The Journal of Geometric Analysis
Volume 13, Number 4, 2003

A Local Extension Theorem for Proper Holomorphic Mappings in $\mathbb{C}^2$

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ABSTRACT. *Let $f : \mathcal{D} \to \mathcal{D}'$ be a proper holomorphic mapping between bounded domains $\mathcal{D}, \mathcal{D}'$ in $\mathbb{C}^2$. Let M, M' be open pieces on $\partial\mathcal{D}, \partial\mathcal{D}'$, respectively that are smooth, real analytic and of finite type. Suppose that the cluster set of M under f is contained in M'. It is shown that f extends holomorphically across M. This can be viewed as a local version of the Diederich–Pinchuk extension result for proper mappings in $\mathbb{C}^2$.*

1. Introduction

Let $\mathcal{D}, \mathcal{D}'$ be smoothly bounded, real analytic domains in $\mathbb{C}^2$. With no further assumptions on the domains such as pseudoconvexity, Diederich and Pinchuk (see [9]) show that any proper holomorphic mapping $f : \mathcal{D} \to \mathcal{D}'$ extends holomorphically across each point of $\partial\mathcal{D}$. The purpose of this article is to propose and prove the following local version of their result.

Theorem 1.1. *Let $\mathcal{D}, \mathcal{D}'$ be bounded domains in $\mathbb{C}^2$ and let $f : \mathcal{D} \to \mathcal{D}'$ be a proper holomorphic mapping. Suppose that M, M' are open pieces of $\partial\mathcal{D}, \partial\mathcal{D}'$, respectively such that*

(i) *$\partial\mathcal{D}$, (respectively $\partial\mathcal{D}'$) is smooth, real analytic and of finite type (in the sense of D'Angelo) in an open neighborhood of $\overline{M}$, (respectively $\overline{M}'$),*
(ii) *the cluster set $cl_f(M) \subset M'$.*

Then f extends holomorphically across each point on M.

Some remarks are in order. First, there is no assumption on the cluster set of M' under f^{-1}. In particular, it is possible that the cluster set of some $z' \in M'$ contains points near which $\partial\mathcal{D}$ may have no regularity at all. Nothing can then be said about extending f^{-1} across z'. The main theorem in [22] can be considered as a weaker version of Theorem 1.1 since it was proved with an additional assumption on the cluster set of M' under f^{-1}. Second, f is not assumed to possess any a priori regularity, such as continuity, near M. Third, a recent result of Diederich–Pinchuk (cf. [11]), which is valid for $n \geq 2$, assumes that f is continuous up to M but not a priori proper. Theorem 1.1 on the other hand assumes properness but not the continuity of f on M.

Math Subject Classifications. Primary 32H40.
Key Words and Phrases. Proper mapping, extension, Segre varieties.

ISSN 1050-6926

Let us briefly recall the salient features of the proof in [9]. There are two main steps: first, to extend f as a proper correspondence and second, to show that extendability as a proper correspondence implies holomorphic extendability. In step one, extension as a proper correspondence is shown first for all strongly pseudoconvex points on $\partial\mathcal{D}$. Building on this they construct proper correspondences, in a step by step approach, that extend f across weakly pseudoconvex points and even the exceptional points in the 'border' (the set $M \setminus (M^+ \cup M^-)$ defined below). Moreover, during all these constructions, it is essential to do the analogous steps for the (multivalued inverse) f^{-1}. This is the main reason why the local version of this result cannot be directly derived from [9]. As mentioned above the cluster set of M' may contain points of $\partial\mathcal{D}$ with no regularity, and therefore, without additional assumptions on the cluster set of M' under f^{-1}, no regularity of f^{-1} near M' can be established.

Many local results on the extendability of holomorphic mappings and correspondences have been obtained by different authors and without mentioning the entire list we refer to [2, 3] and [13] as examples to illustrate the flavor of the techniques used. One common feature in all these results is a 'convexity' assumption, either geometric or of a function theoretic nature, on M, M'. For example, this could be in form of the existence of local plurisubharmonic barriers at points of M, M'. So one possible approach to Theorem 1.1 would be to show the existence of such barrier functions at points of the one dimensional strata of the 'border.' This seems to be unknown as yet. Thus a somewhat different approach has to be used to prove Theorem 1.1.

Let T be the set of points on M where its Levi form vanishes. Then T can be stratified as $T = T_0 \cup T_1 \cup T_2$, where T_k is a locally finite union of smooth real-analytic submanifolds of dimension $k = 0, 1, 2$. Denote by $M_s^{\pm}$ the set of strongly pseudoconvex (resp. strongly pseudoconcave) points on M. Let $M^{\pm}$ be the relative interior, taken with respect to the relative topology on M, of $\overline{M_s^{\pm}}$. Then $M^{\pm}$ is the set of weakly pseudoconvex (resp. weakly pseudoconcave) points of M and the border $M \setminus (M^+ \cup M^-)$ clearly separates M^+ and M^-. It was shown in [8] and [9] that the stratification $T = T_0 \cup T_1 \cup T_2$ can be refined in such a way that the two dimensional strata become maximally totally real manifolds. Let us retain the same notation T_k, $k = 0, 1, 2$ for the various strata in the refined stratification. Further, let $T_k^+ = M^+ \cap T_k$ for all k. Then T_2^+ is the maximally totally real strata near which M is weakly pseudoconvex. The set

$$M_e = \big(M \setminus (M^+ \cup M^-)\big) \cap (T_1 \cup T_0) \tag{1.1}$$

is the exceptional set. It was shown in [6] that

$$\big(M \setminus (M^+ \cup M^-)\big) \cap T_2 \subset M \cap \hat{\mathcal{D}}, \tag{1.2}$$

where $\hat{\mathcal{D}}$ denotes the holomorphic hull of the domain $\mathcal{D}$.

Observe that $M_e \cup T_1^+ \cup T_0^+$ is a locally finite collection of real analytic arcs and points. A similar decomposition exists for M'. Let us note two facts with the hypotheses of Theorem 1.1. First, f clearly extends across points in $M \cap \hat{\mathcal{D}}$. Second, it was shown in [22] that f also extends across $M_s^+ \cup T_2^+$. Thus the verification of Theorem 1.1 will follow once f is shown to extend across $M_e \cup T_0^+ \cup T_1^+$.

