mathcal{E}}$ has the form $z'' = (\bar{P}(w)/w^m)z' + (\bar{Q}(w)/w^{2m})z$. Let $\mathcal{S} = \mathcal{S}^+_{\mathcal{E}}$ be given in a polydisc $\Delta_\delta \times \Delta_\varepsilon$ by $w = \bar\eta e^{i\bar\eta^{m-1}\psi(z\bar\xi,\bar\eta)}$ with ψ as in (3.4), and $\mathcal{S}^*$ be given (in the same polydisc, for simplicity) by $w = \bar\eta e^{-i\bar\eta^{m-1}\psi^*(z\bar\xi,\bar\eta)}$ with ψ^* as in (3.4). Then $\bar{\mathcal{S}}$ is given by $w = \bar\eta e^{-i\bar\eta^{m-1}\bar\psi(z\bar\xi,\bar\eta)}$. According to Corollary 3.14, $\mathcal{E}$ has a real structure if and only if $\bar{P}(w) = P^*(w)$ and $\bar{Q}(w) = Q^*(w)$. It follows from (3.6), (3.7) that the latter conditions are equivalent to

$$\bar\psi_2 = \psi_2^*, \qquad \bar\psi_3 = \psi_3^*, \tag{3.21}$$

so that one has to develop condition (3.21). By the definition of the dual family, one has

$$[\bar\eta = we^{iw^{m-1}\psi(z\bar\xi,w)}] \;\Leftrightarrow\; [w = \bar\eta e^{-i\bar\eta^{m-1}\psi^*(z\bar\xi,\bar\eta)}],$$

and using the expansion (3.4), it is not difficult to deduce that

$$\begin{aligned} z\bar\xi + \psi_2(w)z^2\bar\xi^2 + \psi_3(w)z^3\bar\xi^3 + O(z^4\bar\xi^4) &= \big(z\bar\xi + \psi_2^*(\bar\eta)z^2\bar\xi^2 + \psi_3^*(w)z^3\bar\xi^3 + O(z^4\bar\xi^4)\big) \\ &\quad \times e^{i(m-1)w^{m-1}(z\bar\xi+\psi_2(w)z^2\bar\xi^2+O(z^3\bar\xi^3))}\big|_{\bar\eta=w+iw^m z\bar\xi+O(z^2\bar\xi^2)}. \end{aligned} \tag{3.22}$$

Gathering in (3.22) terms with $z^2\bar\xi^2$ and $z^3\bar\xi^3$ respectively, one gets

$$\begin{aligned} &\psi_2 = \psi_2^* + i(m-1)w^{m-1}, \\ &\psi_3 = \psi_3^* + iw^m(\psi_2^*)' + i(m-1)w^{m-1}\psi_2 - \tfrac{1}{2}(m-1)^2w^{2m-2} + i(m-1)w^{m-1}\psi_2^*. \end{aligned}$$

In view of these identities, (3.21) can be rewritten as

$$\begin{aligned} \psi_2(w) &= \lambda(w) + i\frac{m-1}{2}w^{m-1}, \\ \psi_3(w) &= \mu(w) + \frac{i}{2}w^m\lambda'(w) + i(m-1)w^{m-1}\lambda(w), \end{aligned} \tag{3.23}$$

where $\lambda(w), \mu(w)$ are convergent power series in Δ_r with *real* coefficients. Applying (3.6), (3.7) again, we conclude that (3.23) is equivalent to

$$\begin{aligned} P(w) &= 2i\lambda(w) - mw^{m-1}, \\ Q(w) &= 6\mu(w) - 8\lambda(w)^2 + iw^m\lambda'(w) + 2(m-1)^2w^{2m-2}, \end{aligned} \tag{3.24}$$

which is already equivalent to (3.20) after setting

$$a(w) := \lambda(w), \qquad b(w) := 6\mu(w) - 8\lambda(w)^2 + 2(m-1)^2 w^{2m-2}. \tag{3.25}$$

It remains to prove that if $\mathcal{E}$ has a real structure, then the associated nonminimal real hypersurface $M \subset \mathbb{C}^2$ is Levi-nondegenerate and spherical in $M \setminus X$, where X is the singular locus of the Segre family $\mathcal{S}$ (and, at the same time, the nonminimal locus of M). Fix $p \in M \setminus X$ and its small neighbourhood V. It follows from Proposition 3.12 that if two Segre varieties of M intersect at a point $r \in V$, then the intersection is transverse. Accordingly, any Segre variety of M near p is uniquely determined by its 1-jet at a given point. This implies (see, e.g., [15], [5]) that M is Levi-nondegenerate at p.

Finally, to prove that M is spherical at $p = (z_0, w_0)$, $w_0 \neq 0$, we argue as in the proof of Proposition 3.13: Fix two linearly independent solutions $h_1(w), h_2(w)$ of $\mathcal{E}$ in Δ^*_ε. Then each Q_q with $q = (\xi, \eta) \notin X$ is contained in the graph

$$z\bar{\xi} = \tau_1(\bar\eta)h_1(w) + \tau_2(\bar\eta)h_2(w) \tag{3.26}$$

for some (possibly multiple-valued) analytic functions $\tau_1(\bar\eta), \tau_2(\bar\eta)$ in Δ^*_ε. We then use slightly modified arguments from [1] to construct the desired mapping into a sphere: Since the Wronskian $d(w) = \begin{vmatrix} h_1(w) & h_2(w) \\ h_1'(w) & h_2'(w) \end{vmatrix}$ is nonzero in Δ^*_ε, we may suppose that either $h_1(w_0) \neq 0$ or $h_2(w_0) \neq 0$ (for some fixed analytic elements of h_1, h_2 in V). If, for example, $h_1(w_0) \neq 0$, consider in V the mapping

$$\Lambda : (z, w) \mapsto \left(\frac{z}{h_1(w)}, \frac{h_2(w)}{h_1(w)}\right). \tag{3.27}$$

As the Wronskian $d(w)$ is nonzero in V, we may assume that Λ is biholomorphic there. By the definition of Λ, the graphs (3.26) are the preimages of complex lines under the map Λ, so that Λ maps Segre varieties of M to complex lines. It is not difficult to deduce that $\Lambda(M)$ is contained in a quadric $\mathcal{Q} \subset \mathbb{CP}^2$ (see, for example, [31, proof of Theorem 6.1]), which implies that M is spherical at p. ◻

Note that formula (3.27) gives an effective map of M into a quadric, by using the solutions of the associated ODE. We will use this fact in the next section.

Remark 3.16. It is also possible to give a characterization of the ODEs with a negative real structure: these are obtained by conjugating ODEs with a positive real structure.

Remark 3.17. It follows from (3.20) that a complex linear ODE with an isolated meromorphic singularity at the origin is m-admissible with a positive real structure for at most one $m \in \mathbb{Z}_+$.

Remark 3.18. Theorem 3.15, combined with the proof of Proposition 3.5, gives an effective algorithm for obtaining real hypersurfaces $M \subset \mathbb{C}^2$ nonminimal at the origin with prescribed nonminimality order $m \geq 1$, Levi-nondegenerate and spherical outside the nonminimal locus $X \subset M$, and invariant under the group $z \mapsto e^{it}z$ of rotational symmetries. Moreover, one can prescribe an essentially arbitrary 6-jet to the hypersurface M. For the reader's convenience we summarize this algorithm below.