Removability of real analytic arcs and points was also considered in [8] with the assumption that $f : \mathcal{D} \to \mathcal{D}'$ is biholomorphic and has an extension that is continuous up to M. A related result was obtained in $\mathbb{C}^n$, $n \geq 2$, (see [14]) with the assumption that M, M' are pseudoconvex and f is continuous up to M.

To conclude let us note one consequence of Theorem 1.1.

Theorem 1.2. *With the hypothesis of Theorem 1.1, the extended mapping $f : M \to M'$ satisfies the additional properties: $f(M_s^+) \subset M_s'^+$, $f(T_2^+) \subset T_2'^+$, $f(T_1^+ \cup T_0^+) \subset M_s'^+ \cup T_1'^+ \cup$*

$T_0'^{+}$, $f(M_e) \subset M'_e$ *and* $f(M \cap \hat{\mathcal{D}}) \subset M' \cap \hat{\mathcal{D}}'$. *Moreover, if* $z_0 \in T_0^+$ *is an isolated point of* T, *then* $f(z_0) \in T_0'^{+}$ *is also an isolated point of* T'.

The conclusion $f(T_1^+ \cup T_0^+) \subset M_s'^{+} \cup T_1'^{+} \cup T_0'^{+}$ cannot be strengthened to $f(T_1^+) \subset T_1'^{+}$ as the following example shows. Consider the pseudoconvex domain $\Omega = \{(z, w) : |z|^2 + |w|^4 < 1\}$ and the proper mapping $\eta(z, w) : \Omega \to \mathcal{B}^2$ from Ω to $\mathcal{B}^2$ the unit ball in $\mathbf{C}^2$ defined as $\eta(z, w) = (z, w^2)$. Clearly $T_1^+ \subset \partial\Omega$ is defined by the real analytic arc $\{(e^{i\theta}, 0)\}$ and such points are mapped by η to strongly pseudoconvex points.

2. Notation and preliminaries

The notion of finite type will be in the sense of D'Angelo which means that none of M, M' can contain positive dimensional germs of complex analytic sets. There are other notions such as finite type in the sense of Bloom–Graham and essential finiteness. The reader is referred to [1] for definitions and details. However, all these notions are equivalent in $\mathbb{C}^2$.

Segre varieties have played an important role in the study of boundary regularity of analytic sets and mappings when the obstructions are real analytic. The word 'analytic' will always mean complex analytic unless stated otherwise. Here are a few of their properties that will be used in this article. For a more detailed discussion and complete proofs the reader is referred to [7, 12] and [1]. Let us restrict ourselves to $\mathbb{C}^2$ as the case for $n > 2$ is no different. We will write $z = (z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ for a point $z \in \mathbb{C}^2$.

Pick $\zeta \in M$ and move it to the origin after a translation of coordinates. Let $r = r(z, \bar{z})$ be the defining function of M in a neighborhood of the origin, say U, and suppose that $\partial r/\partial z_2(0) \neq 0$. If U is small enough, the complexification $r(z, \overline{w})$ of r is well defined by means of a convergent power series in $U \times U$. Note that $r(z, \overline{w})$ is holomorphic in z and antiholomorphic in w. For any $w \in U$, the associated Segre variety is defined as

$$Q_w = \left\{z \in U : r(z, \overline{w}) = 0\right\}. \tag{2.1}$$

By the implicit function theorem Q_w can be written as a graph. In fact, it is possible to choose neighborhoods $U_1 \Subset U_2$ of the origin such that for any $w \in U_1$, Q_w is a closed, complex hypersurface in U_2 and

$$Q_w = \left\{z = (z_1, z_2) \in U_2 : z_2 = h(z_1, \overline{w})\right\}, \tag{2.2}$$

where $h(z_1, \overline{w})$ is holomorphic in z and antiholomorphic in w. Such neighborhoods will be called a *standard pair* of neighborhoods and they can be chosen to be polydisks centered at the origin. Note that Q_w is independent of the choice of r. For $\zeta \in Q_w$, the germ of Q_w at ζ will be denoted by ${}_\zeta Q_w$. Let $\mathcal{S} := \{Q_w : w \in U_1\}$ be the set of all Segre varieties, and let $\lambda : w \mapsto Q_w$ be the so-called Segre map. Then $\mathcal{S}$ admits the structure of a complex analytic set on a finite dimensional complex manifold. Consider the complex analytic set

$$I_w := \lambda^{-1}(\lambda(w)) = \{z : Q_z = Q_w\}. \tag{2.3}$$

If $w \in M$, then $I_w \subset M$ and the finite type assumption on M forces I_w to be a finite collection of points. Thus λ is a proper map in a small neighborhood of each point on M. Also, note that $z \in Q_w \iff w \in Q_z$ and $z \in Q_z \iff z \in M$. We shall also have occasion to use the notion of the symmetric point that was introduced in [9]. This is defined as follows: for w close enough to M, the complex line l_w containing the real line through w and orthogonal to M intersects Q_w

at a unique point. This is the symmetric point of w and is denoted by ${}^s w$. It can be checked that for w outside $\mathcal{D}$, the symmetric point ${}^s w \in \mathcal{D}$ and vice-versa. Moreover, for $w \in M$, ${}^s w = w$.

For $z \in \partial\mathcal{D}$, the cluster set $cl_f(z)$ is defined as:

$$cl_f(z) = \left\{ w \in \mathbb{C}^2 : \text{ there exists } (z_j)_{j=1}^{\infty} \subset \mathcal{D}, z_j \to z, \text{ such that } f(z_j) \to w \right\} \tag{2.4}$$

If $K \subset \partial\mathcal{D}$, then $cl_f(K)$ is defined to be the union of the cluster sets of all possible $z \in K$.

For all the notions and terminology introduced here, we simply add a prime to consider the corresponding notions in the target space. For example, $M_s'^{+}$ is the set of all strongly pseudoconvex points in M' and for w' close to M', $Q'_{w'}$ is the corresponding Segre variety.

Finally, we recall that for an analytic set $A \subset U \times U' \subset \mathbb{C}^2 \times \mathbb{C}^2$ of pure dimension two with proper projection to the first component U, there exists a system of canonical defining functions

$$\Phi_I\left(z, z'\right) = \sum_{|J| \le m} \phi_{IJ}(z) z'^{J}, \ |I| = m, \ \left(z, z'\right) \in U \times U' \subset \mathbb{C}^2 \times \mathbb{C}^2 . \tag{2.5}$$

Here $\phi_{IJ}(z) \in \mathcal{O}(U)$ and A is precisely the set of common zeros of the functions $\Phi_I(z, z')$. For details see e.g., [4]. Analytic set A with such properties is usually called a *holomorphic correspondence.* The set A is called a *proper* holomorphic correspondence if both coordinate projections are proper.