Algorithm for obtaining nonminimal spherical real hypersurfaces

1. Take any power series $a(w)$, $b(w)$ with real coefficients, convergent in some disc centred at the origin, and compute $P(w)$, $Q(w)$ by (3.20). This gives an m-admissible ODE (3.1).
2. Solve the holomorphic ODE (3.9) with a holomorphic parameter $\bar\eta$ and the initial data $y(0) = 0,\ y'(0) = 1$. This gives a function $\psi(t, \bar\eta)$ holomorphic near the origin in $\mathbb{C}^2$.
3. Then the equation $w = \bar w e^{i\bar w^{m-1}\psi(z\bar z,\bar w)}$ determines a real hypersurface $M \subset \mathbb{C}^2$ nonminimal at the origin of nonminimality order m, invariant under the group of rotational symmetries, Levi-nondegenerate and spherical outside the nonminimal locus $X = \{w = 0\}$. The 6-jet of M is determined by finding $\lambda(w)$, $\mu(w)$ using (3.25), and then ψ_2, ψ_3 from (3.23).

Remark 3.19. Theorem 3.15 and the algorithm above provide in fact a *complete* description of real-analytic Levi nonflat hypersurfaces $M \subset \mathbb{C}^2$, nonminimal at the origin, Levi-nondegenerate and spherical outside the complex locus, such that $iz\frac{\partial}{\partial z} \in \mathfrak{aut}(M, 0)$. To prove that, one needs to associate to each M as above a second order m-admissible ODE. The fact that every nonminimal spherical M admits an associated ODE is proved in [32].

4. Formally but not holomorphically equivalent real hypersurfaces

In this section we will use the explicit description of linear meromorphic ODEs with a real structure given by Theorem 3.15 to construct, for each fixed nonminimality order $m \geq 2$, a family of pairwise formally equivalent real hypersurfaces m-nonminimal at the origin, Levi-nondegenerate and spherical outside the nonminimal locus, which are, however, generically pairwise holomorphically inequivalent at the origin. The construction is based on existence of families of linear ODEs with a meromorphic singularity at the origin and a positive real structure, with the ODEs in the family pairwise formally but not holomorphically equivalent.

The desired ODEs and the associated real hypersurfaces are introduced as follows. Fix an integer $m \geq 2$ and set $a(w) \equiv 1$ and $b(w) = \beta w^{2m-2}$, where $\beta \in \mathbb{R}$ is a real constant. Applying (3.20), we obtain the following 1-parameter family $\mathcal{E}^m_\beta$ of complex linear ODEs with a meromorphic singularity at the origin, which are m-admissible and have a positive real structure:

$$z'' = \left(\frac{2i}{w^m} - \frac{m}{w}\right)z' + \frac{\beta}{w^2}z. \tag{4.1}$$

As $m \geq 2$, each $\mathcal{E}^m_\beta$ has a *non-Fuchsian* singularity at the origin, which plays a crucial role in our construction. We denote by M^m_β the real hypersurfaces m-nonminimal at the origin, associated with $\mathcal{E}^m_\beta$. Each M^m_β is Levi-nondegenerate and spherical outside the complex locus $X = \{w = 0\}$.

Introducing a new dependent variable $u := z'w$, one can rewrite (4.1) as a first order system

$$\begin{pmatrix} z \\ u \end{pmatrix}' = \left[\frac{1}{w^m}\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} + \frac{1}{w}\begin{pmatrix} 0 & 1 \\ \beta & 1-m \end{pmatrix}\right]\begin{pmatrix} z \\ u \end{pmatrix} \tag{4.2}$$

with a non-Fuchsian singularity at the origin.

Definition 4.1. A (*formal*) *gauge transformation* is a (formally) invertible local transformation $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of the form

$$(z, w) \mapsto (zf(w),\ g(w)), \tag{4.3}$$

where $f(w)$ and $g(w)$ are two (formal) power series with $f(0) \neq 0$, $g(0) = 0$, $g'(0) \neq 0$. A (*formal*) *special gauge transformation* is a (formally) invertible local transformation of the form (4.3), where $f(w)$ and $g(w)$ are (formal) power series that satisfy an additional normalization $f(0) = 1,\ g(w) = w + O(w^{m+1})$.

Clearly, the set of (formal) gauge transformations, as well as the set of (formal) special gauge transformations, form a group. We also note that for a formal gauge transformation the formal recalculation of derivatives is well-defined (see Section 2), so that one can correctly define, in the natural way, formal equivalence of m-admissible linear ODEs by means of gauge transformations.

Proposition 4.2. *For any $m \geq 2$ and $\beta \in \mathbb{R}$ the ODE $\mathcal{E}^m_\beta$ is formally equivalent to the ODE $\mathcal{E}^m_0$ by means of a formal special gauge transformation.*

Proof. The strategy of the proof is based on finding the fundamental system of formal solutions of the ODE $\mathcal{E}^m_\beta$ (we refer to [24], [2], [47], [13] for more information on the concepts of a formal normal form and a fundamental system of formal solutions). It is straightforward to verify that the function $\exp\left(\frac{2i}{1-m} w^{1-m}\right)$ is a solution of the ODE $\mathcal{E}^m_0$, so that the fundamental system of solutions for $\mathcal{E}^m_0$ is $\left\{1, \exp\left(\frac{2i}{1-m} w^{1-m}\right)\right\}$. For the system $\mathcal{E}^m_\beta$ with $\beta \neq 0$ we consider the corresponding system (4.2) and note that the principal matrix $A_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2i \end{smallmatrix}\right)$ is diagonal and its eigenvalues are distinct, hence the system is nonresonant. We first perform a transformation $\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right) \mapsto (I + w^{m-1}H)\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right)$, where I is the identity matrix and H is a constant 2×2 matrix, and obtain the system $\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right)' = \frac{1}{w^m} A(w)\left(\begin{smallmatrix} z \\ u \end{smallmatrix}\right)$, where $A(w)$ is a holomorphic matrix-valued function of the form $A_0 + A_{m-1}w^{m-1} + O(w^m)$. Here A_0 is the same as for the initial system, and

$$A_{m-1} = \begin{pmatrix} 0 & 1 \\ \beta & 1-m \end{pmatrix} + A_0 H - H A_0.$$

By choosing $H = \frac{1}{2i}\left(\begin{smallmatrix} 0 & 1 \\ -\beta & 0 \end{smallmatrix}\right)$ we may eliminate the nondiagonal elements, and so $A_{m-1} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1-m \end{smallmatrix}\right)$. We now follow the Poincaré–Dulac formal normalization procedure for non-Fuchsian systems (see, e.g., [24, Thm. 20.7]), and using the fact that the system is nonresonant, bring it to a polynomial diagonal normal form with the $(m-1)$-jet being equal to $A_0 + w^{m-1}A_{m-1}$. As all terms of the form $O(w^m)$ can be removed in the nonresonant case, the formal normal form of system (4.2) becomes

$$\begin{pmatrix} z \\ u \end{pmatrix}' = \left[\frac{1}{w^m}\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} + \frac{1}{w}\begin{pmatrix} 0 & 0 \\ 0 & 1-m \end{pmatrix}\right]\begin{pmatrix} z \\ u \end{pmatrix}. \tag{4.4}$$

This implies that systems (4.2) for different β are formally gauge equivalent; however, our goal is to deduce the equivalence of the ODEs (4.1), which is a different issue. The normal form (4.4) admits the fundamental matrix of solutions

$$e^{\frac{1}{1-m}w^{1-m}\begin{pmatrix}0&0\\0&2i\end{pmatrix}}\cdot w^{\begin{pmatrix}0&0\\0&1-m\end{pmatrix}}.$$