3. Strategy for the proof of Theorem 1.1

As noted earlier, f extends across $M \setminus (M_e \cup T_1^+ \cup T_0^+)$. We will first consider the one dimensional components of $(M_e \cup T_1^+)$. So let γ be a connected, real analytic arc in $M_e \cup T_1^+$ and suppose $0 \in \gamma$. Let $0 \in U_1 \Subset U_2$ be a standard pair of neighborhoods small enough so that f extends across $(M \cap U_2) \setminus \gamma$. This is possible due to the fact that $M_e \cup T_1^+ \cup T_0^+$ is a locally finite union of connected components. Consider

$$C := \{w \in U_1 : \gamma \cap U_1 \subset Q_w\}, \tag{3.1}$$

which is a finite set (see Lemma 2.3 in [8]). Indeed, $\gamma \cap U_1 \subset Q_w$ implies that Q_w is the unique complexification of $\gamma \cap U_1$. The Segre map λ is locally proper near the origin and hence the finiteness follows.

We need to show that f extends: (i) across $(\gamma \cap U_1) \setminus C$ and (ii) across the discrete set $\gamma \cap C$. The latter case will follow immediately from the former. Indeed, for $z \in C$ choose a smooth, real analytic arc $\tilde{\gamma} \subset M$ containing z such that $\tilde{\gamma}$ is transverse to $Q_z \cap M$ at z. Then the (unique) complexification of $\tilde{\gamma}$ is distinct from Q_z. Therefore the argument of case (i) can be applied to prove (ii). Thus without loss of generality we may assume that $0 \notin C$. To show that f extends across the zero dimensional strata of $M_e \cup T_1^+ \cup T_0^+$ it suffices to invoke the argument of case (ii), which in turn depends on case (i). All the above reductions of the problem can be summarized as follows.

General Situation. *The set γ is a one dimensional stratum of $M_e \cup T_1^+$ and $0 \in \gamma$. Neighborhoods U_1 and U_2 are chosen in such a way that for all $z \in \gamma \cap U_2$, $z \neq 0$, we have $Q_z \neq Q_0$ and $z \notin Q_0$, and therefore, $(Q_0 \setminus \{0\}) \cap M \cap U_2 \subset M \setminus (M_e \cup T_1^+)$. Let U' be an open neighborhood in $\mathbb{C}^2$ such that $M' \subset U'$ and U' is small enough so that for $w' \in U' \setminus \overline{\mathcal{D}}'$, $Q'_{w'}$ is well defined. Does f extend to a neighborhood of the origin?*

The setup of the General Situation will be henceforth assumed, unless otherwise stated. Following [9] define:

Definition 3.1. $V = \{(w, w') \in (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}') : f(Q_w \cap \mathcal{D}) \supset {}_{{}^s w'} Q'_{w'}\}$

and for all $(w, w') \in V$ consider

$$E = \left\{z \in \mathcal{D} \cap U_2 \cap Q_w : f(z) = {}^s w',\ f({}_z Q_w) \supset {}_{{}^s w'} Q'_{w'}\right\} . \tag{3.2}$$

Lemma 12.2 of [9] can be applied since f extends across $Q_0 \setminus \{0\} \subset M \setminus (M_e \cup T_1^+)$ to conclude that V is a closed, complex analytic set in $(U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ and dim $V \equiv 2$. Furthermore, V has no limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times \partial U'$. Following [9] the goal will be to show that V is contained in a closed, complex analytic set $\hat{V} \subset (U_1 \setminus \overline{\mathcal{D}}) \times U'$ with no limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times \partial U'$. Then $\hat{V}$ will define a correspondence that extends f across the origin and thus we have holomorphic extension as well. What are the possible obstructions in this strategy? Clearly, the only difficulty is that V may have limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times M'$. So suppose that $(w_0, w_0') \in \overline{V} \cap ((U_1 \setminus \overline{\mathcal{D}}) \times M')$ and let $(w_j, w_j') \in V$ be a sequence of points converging to (w_0, w_0'). Pick $\zeta_j \in Q_{w_j} \cap U_2 \cap \mathcal{D}$ such that $f(\zeta_j) = {}^s w_j'$. Since $w_j' \to w_0' \Rightarrow {}^s w_j' \to {}^s w_0' = w_0'$, the properness of f implies that ζ_j converges to $\partial\mathcal{D}$. By Lemma 12.2 of [9], $E \Subset U_2$ and hence $\zeta_j \to \zeta_0 \in M \cap U_2$, after perhaps passing to a subsequence. The following cases arise:

(a) If $\zeta_0 \notin \gamma \cap U_2$, then f extends across ζ_0 and $f(\zeta_0) = w_0'$. Select $z_j \in U_2 \setminus \mathcal{D}$ close to ζ_0 such that $f(z_j) = w_j'$ and $z_j \to \zeta_0$. The invariance property shows that

$$f(Q_{z_j} \cap \mathcal{D}) \supset {}_{{}^s w_j'} Q'_{w_j'}$$

for all j. Since $(w_j, w_j') \in V$,

$$f(Q_{w_j} \cap \mathcal{D}) \supset {}_{{}^s w_j'} Q'_{w_j'}$$

for all j. Combining these $Q_{w_j} = Q_{z_j}$ and passing to the limit gives $Q_{w_0} = Q_{\zeta_0}$ and this is a contradiction since $I_{\zeta_0} \subset M$.

(b) If $w_0' \in M' \cap \hat{\mathcal{D}}'$, then Lemma 3.1 in [9] shows that $\zeta_0 \in M \cap \hat{\mathcal{D}}$ and therefore f extends across ζ_0. The same argument as in case (a) yields a contradiction.

(c) If $\zeta_0 \in \gamma \cap U_2$, then f is not a priori known to extend to a neighborhood of ζ_0. The following possibilities may occur:

(i) $cl_f(\zeta_0) \cap M_s'^+ \neq \emptyset$,

(ii) there exists $w_0' \in cl_f(\zeta_0) \cap (T_2'^+ \cup T_1'^+ \cup T_0'^+ \cup M_e')$

To prove Theorem 1.1 it is enough to consider case (c). Section 5 contains the proof of the following proposition.