We conclude from this that the fundamental system of formal solutions for (4.2) is of the form

$$\hat{F}_\beta(w)\cdot e^{\frac{1}{1-m}w^{1-m}\begin{pmatrix}0&0\\0&2i\end{pmatrix}}\cdot w^{\begin{pmatrix}0&0\\0&1-m\end{pmatrix}} \tag{4.5}$$

where $\hat{F}_\beta(w)=\begin{pmatrix}f_\beta(w)&g_\beta(w)\\h_\beta(w)&s_\beta(w)\end{pmatrix}$ is a matrix-valued formal power series of the form $I+\sum_{k\geq 2}F_kw^k$ (I denotes the unit 2×2 matrix). This means that the columns of (4.5) are formally linearly independent and their formal substitution into (4.2) gives the identity. Representation (4.5) implies that equation (4.1) has a formal fundamental system of solutions $\left\{f_\beta(w),\, g_\beta(w)\cdot\exp\left(\frac{2i}{1-m}w^{1-m}\right)\cdot w^{1-m}\right\}$ with formal power series

$$f_\beta(w)=1+O(w),\qquad g_\beta(w)=w^{m-1}+O(w^m) \tag{4.6}$$

(the expansion of g_β follows from the fact that, in view of (4.2),

$$\left(g_\beta(w)\exp\left(\frac{2i}{1-m}w^{1-m}\right)\right)'=\frac{1}{w}s_\beta(w)\exp\left(\frac{2i}{1-m}w^{1-m}\right),$$

and $s_\beta(w)=1+O(w)$, so that $\operatorname{ord}_0 g_\beta=m-1$, and after scaling we get $g_\beta(w)=w^{m-1}+O(w^m)$).

We set

$$\chi(w):=\frac{1}{f_\beta(w)},\qquad \tau(w):=w\left(1+\frac{1-m}{2i}w^{m-1}\ln\frac{g_\beta(w)}{w^{m-1}f_\beta(w)}\right)^{\frac{1}{1-m}}.$$

In view of (4.6), $\tau(w)$ is a well-defined formal power series of the form $w+O(w^{m+1})$, and $\chi(w)$ is a well-defined formal power series of the form $1+O(w)$. We claim that

$$(z,w)\mapsto(\chi(w)z,\tau(w)) \tag{4.7}$$

is the desired formal special gauge transformation sending $\mathcal{E}^m_\beta$ into $\mathcal{E}^m_0$. This can be seen either from a straightforward computation (one has to perform the substitution (4.7) in $\mathcal{E}^m_\beta$ and use the fact that $\left\{f_\beta(w),\, g_\beta(w)\cdot\exp\left(\frac{2i}{1-m}w^{1-m}\right)\cdot w^{1-m}\right\}$ is the fundamental system of solutions for $\mathcal{E}^m_\beta$), or as follows. As is shown in [1], if $z_1(w), z_2(w)$ are linearly independent holomorphic solutions of a second order linear ODE $z''=p(w)z'+q(w)z$, then the transformation $z\mapsto(1/z_1(w))z$, $w\mapsto z_2(w)/z_1(w)$ transfers the initial ODE into the simplest ODE $z''=0$. The same fact can be verified, by a simple computation, for more general classes of functions, for example, for series of type $h(w)\cdot\exp(aw^\alpha)$, where $h(w)$ is a formal Laurent series with a finite principal part, and $a,\alpha\in\mathbb{C}$ are fixed constants. Then

$$z\mapsto\frac{1}{f_\beta(w)}z,\qquad w\mapsto\frac{g_\beta(w)}{w^{m-1}f_\beta(w)}\exp\left(\frac{2i}{1-m}w^{1-m}\right) \tag{4.8}$$

transforms formally $\mathcal{E}^m_\beta$ into $z''=0$, and

$$z \mapsto z, \qquad w \mapsto \exp\left(\frac{2i}{1-m}w^{1-m}\right) \tag{4.9}$$

transforms $\mathcal{E}^m_0$ into $z''=0$. It follows that the formal substitution of (4.7) into (4.9) gives (4.8). Since the chain rule agrees with the above formal substitutions, this shows that (4.7) transfers $\mathcal{E}^m_\beta$ into $\mathcal{E}^m_0$. This proves the proposition. □

On the other hand, the ODEs $\mathcal{E}^m_\beta$ and $\mathcal{E}^m_0$ are holomorphically inequivalent for a generic β, as the following proposition shows.

Proposition 4.3. *For any $m\geq 2$ and $\beta\neq l(l-m+1)$, $l\in\mathbb{Z}$, the ODE $\mathcal{E}^m_\beta$ has nontrivial monodromy, while the ODE $\mathcal{E}^m_0$ has trivial one.*

Proof. For the ODE $\mathcal{E}^m_0$ the fundamental system of holomorphic solutions is given in $\mathbb{C}\setminus\{0\}$ by $\left\{1, \exp\left(\frac{2i}{1-m}w^{1-m}\right)\right\}$, so that all solutions of $\mathcal{E}^m_0$ are single-valued in $\mathbb{C}\setminus\{0\}$. Hence, its monodromy is trivial. One now needs to obtain the monodromy matrix for a generic system (4.2). To this end we consider ∞ as an isolated singular point for (4.2) and perform the change of variables $t:=1/w$. We obtain the system

$$\begin{pmatrix} y\\ u\end{pmatrix}' = \left[t^{m-2}\begin{pmatrix}0 & 0\\ 0 & -2i\end{pmatrix} + \frac{1}{t}\begin{pmatrix}0 & -1\\ -\beta & m-1\end{pmatrix}\right]\begin{pmatrix} y\\ u\end{pmatrix} \tag{4.10}$$

with an isolated *Fuchsian* singularity at $t=0$. As (4.2) does not have any more singular points in $\bar{\mathbb{C}}$ besides $w=0$ and $w=\infty$, it is sufficient to prove nontriviality of the monodromy matrices at $t=0$ for systems (4.10) with $\beta\neq l(l-m+1)$, $l\in\mathbb{Z}$. For the residue matrix $R_\beta=\left(\begin{smallmatrix}0 & -1\\ -\beta & m-1\end{smallmatrix}\right)$ of (4.10) at $t=0$, denote by λ_1,λ_2 its eigenvalues. The Poincaré–Dulac procedure for Fuchsian systems implies (see, e.g., [24, Corollary 16.20]) that the eigenvalues of the monodromy operator for (4.10) are $\{e^{2\pi i\lambda_1}, e^{2\pi i\lambda_2}\}$. In particular, if one of the eigenvalues is not an integer, the system (4.2) (and the corresponding ODE $\mathcal{E}^m_\beta$) has nontrivial monodromy. Applying the relations $\lambda_1+\lambda_2=m-1$, $\lambda_1\lambda_2=-\beta$, we obtain the claim of the proposition. □

Next we need to establish a connection between equivalences of the m-admissible ODEs $\mathcal{E}^m_\beta$ and the associated real hypersurfaces. We start with

Proposition 4.4. *The only formal special gauge transformation preserving the ODE $\mathcal{E}^m_0$ is the identity. In particular, the only formal special gauge transformation transferring $\mathcal{E}^m_\beta$ into $\mathcal{E}^m_0$ is given by (4.7).*

Proof. Let $F: z^*=zf(w),\ w^*=g(w),\ f=1+O(w),\ g=w+O(w^{m+1})$ be a formal special gauge self-transformation of $\mathcal{E}^m_0$. It is not difficult to calculate that F^{-1} transforms $\mathcal{E}^m_0$ into a well-defined formal meromorphic second order linear ODE

$$\frac{f}{(g')^2}z'' + \left(2\frac{f'}{(g')^2} - \frac{fg''}{(g')^3}\right)z' + \left(\frac{f''}{(g')^2} - \frac{f'g''}{(g')^3}\right)z = \left(\frac{2i}{g^m} - \frac{m}{g}\right)\left(\frac{f}{g'}z' + \frac{f'}{g'}z\right).$$