Proposition 3.2. *Let* $0 \in (\gamma \cap U_1) \setminus C$ *and suppose that* $0' \in cl_f(0) \cap M_s'^+$. *Then* f *extends holomorphically to a neighborhood of the origin and* $f(0) = 0'$.

By this proposition, it follows that f will extend across ζ_0 in case (c) (i) and therefore the argument used in case (a) applies again. Hence V cannot have any limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times M_s'^+$.

After that the only remaining possibility is $\zeta_0 \in \gamma \cap U_2$ and $w_0' \in cl_f(\zeta_0) \cap (T_2'^+ \cup T_1'^+ \cup T_0'^+ \cup M_e')$. Suppose that $w_0' \in T_2'^+$. Observe that $\zeta_j \in Q_{w_j}$ for all j and hence $\zeta_0 \in Q_{w_0}$ in the limit. Therefore $w_0 \in Q_{\zeta_0}$, where $\zeta_0 \in \gamma \cap U_2$. This motivates the consideration of

$$\mathcal{L} = \bigcup_{z \in \gamma \cap U_1} (Q_z \cap U_1) \tag{3.3}$$

which is a real analytic set in U_1 of real dimension 3. $\mathcal{L}$ is locally foliated by open pieces of Segre varieties at all of its regular points. Note that $\mathcal{L} \cap M$ has real dimension at most two since M is of finite type. Section 6 studies the limit points of V on $\mathcal{L} \times T_2'^{+}$ and shows that all of them are removable. More precisely, we show:

Proposition 3.3. *Let $(p, p') \in \mathcal{L} \times T_2'^{+}$ be a limit point for V. Then $\overline{V}$ is analytic near (p, p').*

Finally, as in [9], Bishop's lemma can be applied to handle the case when $w_0' \in M_e' \cup T_1'^{+} \cup T_0'^{+}$.

Proposition 3.4. *Let $(p, p') \in \mathcal{L} \times (M_e' \cup T_1'^{+} \cup T_0'^{+})$ be a limit point for V. Then $\overline{V}$ is analytic near (p, p'). As a consequence, f holomorphically extends across the origin.*

Thus the proof of Theorem 1.1 is complete once Propositions 3.2–3.4 are proved. Theorem 1.2 is proved in Section 7.

4. Properties of the set V

In this section we prove some technical results concerning $cl_f(0)$ and derive additional properties of the set V. These results will be used in the subsequent sections.

Proposition 4.1. *Assume the conditions described in the general situation, and let $w' \in cl_f(0) \cap M'$. Then there exists a sequence $\{w_j\} \in M \setminus \gamma$, $w_j \to 0$ such that $f(w_j) \to w'$. In particular, we may choose the sequence $\{w_j\} \in M \setminus (\mathcal{L} \cap M)$.*

Proof. Fix arbitrarily small neighborhoods $0 \in U$ and $w' \in U'$ and consider the non-empty locally complex analytic set $A := \Gamma_f \cap (U \times U')$. Without loss of generality, $0 \notin M \cap \hat{\mathcal{D}}$. If the proposition were false, the properness of f implies that $(\overline{A} \setminus A) \cap (U \times U') \subset (\gamma \cap U) \times (M' \cap U')$. Clearly $(\gamma \cap U) \times (M' \cap U')$ is a smooth, real analytic manifold with CR dimension 1. A result of Sibony (cf. Theorem 3.1 and Corollary 3.2 in [19]; see [20] also) shows that A has locally finite volume in $U \times U'$. Let $\gamma^{\mathbb{C}}$ be the (local) complexification of γ. Shrink U if necessary to ensure that $\gamma^{\mathbb{C}}$ is a closed analytic set in U. Then $A \setminus (\gamma^{\mathbb{C}} \times U')$ is a non-empty analytic set in $(U \times U') \setminus (\gamma^{\mathbb{C}} \times U')$ with locally finite volume. Now $\gamma^{\mathbb{C}} \times U'$ is pluripolar and hence Bishop's theorem (see Section 18.3 in cf. [4]) shows that all the limit points of $A \setminus (\gamma^{\mathbb{C}} \times U')$ are removable singularities. Thus $\overline{A}$ is analytic in $U \times U'$ and $\overline{A} \subset (\overline{U \cap \mathcal{D}}) \times (\overline{U' \cap \mathcal{D}'})$.

Next we show that $(0, 0')$ is in the envelope of holomorphy of $(\overline{U \cap \mathcal{D}}) \times U'$ (cf. [17]). Let $\pi : \overline{A} \to U$ be the natural projection. Then there are points $z \in \pi(\overline{A})$ such that $\pi^{-1}(z)$ is discrete. Indeed, if not, then each fiber $\pi^{-1}(z)$ is at least 1 dimensional for all $z \in \pi(\overline{A})$ and hence by Theorem 2 in Section V.3.2 of [15] it follows that

$$\dim \overline{A} \geq 1 + \dim \pi\left(\overline{A}\right),$$

and this implies that $\dim \pi(\overline{A}) \leq 1$. This is a contradiction since $\overline{A}$ contains Γ_f over a non-empty subset of U. For $(z, z') \in \overline{A}$, let $(\pi^{-1}(z))_{(z,z')}$ denote the germ of the fiber $\pi^{-1}(z)$ at (z, z'). Let

$$S = \left\{(z, z') : \dim \left(\pi^{-1}(z)\right)_{(z,z')} \geq 1\right\} .$$

By a theorem of Cartan–Remmert, S is known to be analytic and the above reasoning shows that $\dim S \leq 1$. Without loss of generality $(0, 0') \in S$ as otherwise $\pi : \overline{A} \to U$ defines a correspondence near $(0, 0')$ and by Theorem 7.4 in [9] f would extend across 0. Select $L \subset \overline{A}$, an analytic subset in $U \times U'$ with the following properties: it contains $(0, 0')$, has pure dimension 1 and is distinct from S. Note that $L \subset \overline{U \cap \mathcal{D}} \times U'$. If $L \cap ((M \cap U) \times U')$ is discrete, then the continuity principle forces $(0, 0')$ to be in the envelope of holomorphy of $(\overline{U \cap \mathcal{D}}) \times U'$. Since M is assumed to be of finite type and since $L \neq S$, no open subset of L can be contained in $L \cap ((M \cap U) \times U')$. The strong disk theorem in [23] shows that $(0, 0')$ is again in the envelope of holomorphy of $(\overline{U \cap \mathcal{D}}) \times U'$.