Comparing the above identity with (4.1) with $\beta = 0$ gives

$$\left(\frac{2i}{g^m} - \frac{m}{g}\right)\frac{f'}{g'} - \left(\frac{f''}{(g')^2} - \frac{f'g''}{(g')^3}\right) = 0, \tag{4.11}$$

$$g'\left(\frac{2i}{g^m} - \frac{m}{g}\right) - 2\frac{f'}{f} + \frac{g''}{g'} = \frac{2i}{w^m} - \frac{m}{w}. \tag{4.12}$$

If $f \not\equiv 1$, then (4.11) gives $\frac{f''}{f'} = g'\left(\frac{2i}{g^m} - \frac{m}{g}\right) + \frac{g''}{g'}$, and comparing with (4.12), we obtain $\frac{f''}{f'} = 2\frac{f'}{f} + \frac{2i}{w^m} - \frac{m}{w}$. Substituting $h := 1/f$ (note that $h(w)$ is also a well-defined formal power series with $h(w) = 1 + O(w)$) it is not difficult to deduce that $h'' = \left(\frac{2i}{w^m} - \frac{m}{w}\right)h'$, so that h satisfies the initial ODE $\mathcal{E}_0^m$. But any (formal) power series solution for $\mathcal{E}_0^m$ is constant (as can be seen, for example, from the fact that the fundamental system of solutions for $\mathcal{E}_0^m$ is $\left\{1, \exp\left(\frac{2i}{1-m}w^{1-m}\right)\right\}$), which contradicts $h \not\equiv 1$, $f \not\equiv 1$.

Suppose now that $f \equiv 1$. Then (4.11) holds trivially, and we examine (4.12). Assuming that $g(w) \not\equiv w$, (4.12) can be rewritten as a *well-defined* differential relation

$$2i\left(\frac{1}{g^{m-1}(1-m)}\right)' - 2i\left(\frac{1}{w^{m-1}(1-m)}\right)' + (\ln g')' - m\left(\ln\frac{g}{w}\right)' = 0,$$

which gives $\frac{2i}{1-m}\left(\frac{1}{g^{m-1}} - \frac{1}{w^{m-1}}\right) + \ln g' - m\ln\frac{g}{w} = C_1$ for some constant $C_1 \in \mathbb{C}$. It follows that the formal meromorphic Laurent series $\frac{1}{1-m}\left(\frac{1}{g^{m-1}} - \frac{1}{w^{m-1}}\right)$ is in fact a formal power series, and a straightforward computation shows that the substitution $\frac{1}{1-m}\left(\frac{1}{g^{m-1}} - \frac{1}{w^{m-1}}\right) =: u$, where $u(w)$ is a formal power series, transforms the latter equation for g into $2iu + \ln(w^m u' + 1) = C_1$. Shifting u, we get the equation $2iu + \ln(w^m u' + 1) = 0$, where $u(0) = 0$. Hence we finally obtain the following meromorphic first order ODE for the formal power series $u(w)$:

$$u' = \frac{1}{w^m}(e^{-2iu} - 1). \tag{4.13}$$

However, (4.13) has no nonzero formal power series solutions. To see this, we note that for $u \not\equiv 0$, $u(0) = 0$, equation (4.13) can be represented as $-\frac{1}{2i}u'\left(\frac{1}{u} + H(u)\right) = \frac{1}{w^m}$, where $H(t)$ is a function holomorphic at the origin. Hence, the logarithmic derivative u'/u has the expansion $-2i/w^m + \cdots$, where the dots denote a formal power series in w. But this clearly cannot happen for a formal power series $u(w)$. Hence $u \equiv 0$, and returning to the unknown function g, we get $\frac{1}{g^{m-1}} - \frac{1}{w^{m-1}} = C$ for some constant $C \in \mathbb{C}$, so that

$$g(w) = \frac{w}{(1 + Cw^{m-1})^{1/(m-1)}}.$$

Taking into account $g(w) = w + O(w^{m+1})$, we conclude that $C = 0$ and $g(w) = w$. This proves the proposition. □

Let now $\mathcal{S} = \{w = \rho(z, \bar{\xi}, \bar{\eta})\}$ be a (general) Segre family in a polydisc $\Delta_\delta \times \Delta_\varepsilon$. We consider the *complex* submanifold

$$\mathcal{M}_{\mathcal{S}} = \{(z, w, \xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon \times \Delta_\delta \times \Delta_\varepsilon : w = \rho(z, \xi, \eta)\} \subset \mathbb{C}^4, \tag{4.14}$$

and call it the *associated foliated submanifold* of the family $\mathcal{S}$. If $\mathcal{S}$ is associated with an m-admissible ODE $\mathcal{E}$, we call $\mathcal{M}_{\mathcal{S}}$ the *associated foliated submanifold* of $\mathcal{E}$. We call $\mathcal{M}_{\mathcal{S}}$ *m-admissible* if $\mathcal{S}$ is m-admissible. If S is the Segre family of a real hypersurface $M \subset \mathbb{C}^2$, then the associated foliated submanifold is simply the complexification of M. The concept of the associated foliated submanifold is somewhat analogous to that of the *submanifold of solutions* of a nonsingular completely integrable PDE system (see, e.g., [11], [38], [19], [34]). Here we consider the case of *singular* differential equations and formal mappings between them.

The foliated submanifold $\mathcal{M}_{\mathcal{S}}$ admits two natural foliations. The first one is the initial foliation $\mathcal{S}$ with leaves $\{(z, w, \xi, \eta) \in \mathcal{M}_{\mathcal{S}} : \xi = \text{const}, \ \eta = \text{const}\}$. The second one is the family of dual Segre varieties with leaves $\{(z, w, \xi, \eta) \in \mathcal{M}_{\mathcal{S}} : z = \text{const}, \ w = \text{const}\}$. If now $\mathcal{E}_1, \mathcal{E}_2$ are two m-admissible ODEs, then it is crucial for the study of (formal) biholomorphisms between them to consider (formal) biholomorphisms between the associated foliated submanifolds $\mathcal{M}_{\mathcal{S}_1}, \mathcal{M}_{\mathcal{S}_2}$, preserving the origin and both foliations. Clearly, any such biholomorphism has the form

$$(z, w, \xi, \eta) \mapsto (F(z, w), G(\xi, \eta)), \tag{4.15}$$

where $F(z, w), G(\xi, \eta)$ are (formal) biholomorphisms $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. In this case we call the transformation $(F(z, w), G(\xi, \eta)) : (\mathcal{M}_{\mathcal{S}_1}, 0) \to (\mathcal{M}_{\mathcal{S}_2}, 0)$ a (*formal*) *coupled transformation* of $\mathcal{M}_{\mathcal{S}_1}$ into $\mathcal{M}_{\mathcal{S}_2}$.

Using the notion of associated foliated submanifolds, one can push the concept of a Segre family to the formal level. Namely, let $\rho(z, \xi, \eta)$ be a formal power series without a constant term and the linear part equal to η. We then call the formal complex submanifold $\mathcal{M} = \{w = \rho(z, \xi, \eta)\}$ of $\mathbb{C}^4$ a *formal foliated submanifold.* A formal foliated submanifold can be identified with its formal defining function ρ. If, in addition, ρ is as in (3.4), we call $\mathcal{M}$ *m-admissible.* If $\mathcal{M} = \{w = \rho(z, \xi, \bar\eta)\}$ is a formal foliated submanifold such that the defining function $\rho(z, \xi, \eta)$ contains η as a factor (for example, all m-admissible formal foliated submanifolds have this property), and $\mathcal{E}$ is an m-admissible ODE, then the derivatives $\rho_z(z, \xi, \bar\eta)$ and $\rho_{zz}(z, \xi, \eta)$ are well-defined power series, and we say that $\mathcal{M}$ is *formally associated* with the ODE $\mathcal{E}$ if the well-defined substitution of the power series $\rho(z, \xi, \eta)$ into the inverse ODE to $\mathcal{E}$ gives the identity of the formal power series in z, ξ, η on both sides of the equation.

Let now $\mathcal{E}_1, \mathcal{E}_2$ be two m-admissible ODEs, let $\mathcal{M}_1$ be a foliated submanifold associated with $\mathcal{E}_1$, and $F(z, w) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a formal invertible mapping tangent to the identity map at the origin. Then the formal recalculation of the derivatives z_w, z_{ww}, w_z, w_{zz} is well-defined (see Section 2), and one can correctly define the formal equivalence of $\mathcal{E}_1, \mathcal{E}_2$ by means of F. In addition, consider a similar formal transformation $G(\xi, \eta) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of the space of parameters ξ, η. One can then correctly define the image of the foliated submanifold $\mathcal{M}_1$ under the formal direct product $(F(z, w), G(\xi, \eta)) : (\mathbb{C}^4, 0) \to (\mathbb{C}^4, 0)$ and obtain a unique formal foliated submanifold $\mathcal{M}$ (one has to substitute (F^{-1}, G^{-1}) into $\mathcal{M}_1$ and apply the implicit function theorem in the category of formal power series). It is then immediate that for any formal invertible transformation $F(z, w)$ transferring $\mathcal{E}_1$ into $\mathcal{E}_2$, and any formal invertible transformation $G(\xi, \eta)$ in the space of parameters, where both F and G are tangent to the identity at zero, the image of $\mathcal{M}_1$ under the direct product $(F(z,w), G(\xi,\eta))$ is a foliated submanifold $\mathcal{M}_2$ associated with $\mathcal{E}_2$.

Consider then a (formal) special gauge transformation $(z,w) \mapsto F(z,w) = (zf(w), g(w))$ transforming an m-admissible ODE $\mathcal{E}_1$ into an m-admissible ODE $\mathcal{E}_2$. Let $\mathcal{S}_1, \mathcal{S}_2$ be the associated positive m-admissible Segre families and $\mathcal{M}_1, \mathcal{M}_2$ the associated foliated submanifolds. We claim that there exists a (formal) special gauge transformation $(\xi,\eta) \mapsto G(\xi,\eta) = (\xi\lambda(\eta), \mu(\eta))$ such that $(z,w,\xi,\eta) \mapsto (F(z,w), G(\xi,\eta))$ is a (formal) coupled transformation of $\mathcal{M}_1$ into $\mathcal{M}_2$. Indeed, let us first prove

Lemma 4.5. *There exists a unique* (*formal*) *special gauge transformation*

$$(\xi,\eta) \mapsto G(\xi,\eta) = (\xi\lambda(\eta), \mu(\eta))$$

such that the (*formal*) *transformation* $(z,w,\xi,\eta) \mapsto (F(z,w), G(\xi,\eta))$ *sends* $\mathcal{M}_1$ *to an* m*-admissible* (*formal*) *foliated submanifold* $\mathcal{M}$.

Proof. To simplify notation we will prove the same statement for the special gauge mapping F^{-1} of the ODE $\mathcal{E}_2$. Let $\mathcal{M}_2$ be given by (4.14) with ψ as in (3.4). Our goal is to uniquely determine two (formal) power series $\lambda(\eta), \mu(\eta)$ with $\lambda(\eta) = 1 + O(\eta)$, $\mu(\eta) = \eta + O(\eta^{m+1})$ such that

$$g(w) = \mu(\eta) e^{i\mu(\eta)^{m-1}\psi(zf(w), \xi\lambda(\eta))} \tag{4.16}$$

defines an m-admissible foliated submanifold. Note that (4.16) can be represented as

$$g(w) = \mu(\eta) + i\bar{\mu}(\eta)^m z\xi f(w)\lambda(\eta) + O(z^2\xi^2\eta^m), \tag{4.17}$$

from which we conclude that (4.16) defines a formal foliated submanifold of the form $w = \sum_{j\ge 0}\varphi_j(\eta)(z\xi)^j$ with $\varphi_0(\eta) = O(\eta)$ and $\varphi_j(\eta) = O(\eta^m)$ for $j \ge 1$. Hence we are interested in the choice of $\lambda(\eta), \mu(\eta)$ which gives $\varphi_0(\eta) = \eta$, $\varphi_1(\eta) = i\eta^m$. The latter is equivalent to the fact that the substitution $w = \eta + i\eta^m z\xi + O(z^2\xi^2\eta^m)$ (corresponding to the desired target foliated submanifold $\mathcal{M}$) into (4.17) makes (4.17) an identity modulo $z^2\bar{\xi}^2$. Thus we get $g(\eta) + i\eta^m g'(\eta) z\xi = \mu(\eta) + i\mu(\eta)^m z\xi f(\eta)\lambda(\eta) + O(z^2\xi^2)$, which is equivalent to

$$g(\eta) = \mu(\eta), \qquad \eta^m g'(\eta) = \mu(\eta)^m f(\eta)\lambda(\eta). \tag{4.18}$$

Equations (4.18) enable one to determine $\lambda(\eta), \mu(\eta)$ with the desired properties uniquely, and this proves the lemma. □

If now $G(\xi,\eta)$ is the special gauge transformation provided by Lemma 4.5, it follows from the above arguments that the (formal) image of $\mathcal{M}_1$ under the direct product $(F(z,w), G(\xi,\eta))$ is a (formal) *m-admissible* foliated submanifold $\mathcal{M}$ associated with $\mathcal{E}_2$. However, it is not difficult to show, in the same manner as in the proof of Proposition 3.5, that even in the formal category the associated m-admissible foliated submanifold is unique (since the uniqueness follows from the uniqueness of the solution of the Cauchy problem for the holomorphic ODE (3.9), which holds true in the formal category as well, see [23]). Thus we conclude that $\mathcal{M} = \mathcal{M}_2$, and this proves the existence of the special gauge transformation G in both holomorphic and formal settings.

Conversely, let $(z, w, \xi, \eta) \mapsto (F(z, w), G(\xi, \eta))$ be a (formal) coupled transformation, sending $\mathcal{M}_1$ to $\mathcal{M}_2$, where both F and G are special gauges. It is easy to check, by a computation similar to those in Proposition 4.4, that $F(z, w)$ transfers $\mathcal{E}_1$ into some (formal) m-admissible ODE $\mathcal{E}$. On the other hand, $(F(z, w), G(\xi, \eta))$ (formally) transfers $\mathcal{M}_1$ into $\mathcal{M}_2$, so that $\mathcal{M}_2$ is (formally) associated with $\mathcal{E}$. This shows that $\mathcal{E} = \mathcal{E}_2$ in the case of a holomorphic coupled transformation. To treat the formal case we note that relations (3.6), (3.7) similarly hold for formal m-admissible families, associated with formal m-admissible ODEs (the proof does not change), so that the conclusion $\mathcal{E} = \mathcal{E}_2$ holds true in the formal case as well.