To conclude, given an arbitrary $g \in \mathcal{O}(U \cap \mathcal{D})$, we may regard $g \in \mathcal{O}((U \cap \mathcal{D}) \times U')$, i.e., independent of the z' variables. Then g extends across $(0, 0')$ and the uniqueness theorem shows that the extension of g, say $\tilde{g}$, is also independent of z'. Thus g extends across 0. In particular, f extends across 0 and this is a contradiction.

Finally, we can choose $\{w_j\} \in M \setminus (\mathcal{L} \cap M)$ because $\mathcal{L} \cap M$ is nowhere dense in M. □

Remarks.

1. First, the cluster set of the origin may be defined in two possible ways. One way is considered in Section 2, where the approaching sequence $\{z_j\} \subset \mathcal{D}$. The other possibility is to consider $\{z_j\} \subset M \setminus (M_e \cup T_1^+)$. This is well defined since f extends across $M \setminus (M_e \cup T_1^+)$. Clearly, the former set contains the latter in general. However, in the special case of the General Situation of Section 3, Proposition 4.1 shows that these two notions of the cluster set coincide.
2. If w' in Proposition 4.1 is also known to belong to $M_s'^{+}$, then each point in the sequence $\{w_j\}$ can be chosen from M_s^+. Indeed, two cases have to be considered. First, if $\gamma \subset T_1^+$, then for a small enough neighborhood $0 \in U$ it follows that $M_s^+ \cap U$ is dense in $M \cap U$ and hence the sequence $\{w_j\}$ can be chosen to belong to M_s^+. Second, if $\gamma \subset M_e$, then every neighborhood $0 \in U$ contains points from both M_s^+ and $M \cap \hat{\mathcal{D}}$. It follows that for large j, w_j must be strongly pseudoconvex since $f(w_j)$ is so; w_j cannot belong to any other strata of M as otherwise it will be possible to choose a strongly pseudoconcave point near the origin that is mapped locally biholomorphically to a strongly pseudoconvex point and this violates the invariance of the Levi form.

With V as in Definition 3.1 and $0' \in cl_f(0) \cap M_s'^{+}$, pick a standard pair of neighborhoods $0 \in U_1 \Subset U_2$ and $0' \in U_1' \Subset U_2'$ so that $M' \cap U_2'$ is strongly pseudoconvex. Choose another neighborhood U', $0' \in U' \Subset U_1' \Subset U_2'$ which has the additional property that for all $w' \in U' \setminus \overline{\mathcal{D}}'$, $Q'_{w'} \cap M' \Subset U_1'$. To see this, combine the fact that locally $Q'_{0'} \cap \overline{\mathcal{D}}' = \{0'\}$ with the holomorphic dependence of $Q'_{w'}$ on w'. Shrink U_1', U_2' so that for all $w' \in M' \cap U'$, $Q'_{w'} \cap M' = w'$. Proposition 4.1 shows the existence of $(w_0, w_0') \in ((M \setminus \gamma) \cap U_1) \times (M' \cap U')$ such that $w_0' = f(w_0)$. The invariance property

$$f(Q_w \cap \mathcal{D}) \supset {}_{s_{w'}} Q'_{w'}$$

holds for all $(w, w') \in (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ close to (w_0, w_0') and hence $V \cap ((U_1 \setminus \overline{\mathcal{D}}) \times U') \neq \emptyset$. Now $V \cap ((U_1 \setminus \overline{\mathcal{D}}) \times U')$ may have several irreducible components but we retain only those that contain the extension of Γ_f across points in $(M \cap U_1) \times (M' \cap U')$. Take the union of all such components and call the resulting analytic set $V_{\rm loc} \subset (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$.

Proposition 4.2. *$V_{\rm loc}$ consists of those points $(w, w') \in (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ that satisfy:*

$$f(Q_w \cap \mathcal{D}) \supset {}^{s}{}_{w'}Q'_{w'} \text{ and } f({}^{s}{}_{w}Q_w) \subset Q'_{w'} \cap \mathcal{D}' .$$

Proof. We only need to show that for $(w, w') \in V$ the second inclusion holds. Define

$$W = \left\{(w, w') \in \left(U_1 \setminus \overline{\mathcal{D}}\right) \times \left(U' \setminus \overline{\mathcal{D}}'\right) : f({}^{s}{}_{w}Q_w) \subset Q'_{w'} \cap \mathcal{D}'\right\} .$$

Pick $(\zeta_0, f(\zeta_0)) \in ((M \setminus \gamma) \cap U_1) \times (M' \cap U')$. Then for all $(w, w') \in (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ close to $(\zeta_0, f(\zeta_0))$

$$f({}^{s}{}_{w}Q_w) \subset Q'_{w'} \cap \mathcal{D}'$$

holds true. Thus W is non-empty. Denote the irreducible component of $V_{\rm loc}$ which contains the extension of Γ_f near $(\zeta_0, f(\zeta_0))$ by $\tilde{V}$. It will suffice to show that $\tilde{V} \subset W$.

A similar argument as in [18], Proposition 3.2 shows that W is locally analytic in $(U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ and $\dim W \equiv 2$. To show that W is closed, pick a sequence $(w_j, w'_j) \in W$ converging to $(w_0, w'_0) \in (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$. Then

$$f\left({}^{s}{}_{w_j}Q_{w_j}\right) \subset Q'_{w'_j} \cap \mathcal{D}'$$

holds for all j. By construction $Q'_{w'_j} \cap M' \Subset U'_1$ and since $f({}^{s}w_j) \in Q'_{w'_j} \cap \mathcal{D}'$ we may pass to the limit to get

$$f\left({}^{s}{}_{w_0}Q_{w_0}\right) \subset Q'_{w'_0} \cap \mathcal{D}' .$$

Clearly $\tilde{V} \subset W$ locally near $(\zeta_0, f(\zeta_0))$ and hence it follows that $\tilde{V} \subset W$. □

Remarks. First, without a priori regularity of f near 0, such as continuity, it seems difficult to control the images $f({}^{s}{}_{w}Q_w)$ as w moves in $U_1 \setminus \overline{\mathcal{D}}$. In particular, they may escape from U'_2. Thus W may not be closed in general. Second, the two conditions specified in the above proposition are not the same since $Q_w \cap \mathcal{D}$ may have many components in general.

Proposition 4.3. *The set $V_{\rm loc}$ satisfies the following properties:*

(i) *$V_{\rm loc}$ has no limit points on $(M \cap U_1) \times (U' \setminus \overline{\mathcal{D}}')$,*
(ii) *$V_{\rm loc}$ has no limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times (M' \cap U')$,*
(iii) *for $w_0 \in (M \setminus \gamma) \cap U_1$, $\#\pi^{-1}(w_0) \cap (\overline{V}_{\rm loc} \setminus V_{\rm loc}) \cap \left((M \setminus \gamma) \times (M' \cap U')\right) \leq 1$.*

Proof. (i) Suppose that $(w_0, w'_0) \in (M \cap U_1) \times (U' \setminus \overline{\mathcal{D}}')$ is a limit point for $V_{\rm loc}$. There are two cases to consider.

Case (a): Let $(w_0, w'_0) \in ((M \setminus \gamma) \cap U_1) \times (U' \setminus \overline{\mathcal{D}}')$ and choose a sequence $(w_j, w'_j) \in V_{\rm loc}$ converging to (w_0, w'_0). Then

$$f\left({}^{s}{}_{w_j}Q_{w_j}\right) \subset Q'_{w'_j} \cap \mathcal{D}'$$

holds for all j. Note that $w_j \to w_0 \Rightarrow {}^s w_j \to {}^s w_0 = w_0$. Since $f({}^s w_j) \subset Q'_{w'_j} \cap \mathcal{D}'$ and $Q'_{w'_j} \cap M' \Subset U'_1$, it follows that $f({}^s w_j) \to \zeta'_0 \in Q'_{w'_0} \cap M' \cap U'_1$ after perhaps passing to a subsequence. Also, f extends across w_0 and $f(w_0) = \zeta'_0$. By the invariance property applied near (w_0, ζ'_0) we get

$$f\left({}_{{}^s w_j} Q_{w_j}\right) \subset Q'_{f(w_j)} \cap \mathcal{D}'$$

for all j. Combining these inclusions gives $Q'_{f(w_j)} = Q'_{w'_j}$ for all j and hence $Q'_{\zeta'_0} = Q'_{w'_0}$ in the limit. This is a contradiction to $I_{\zeta'_0} \subset M'$.

Case (b): Let $(w_0, w'_0) \in (\gamma \cap U_1) \times (U' \setminus \overline{\mathcal{D}}')$ and choose a sequence $(w_j, w'_j) \in V_{\text{loc}}$ converging to (w_0, w'_0). Then

$$f\left(Q_{w_j} \cap \mathcal{D}\right) \supset {}_{{}^s w'_j} Q'_{w'_j}$$

holds for all j. Select $\zeta_j \in Q_{w_j} \cap \mathcal{D}$ such that $f(\zeta_j) = {}^s w'_j$ and $f({}_{\zeta_j} Q_{w_j}) \supset {}_{{}^s w'_j} Q'_{w'_j}$. Lemma 12.2 in [9] shows that $\{\zeta_j\} \Subset U_2$ and hence $\zeta_j \to \zeta_0 \in Q_{w_0}$ after passing to a subsequence. By continuity

$$f\left({}_{\zeta_0} Q_{w_0}\right) \supset {}_{{}^s w'_0} Q'_{w'_0} .$$

Note that $\zeta_0 \in \mathcal{D}$ since ${}^s w'_0 \in \mathcal{D}'$ and f is proper.

Let us show that (w_0, w'_0) is isolated in $\pi^{-1}(w_0) \cap (\overline{V}_{\text{loc}} \setminus V_{\text{loc}})$, where π is the projection to U_1. Suppose not. Then there exist infinitely many $\{\tilde{w}'_j\}$ such that $(w_0, \tilde{w}'_j) \in \overline{V}_{\text{loc}} \setminus V_{loc}$ and $\tilde{w}'_j \to w'_0$. The same argument as above can be repeated for each $\tilde{w}'_j$, i.e., selecting $\zeta_{0,j} \in Q_{w_0}$ etc., so that in the limit we have

$$f\left({}_{\zeta_{0,j}} Q_{w_0}\right) \supset {}_{{}^s \tilde{w}'_j} Q'_{\tilde{w}'_j} . \tag{4.1}$$

for all j. Note that $Q_{w_0} \cap U_2$ will have only finitely many components, after perhaps shrinking U_2 slightly, while (4.1) shows that at least one component of $Q_{w_0} \cap \mathcal{D}$ must be mapped onto infinitely many $Q'_{\tilde{w}'_j}$. This contradicts the properness of the Segre map $\lambda' : U' \to \mathcal{S}'$ and the claim follows.

Without loss of generality, we may assume that $w_0 \notin M \cap \hat{\mathcal{D}}$ as otherwise the same argument in case (a) applies to yield a contradiction. Since w'_0 is isolated in the fiber of $\overline{V}_{\text{loc}} \setminus V_{\text{loc}}$ over w_0, it is possible to choose small balls B_ϵ, B'_ϵ centered at w_0, w'_0, respectively and of radius $\epsilon > 0$ so that the projection

$$\pi : V_{\text{loc}} \cap \left(\left(B_\epsilon \cap \left(U \setminus \overline{\mathcal{D}}\right)\right) \times B'_\epsilon\right) \to B_\epsilon \cap \left(U \setminus \overline{\mathcal{D}}\right) \tag{4.2}$$

is proper, and hence a finite branched covering. The canonical defining pseudo-polynomials of this cover defined as in (2.5) are monic in z'_1, z'_2 with coefficients that are holomorphic in $B_\epsilon \cap (U \setminus \overline{\mathcal{D}})$. Since $w_0 \notin M \cap \hat{\mathcal{D}}$, Trepreau's theorem (cf. [21]) shows that all the coefficients extend to all of B_ϵ, with a slightly smaller ϵ perhaps. Hence there exists an analytic set $V^{\text{ext}} \subset B_\epsilon \times B'_\epsilon$ that contains V_{loc} near (w_0, w'_0). Note that the projection

$$\pi : V^{\text{ext}} \to B_\epsilon \tag{4.3}$$

is still proper. Let $(w_\nu, w'_\nu) \in V_{\text{loc}}$ be a sequence that converges to $(\tilde{w}, \tilde{w}') \in V^{\text{ext}}$ where $\tilde{w} \in (M \setminus \gamma) \cap U_1$. Such a choice exists by (4.2) and (4.3). But this is precisely the situation of case (a) and the same arguments there lead to a contradiction.