We summarize the above arguments in the following

Proposition 4.6. *Let $\mathcal{E}_1, \mathcal{E}_2$ be two m-admissible ODEs, and $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{C}^4$ be the associated foliated submanifolds. There is a one-to-one correspondence $F(z, w) \mapsto (F(z, w), G(\xi, \eta))$ between (formal) special gauge equivalences $F(z, w)$, transforming $\mathcal{E}_1$ into $\mathcal{E}_2$, and (formal) coupled transformations $(F(z, w), G(\xi, \eta))$ sending $\mathcal{M}_1$ to $\mathcal{M}_2$.*

We are now ready to prove the main result of this section. It is a more detailed version of Theorem A.

Theorem 4.7. *For any $m \geq 2$ and $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$, the real hypersurface $M^m_\beta \subset \mathbb{C}^2$, nonminimal at the origin, associated with the ODE $\mathcal{E}^m_\beta$ as in (4.1), is formally equivalent at the origin to the hypersurface M^m_0 by means of the formal special gauge transformation (4.7), but is locally biholomorphically inequivalent to M^m_0.*

Proof. Consider the foliated submanifolds $\mathcal{M}^m_\beta$ associated with $\mathcal{E}^m_\beta$. It follows from the definitions of the associated real submanifold and the associated foliated submanifold that $\mathcal{M}^m_\beta$ is the complexification of M^m_β. Considering now the reality condition (2.2) for M^m_β and complexifying it, we conclude that $\mathcal{M}^m_\beta$ is invariant under the antiholomorphic linear mapping $\sigma : \mathbb{C}^4 \to \mathbb{C}^4$ given by

$$(z, w, \xi, \eta) \mapsto (\bar{\xi}, \bar{\eta}, \bar{z}, \bar{w}). \tag{4.19}$$

Let now $F(z, w)$ be the formal special gauge equivalence provided by Proposition 4.2, and $(F(z, w), G(\xi, \eta))$ the formal coupled special gauge transformation between $\mathcal{M}^m_\beta$ and $\mathcal{M}^m_0$ provided by Proposition 4.6. Then $\sigma \circ (F(z, w), G(\xi, \eta)) \circ \sigma = (\bar{G}(z, w), \bar{F}(\xi, \eta)$ is also a formal coupled special gauge transformation between $\mathcal{M}^m_\beta$ and $\mathcal{M}^m_0$. Applying now Proposition 4.4, we conclude that $G(\xi, \eta) = \bar{F}(\xi, \eta)$. This immediately implies that the transformation $(F(z, w), G(\xi, \eta))$ is the complexification of $F(z, w)$ (see Section 2), so that $F(z, w)$ maps M^m_β into M^m_0 formally.

Finally, to prove the nonequivalence of M^m_β and M^m_0 for $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$, we use the fact that each M^m_β is Levi-nondegenerate and spherical in $M^m_\beta \setminus X$, where $X = \{w = 0\}$ is the complex locus. As was explained in the proof of Theorem 3.15, for a fixed point $p = (z_0, w_0) \in M^m_\beta \setminus X$ and two fixed solutions $h_1(w), h_2(w)$ of $\mathcal{E}^m_\beta$ near p

with $h_1(w_0) \neq 0$, one of the possible mappings Λ of M^m_β into a quadric $\mathcal{Q} \subset \mathbb{CP}^2$ is given by (3.27). Clearly, Λ has trivial monodromy about the complex locus X if and only if both $h_1(w), h_2(w)$ have trivial monodromy about the origin, and the latter is equivalent to the ODE $\mathcal{E}^m_\beta$ having trivial monodromy at $w = 0$. Now the desired statement follows from Proposition 4.3 and the fact that the monodromy of a mapping into a quadric for a nonminimal hypersurface, Levi-nondegenerate and spherical outside the complex locus, is a biholomorphic invariant (see [31]). This completes the proof of the theorem. □

Proof of statement (a) of Theorem C. The main step of the proof is the generalization of the constructions of Theorem 4.7 to hypersurfaces in $\mathbb{C}^N$ with $N \geq 3$. Fix $m \geq 2$ and $\beta \neq l(l-m+1)$, $l \in \mathbb{Z}$, and suppose that $M^m_\beta, M^m_0 \subset \mathbb{C}^2$ are given near the origin by the defining equations $\operatorname{Im} w = \theta(z\bar{z}, \operatorname{Re} w)$ and $\operatorname{Im} w = \theta'(z\bar{z}, \operatorname{Re} w)$. We also denote the mapping (4.7) by $F(z, w) = (zf(w), g(w))$ and the coordinates in $\mathbb{C}^N$ by $z_1, \dots, z_{N-1}, w$. Then it is not difficult to see that the formal invertible mapping H : $(z_1, \dots, z_{N-1}, w) \mapsto (z_1 f(w), \dots, z_{N-1} f(w), g(w))$ transfers the smooth real-analytic hypersurface $M = \{\operatorname{Im} w = \theta(z_1\bar{z}_1 + \cdots + z_{N-1}\bar{z}_{N-1}, \operatorname{Re} w)\}$ nonminimal at the origin formally into the smooth real-analytic hypersurface $M' = \{\operatorname{Im} w = \theta'(z_1\bar{z}_1 + \cdots + z_{N-1}\bar{z}_{N-1}, \operatorname{Re} w)\}$ nonminimal at the origin. Since M^m_β and M^m_0 are Levi-nondegenerate outside the complex locus $\{w = 0\}$, the same holds true for M and M', so that M and M' are holomorphically nondegenerate.

It can be seen from the proof of Theorem 4.7 that for any choice of a single-valued branch of the mapping Λ, the target quadric $\mathcal{Q}$, considered in the affine chart $\mathbb{C}^2 \subset \mathbb{CP}^2$, is invariant under the rotations $z^* \mapsto e^{it} z^*$, $t \in \mathbb{R}$. Thus one can argue as in the proof of Theorem 4.7 and consider, in the spirit of (3.27), the mapping

$$\Lambda_n : (z_1, \dots, z_{N-1}, w) \mapsto \left(\frac{z_1}{h_1(w)}, \dots, \frac{z_{N-1}}{h_1(w)}, \frac{h_2(w)}{h_1(w)} \right),$$

where $h_1(w)$ and $h_2(w)$ are some linearly independent analytic solutions of the ODE $\mathcal{E}^m_\beta$ in $\mathbb{C} \setminus \{0\}$. Since Λ sends a germ of M^m_β at a Levi-nondegenerate point to a quadric $\mathcal{Q} \subset \mathbb{CP}^2$, the mapping Λ_n transfers a germ of M at a Levi-nondegenerate point into a *nondegenerate* quadric $\mathcal{Q}_N \subset \mathbb{CP}^N$, obtained from $\mathcal{Q}$ by the substitution of $z_1\bar{z}_1 + \cdots + z_{N-1}\bar{z}_{N-1}$ for $z\bar{z}$. Since Λ has nontrivial monodromy, we conclude that the nonminimal hypersurface M has a nontrivial monodromy operator in the sense of [31]. In a similar way we deduce that the monodromy operator of the nonminimal hypersurface M' is trivial. Hence, M and M' are holomorphically inequivalent at the origin. This proves the theorem in the hypersurface case.