(ii) Suppose that $(w_0, w'_0) \in (U_1 \setminus \overline{\mathcal{D}}) \times (M' \cap U')$ is a limit point for V_{loc}. Select $(w_j, w'_j) \in V_{\text{loc}}$ converging to (w_0, w'_0). Then

$$f\left({}_{{}^s w_j} Q_{w_j}\right) \subset Q'_{w'_j} \cap \mathcal{D}'$$

holds for all j. Observe that $\{f({}^s w_j)\} \Subset U'_1 \cap \mathcal{D}'$ since f is proper. On the other hand, $f({}^s w_j) \in Q'_{w'_j} \cap \mathcal{D}'$ and as $j \to \infty$, $Q'_{w'_j} \cap \mathcal{D}' \to Q'_{w'_0} \cap \mathcal{D}' = \emptyset$ since w'_0 is a strongly pseudoconvex point. This is a contradiction.

(iii) Finally, suppose that $(w_0, w'_{0,1}) \in (\overline{V}_{\text{loc}} \setminus V_{\text{loc}}) \cap ((M \setminus \gamma) \times (M' \cap U'))$. Using the same argument as in case (a), it can be seen that $f(w_0) = w'_{0,1}$. If $(w_0, w'_{0,2}) \in ((M \setminus \gamma) \times (M' \cap U'))$ is another limit point for V_{loc}, then the same argument would show that $f(w_0) = w'_{0,2}$ and this is a contradiction. □

Remark. $V_{\text{loc}} \subset (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ can now be regarded as an analytic set in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$.

5. f extends if $cl_f(\zeta_0) \cap M'^{+}_s \neq \emptyset$

In this section we prove Proposition 3.2. For that we will consider the sequence of points $p_j \to 0$, $f(p_j) \to 0'$, whose existence is guaranteed by Proposition 4.1 and study a certain family of analytic sets $\{\mathcal{C}_{p_j}\}$ associated with $\{p_j\}$. The goal is to derive some properties of the limit set of $\mathcal{C}_{p_j}$. We prove several preparation lemmas first.

For any $z \in M \cap U_1$, $Q_z \times (U' \setminus \overline{\mathcal{D}}')$ is analytic in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$ of pure dimension 3. Since V_{loc} contains the extension of the graph of f across some points close to $(0, 0')$, it follows that $V_{\text{loc}} \cap (Q_z \times (U' \setminus \overline{\mathcal{D}}'))$ is either empty or analytic of pure dimension 1. By Proposition 4.1, pick $a \in M \cap U_1$, a strongly pseudoconvex point across which f extends such that $f(a) \in M' \cap U'$. Shifting a slightly, if needed, ensures that $a \notin \mathcal{L} \cap M$. By the invariance property

$$f({}_a Q_a) \subset {}_{f(a)} Q'_{f(a)} \,.$$

Since both a, $f(a)$ are strongly pseudoconvex, the germs ${}_a Q_a$, ${}_{f(a)} Q'_{f(a)}$ are contained outside $\mathcal{D}$ and $\mathcal{D}'$, respectively. For simplicity we consider representatives of the germs of ${}_a Q_a$ and ${}_{f(a)} Q'_{f(a)}$, that satisfy the above properties. Choose $b \in {}_a Q_a \setminus \overline{\mathcal{D}}$ where f is defined, so that $f(b) \in {}_{f(a)} Q'_{f(a)} \setminus \overline{\mathcal{D}}'$. Consider the graph of the extension of f over ${}_b Q_a$. This is a pure 1 dimensional germ contained in $V_{\text{loc}} \cap (Q_a \times (U' \setminus \overline{\mathcal{D}}'))$. Let

$$\mathcal{C}_a \subset V_{\text{loc}} \cap \left(Q_a \times \left(U' \setminus \overline{\mathcal{D}}'\right)\right) \tag{5.1}$$

be the irreducible component of dimension 1 that contains this germ. Note that $\mathcal{C}_a$ is analytic in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$. Also, the invariance property shows that

$$\mathcal{C}_a \subset V_{\text{loc}} \cap \left(U_1 \times \left(Q'_{f(a)} \setminus \{f(a)\}\right)\right) . \tag{5.2}$$

Lemma 5.1. *$\overline{\mathcal{C}}_a$ is analytic in $U_1 \times U'$.*

Proof. By Proposition 4.3, all the limit points of $\mathcal{C}_a$ are contained in $(Q_a \cap M \cap U_1) \times \{f(a)\}$. Since $a \notin \mathcal{L} \cap M$, it follows that $Q_a \cap M \subset M \setminus \gamma$ and thus f extends to a neighborhood of $Q_a \cap M$. Suppose that $(w_0, f(a)) \in \overline{\mathcal{C}}_a \setminus \mathcal{C}_a$. Now exactly the same arguments used in case (a) of Proposition 4.3 show that $f(w_0) = f(a)$. The global correspondence $f^{-1} : \mathcal{D}' \to \mathcal{D}$ has finite multiplicity and hence there can be only finitely many w_0 so that $(w_0, f(a))$ is a limit point for $\mathcal{C}_a$. Since dim $\mathcal{C}_a \equiv 1$, the Remmert–Stein theorem shows that $\overline{\mathcal{C}}_a$ is analytic in $U_1 \times U'$. □

Remarks. First, it is clear that a different choice of $b \in {}_aQ_a$ will give rise to the same component $\mathcal{C}_a$. Second, it follows by construction that $(a, f(a)) \in \overline{\mathcal{C}}_a$ and $\overline{\mathcal{C}}_a \subset Q_a \times Q'_{f(a)}$.

We will now focus on $cl_g(0')$ near 0, where $g = f^{-1}$ is a proper holomorphic correspondence. If $0 \in cl_g(0')$ is not isolated, it must be a continuum near 0. Note that there can only be finitely many $z \in cl_g(0') \cap (M \setminus \gamma) \cap U_1$ since g has finite multiplicity. Hence we may assume that $cl_g(0') \cap \gamma$ is a continuum in U_1 containing 0 and that no point in it belongs to $M \cap \hat{\mathcal{D}}$. Fix $p \in cl_g(0') \cap (M \setminus \gamma) \cap U_1$ and a sequence $\{p_j\} \subset M_s^+ \setminus \mathcal{L}$ converging to p such that $f(p_j)$ converges to $0'$. Associated with each p_j is the analytic set $\mathcal{C}_{p_j} \subset U_1 \times (U' \setminus \overline{\mathcal{D}}')$ constructed as above such that $\overline{\mathcal{C}}_{p_j} \subset U_1 \times U'$ is analytic. The goal will be to associate a pure 1 dimensional analytic set, say $\mathcal{C}_p$, to the chosen point p by considering the sequence $\mathcal{C}_{p_j}$.