For each class of CR-submanifolds of codimension $k \geq 2$ and CR-dimension $n \geq 1$ we consider the holomorphically nondegenerate CR-submanifolds $P = M \times \Pi_{k-1}$ and $P' = M' \times \Pi_{k-1}$, where $M, M' \subset \mathbb{C}^{n+1}$ are chosen from the hypersurface case and $\Pi_{k-1} \subset \mathbb{C}^{k-1}$ is the totally real plane $\operatorname{Im} W = 0$, $W \in \mathbb{C}^{k-1}$. Then the direct product of the above mapping H and the identity map gives a divergent formal equivalence between P and P'. Finally, to show that P and P' are holomorphically inequivalent, we denote the coordinates in $\mathbb{C}^{n+k}$ by (Z, W), $Z \in \mathbb{C}^{n+1}$, $W \in \mathbb{C}^{k-1}$, and note that since Π

is totally real, for each holomorphic equivalence

$$(\Phi(Z,W),\Psi(Z,W)) : (M\times\Pi_{k-1},0)\to(M'\times\Pi_{k-1},0),$$

one has $\Psi(Z,W)=\Psi(W)$ for a vector power series $\Psi(W)$ with real coefficients and $\Psi(0)=0$. Since the initial mapping $(\Phi(Z,W),\Psi(Z,W))$ is invertible at 0, we conclude that the mapping $\Phi(Z,0):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ is invertible at 0 as well, and since $(\Phi(Z,W),\Psi(W)) : (M\times\Pi_{k-1},0)\to(M'\times\Pi_{k-1},0)$, the map $\Phi(Z,0)$ is a local equivalence between $(M,0)$ and $(M',0)$. Now the desired statement is obtained from the hypersurface case.

5. Real hypersurfaces with divergent CR-automorphisms

As an application of Theorem 4.7 we will show in this section that a generic hypersurface M^m_β from Section 4 with $m\geq 2$ has the following property: there exists a *divergent* formal vector field of the form $L=zA(w)\frac{\partial}{\partial z}+B(w)\frac{\partial}{\partial w}$, vanishing to order m at zero, such that its real part $\operatorname{Re}L=L+\bar L$ is formally tangent to M^m_β. In particular, the formal flow of $\operatorname{Re}L$ provides generically *divergent* formal automorphisms of $(M^m_\beta,0)$.

We start with a detailed study of the real hypersurfaces $M^m_0\subset\mathbb{C}^2$. It turns out that they can be described explicitly using elementary functions. Fix an integer $m\geq 2$ and recall that the fundamental system of holomorphic solutions for the ODE $\mathcal{E}^m_0$ is given in $\mathbb{C}\setminus\{0\}$ by $\left\{1,\exp\left(\frac{2i}{1-m}w^{1-m}\right)\right\}$. Applying (3.27), we find that the locally biholomorphic map

$$\Lambda:(Z,W)=\left(z,\ e^{\frac{2i}{1-m}w^{1-m}}\right) \tag{5.1}$$

maps $\mathcal{E}^m_0$ into the simplest equation $Z_{WW}=0$. Consider now the real hyperquadric

$$\mathcal{Q}=\{2|Z|^2+|W|^2=1\}\subset\mathbb{C}^2,$$

linearly equivalent to the standard sphere $S^3\subset\mathbb{C}^2$. We claim that $\Lambda^{-1}(\mathcal{Q})$ contains the Levi-nondegenerate part of the desired hypersurface M^m_0. Indeed, the set $\Lambda^{-1}(\mathcal{Q})\subset\mathbb{C}^2$ can be described as

$$2|z|^2+e^{\frac{2i}{1-m}w^{1-m}}\cdot e^{\frac{-2i}{1-m}\bar w^{1-m}}=1,$$

so that it contains the set $\frac{2i}{1-m}w^{1-m}=\frac{2i}{1-m}\bar w^{1-m}+\ln(1-2|z|^2)$, $|z|^2<1/2$, and the union of this real-analytic set, considered in a sufficiently small polydisc $U\ni 0$, and the complex line $\{w=0\}$ contains the component

$$w=\bar w\left(1+\frac{i}{2}(1-m)\bar w^{m-1}\ln\frac{1}{1-2|z|^2}\right)^{\frac{1}{1-m}}. \tag{5.2}$$

Since Λ is locally biholomorphic in $\mathbb{C}^2\setminus\{w=0\}$, equation (5.2) defines in the polydisc $U\ni 0$ a smooth real-analytic real hypersurface M nonminimal at the origin. As the right-hand side of (5.2) has the expansion $\bar w+i\bar w^m|z|^2+O(\bar w^m|z|^4)$, we conclude that M is m-admissible. The mapping Λ maps locally biholomorphically each of the two sides $\{\operatorname{Re} w > 0\}$ and $\{\operatorname{Re} w < 0\}$ of M into $\mathcal{Q}$. Since all Segre varieties $Q_{(A,B)}$ of $\mathcal{Q}$ with $A \neq 0$ satisfy the simplest ODE $Z_{WW} = 0$, and Λ transforms the ODE $\mathcal{E}_0^m$ into $Z_{WW} = 0$, we conclude that all Segre varieties $Q_{(a,b)}$ of M with $a, b \neq 0$ satisfy the ODE $\mathcal{E}_0^m$. Hence M is an m-admissible real hypersurface associated with $\mathcal{E}_0^m$, and we finally conclude from Proposition 3.5 that $M = M_0^m$, so that the hypersurfaces M_0^m are given by (5.2) for each $m \geq 2$.

Consider now a holomorphic vector field $X = 2iW\frac{\partial}{\partial W} \in \mathfrak{hol}(\mathcal{Q})$. Computation shows that its pull-back under the mapping Λ near each point with $w \neq 0$ equals $w^m\frac{\partial}{\partial w}$. This holomorphic vector field extends holomorphically to the origin, and we conclude that

$$L_0^m = w^m\frac{\partial}{\partial w} \in \mathfrak{hol}(M_0^m, 0).$$

We may construct the desired divergent formal vector field tangent to a hypersurface M_β^m with $m \geq 2$ and $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$, by pulling back the vector field L_0^m with the invertible formal mapping (4.7) (denoted by Φ in what follows). Since the real flow F^t of the vector field L_0^m preserves $(M_0^m, 0)$, and Φ formally transforms $(M_\beta^m, 0)$ into $(M_0^m, 0)$, the well-defined real flow $H^t := \Phi \circ F^t \circ \Phi^{-1}$ formally preserves $(M_\beta^m, 0)$, and the derivation of H^t at $t = 0$ gives a formal vector field L_β^m whose real part is formally tangent to $(M_\beta^m, 0)$. As follows from the construction, L_β^m can be obtained from L_0^m by applying the usual chain rule. Since L_0^m vanishes to order m, we conclude that the same holds for L_β^m. Using the facts that $\Phi(z, w) = (z\chi(w), \tau(w))$, $\chi(w) = 1 + O(w)$, $\tau(w) = w + O(w^{m+1})$ (see Proposition 4.2), we finally calculate

$$L_\beta^m = -\frac{\chi'\tau^m}{\chi\tau'}z\frac{\partial}{\partial z} + \frac{\tau^m}{\tau'}\frac{\partial}{\partial w} = A(w)z\frac{\partial}{\partial z} + B(w)\frac{\partial}{\partial w}. \tag{5.3}$$

Below we formulate the main result of this section, which is a detailed formulation of Theorem B.

Theorem 5.1. *For any $m \geq 2$ and $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$, the germ $(M_\beta^m, 0)$ admits a divergent formal infinitesimal automorphism L_β^m, vanishing to order m. In fact, L_β^m is given by (5.3), where χ and τ are defined by (4.7). The real formal flow $F_t(z, w)$, generated by L_β^m consists of divergent formal automorphisms of $(M_\beta^m, 0)$ for all $t \in \mathbb{R}\setminus C$, where C is a cyclic subgroup in $(\mathbb{R}, +)$.*

Proof. The proof is based on the detailed analysis of the proof of Proposition 4.2. First, we show that the formal power series $B(w)$ in (5.3) is divergent. We denote by $\mathbb{C}[[w]]$ the algebra of formal power series in w and by Υ the linear space of formal series of the form $f(w)w^{-m}\exp\left(\frac{2i}{1-m}w^{1-m}\right)$, where $f(w) \in \mathbb{C}[[w]]$. Recall that $z_1(w) = f_\beta(w) \in \mathbb{C}[[w]]$ and $z_2(w) = g_\beta(w) \cdot w^{1-m} \cdot \exp\left(\frac{2i}{1-m}w^{1-m}\right) \in \Upsilon$ form the fundamental system of formal solutions for $\mathcal{E}_\beta^m$. It is not difficult to verify, by combining the facts that $z_1(w)$ and $z_2(w)$ satisfy the ODE $\mathcal{E}_\beta^m$, that for the well-defined formal Wronskian $D(w) = z_2'z_1 - z_1'z_2 \in \Upsilon$

the classical Liouville–Ostrogradsky formula holds:

$$D'(w) = \left(\frac{2i}{w^m} - \frac{m}{w}\right) D(w). \tag{5.4}$$

Since $D(w) \in \Upsilon$, we deduce from (5.4) that $D(w) = C_0 w^{-m} \exp\left(\frac{2i}{1-m} w^{1-m}\right)$, $C_0 \in \mathbb{C}$, so that the element $D(w) \in \Upsilon$ is convergent. We claim that the ratio $\frac{g_\beta(w)}{w^{m-1} f_\beta(w)} \in \mathbb{C}[[w]]$ is divergent. Indeed, otherwise $z_2(w)/z_1(w) \in \Upsilon$ is convergent as well, and from the relation $(z_1(w))^2 (z_2(w)/z_1(w))' = D(w)$ it follows that $z_1(w)$ is convergent, and hence the mapping (4.7) is convergent, which contradicts Proposition 4.3. Now, from the definition of $\tau(w)$ we conclude that $\tau(w)$ is divergent, and (5.3) shows that $B(w) = (1-m)/(\tau^{1-m})'$ is divergent, which proves the divergence of the vector field L^m_β.

Finally, to prove the divergence of a generic transformation in the flow of L^m_β, we consider the one-dimensional divergent formal vector field $Y = B(w)\frac{\partial}{\partial w}$, vanishing to order m. We then apply to Y the theory of Ecalle–Voronin (we refer to [24] for details). Denote by $H^t(w)$ the formal flow of Y, and assume that it contains a convergent transformation $H^{t_0}(w)$, $t_0 \neq 0$. In the terminology of [24], the convergent transformations in $H^t(w)$ with $t \neq 0$ are *parabolic germs*, and as the vector field Y is divergent, $H^{t_0}(w)$ is a nonembeddable parabolic germ (its Ecalle–Voronin invariants are nontrivial). As any convergent transformation in $H^t(w)$ commutes with $H^{t_0}(w)$, it necessarily lies in the centralizer of $H^{t_0}(w)$, and it follows from the Ecalle–Voronin theory that the set $\{t \in \mathbb{C} : H^t(w)$ is convergent$\}$ is contained in a cyclic subgroup of $(\mathbb{R}, 0)$, generated by some $c \in \mathbb{R}$. Since the flow of Y is the second coordinate function of the flow of L^m_β, we obtain the desired divergence statement. The theorem is now completely proved. □

Proof of statement (b) of Theorem C. The arguments are similar to those for statement (a) (see Section 4). We fix $m \geq 2$, $\beta \neq l(l-m+1)$, $l \in \mathbb{Z}$, and $N \geq 3$. Arguing as in the proof of statement (a), we construct, using the real hypersurface M^m_β, a smooth real-analytic holomorphically nondegenerate hypersurface $M \subset \mathbb{C}^N$ nonminimal at the origin. Then the real part of the divergent formal vector field $L^m_\beta = A(w)z\frac{\partial}{\partial z} + B(w)\frac{\partial}{\partial w}$ is formally tangent to M^m_β, and it is not difficult to see that the real part of the divergent formal vector field $L = A(w)\left(z_1\frac{\partial}{\partial z_1} + \cdots + z_{N-1}\frac{\partial}{\partial z_{N-1}}\right) + B(w)\frac{\partial}{\partial w}$ is formally tangent to M. The vector field L vanishes to order m. The divergence statement for the elements of the real flow of L can be verified in the same way as in the proof of Theorem 5.1. This completes the proof of Theorem C. □

Note that Corollary 1.1 follows directly from Theorem C.

Remark 5.2. As can be verified, for example, from [18], solutions of the ODEs $\mathcal{E}^m_\beta$ with arbitrary $\beta \in \mathbb{R}$ can be described using Bessel functions. Accordingly, it is possible to follow the above method and describe the real hypersurfaces M^m_β in terms of Bessel functions. However, the required computations are quite involved and we do not provide them here.

In conclusion we formulate some open questions. The first one concerns the holomorphic and formal isotropy dimensions (see Introduction) for a Levi-nonflat hypersurface $M \subset \mathbb{C}^2$. The investigation of these two characteristics of a real hypersurface goes back to Poincaré [39], who proved the bound $\dim \mathfrak{aut}(M,0) \leq 5$ for the holomorphic isotropy dimension of a Levi-nondegenerate hypersurface. Combining the known results in the holomorphic category with the convergence results in [6], [26], one can deduce the bounds $\dim \mathfrak{aut}(M,0) \leq 5$, $\dim \mathfrak{f}(M,0) \leq 5$ for all minimal hypersurfaces, as well as for 1-nonminimal ones. In [32] the authors prove the bound $\dim \mathfrak{aut}(M,0) \leq 5$ for an arbitrary Levi-nonflat hypersurface. Somewhat surprisingly, for the formal isotropy dimension, even its finiteness does not seem to follow from any known results. As Theorem B shows, the formal and holomorphic dimensions do not coincide in general, so that the bound $\dim \mathfrak{f}(M,0) \leq 5$ cannot be deduced from the holomorphic case. This leads to the following

Conjecture 5.3. The bound $\dim \mathfrak{f}(M,0) \leq 5$ holds for an arbitrary real-analytic Levi-nonflat germ $(M,0) \subset \mathbb{C}^2$, in particular, $\dim \mathfrak{f}(M,0) < \infty$.

The above question becomes even more delicate if one considers the isotropy group $\mathrm{Aut}(M,0)$ as well as the formal isotropy group $\mathcal{F}(M,0)$. The group structure results in [25], [26] were obtained in the settings where *a posteriori* $\mathrm{Aut}(M,0) = \mathcal{F}(M,0)$ and $\mathfrak{aut}(M,0) = \mathfrak{f}(M,0)$. Since the m-nonminimal case with $m \geq 2$ is significantly different in the sense that $\mathrm{Aut}(M,0) \subsetneq \mathcal{F}(M,0)$ and $\mathfrak{aut}(M,0) \subsetneq \mathfrak{f}(M,0)$ in general, it is of interest to establish a connection between the objects $\mathfrak{aut}(M,0)$, $\mathfrak{f}(M,0)$, $\mathrm{Aut}(M,0)$ and $\mathcal{F}(M,0)$, as well as the group structures for $\mathrm{Aut}(M,0)$ and $\mathcal{F}(M,0)$ in the case $m \geq 2$.

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