Lemma 5.2. *Fix $p \in cl_g(0') \cap \gamma \cap U_1$ and consider the sequence of analytic sets $\{\mathcal{C}_{p_j}\}$. Then the limit set of this sequence of analytic sets is non-empty in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$.*

Proof. Without loss of generality we may assume that $p = 0$. The following observations can be made. First, ${}_0Q_0 \setminus \{0\}$ intersects both $U_1 \cap \mathcal{D}$ and $U_1 \setminus \overline{\mathcal{D}}$ or else is contained in $U_1 \setminus \overline{\mathcal{D}}$, as otherwise the continuity principle forces $0 \in M \cap \hat{\mathcal{D}}$. Thus it is possible to fix a ball B_ϵ around the origin so that $(Q_0 \cap \partial B_\epsilon) \setminus \overline{\mathcal{D}} \neq \emptyset$. Since Q_z depends smoothly on z, the same will then be true with Q_z for $z \in M$ close to 0. Second, fix a polydisk $\Delta^2 \Subset U'$ centered at $0'$ with its sides parallel to those of U'. Then $Q'_{0'} \cap \partial\Delta^2 \Subset U' \setminus \overline{\mathcal{D}}'$ and the same will be true for $Q'_{z'}$, $z' \in M'$ close to $0'$. Consider the non-empty analytic sets $\overline{\mathcal{C}}_{p_j} \cap (B_\epsilon \times \Delta^2)$ and examine the projection

$$\pi : \overline{\mathcal{C}}_{p_j} \cap \left(B_\epsilon \times \Delta^2\right) \to B_\epsilon\,. \tag{5.3}$$

Case (a): If π is proper for all j, then the image $\pi\big(\overline{\mathcal{C}}_{p_j} \cap (B_\epsilon \times \Delta^2)\big)$ is analytic and by the remark after Lemma 5.1, it follows that $\pi\big(\overline{\mathcal{C}}_{p_j} \cap (B_\epsilon \times \Delta^2)\big) = Q_{p_j} \cap B_\epsilon$. Fix a smaller ball $B_{\epsilon/2}$ around 0, choose $w_0 \in (Q_0 \cap \partial B_{\epsilon/2}) \setminus \overline{\mathcal{D}}$ and let $w_j \in (Q_{p_j} \cap \partial B_{\epsilon/2}) \setminus \overline{\mathcal{D}}$ converge to w_0, as $j \to \infty$. Since π is proper, it is possible to choose $(w_j, w'_j) \in \overline{\mathcal{C}}_{p_j} \cap (B_\epsilon \times \Delta^2)$. After passing to a subsequence, $(w_j, w'_j) \to (w_0, w'_0)$. Since $w_0 \in U_1 \setminus \overline{\mathcal{D}}$, Proposition 4.3 shows that $w'_0 \in U' \setminus \overline{\mathcal{D}}'$. Thus $(w_0, w'_0) \in (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ is a limit point for the sequence of analytic sets $\{\mathcal{C}_{p_j}\}$.

Case (b): For some subsequence, still indexed by j, π in (5.3) is not proper. Then it is possible to choose $(w_j, w'_j) \in \overline{\mathcal{C}}_{p_j} \cap (B_\epsilon \times \Delta^2)$ with $w'_j \in Q'_{f(p_j)} \cap \partial\Delta^2 \Subset U' \setminus \overline{\mathcal{D}}'$. By Proposition 4.3, $w_j \in U_1 \setminus \overline{\mathcal{D}}$ for all j. Passing to a subsequence $w'_j \to w'_0 \in Q'_{0'} \cap \partial\Delta^2 \Subset U' \setminus \overline{\mathcal{D}}'$ and $w_j \to w_0 \in \overline{B}_\epsilon$. Thanks to Proposition 4.3 again, $w_0 \in U_1 \setminus \overline{\mathcal{D}}$. Thus $(w_0, w'_0) \in (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ is a limit point for the sequence of analytic sets $\{\mathcal{C}_{p_j}\}$. □

From (2.1) it follows that $r(z, \overline{p}_j)$ is the defining function for Q_{p_j} in U_1. Regard $r(z, \overline{p}_j) \in \mathcal{O}(U_1 \times U')$, i.e., independent of the z' variables and to emphasize this point, we will write it as

$r(z, z', \overline{p}_j)$. This way $r(z, z', \overline{p}_j)$ is the defining function for $Q_{p_j} \times (U' \setminus \overline{\mathcal{D}}')$ in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$ and denote $Z_{p_j} = \{(z, z') : r(z, z', \overline{p}_j) = 0\}$. Now note that $V_{\text{loc}} \cap (Q_{p_j} \times (U' \setminus \overline{\mathcal{D}}'))$ is a pure 1 dimensional analytic set. This has two consequences.

First, it follows that (for example, see Section 16.3 in [4]) $\log |r(z, z', \overline{p}_j)|$ is locally absolutely integrable on V_{loc}. Hence it defines a current, denoted by $\log |r(z, z', \overline{p}_j)| \cdot [V_{\text{loc}}]$ in the following way:

$$\left\langle \log \left| r\left(z, z', \overline{p}_j\right)\right| \cdot V_{\text{loc}}, \phi \right\rangle = \int_{V_{\text{loc}}} \log \left| r\left(z, z', \overline{p}_j\right)\right| \phi \,, \tag{5.4}$$

where $\phi \in \mathcal{D}^{(4)}(U_1 \times (U' \setminus \overline{\mathcal{D}}'))$, the space of all smooth, complex valued differential forms of total degree 4 with compact support in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$.

Second, let $\{C^j_m\}_{m \ge 0}$ be the various components of $V_{\text{loc}} \cap (Q_{p_j} \times (U' \setminus \overline{\mathcal{D}}'))$, and let k^j_m be the corresponding (positive) intersection multiplicities which are constant along C^j_m. Note that for a fixed j, $\mathcal{C}_{p_j} = C^j_m$ for some m. The wedge product

$$V \wedge Z_{p_j} = \sum_{m \ge 0} k^j_m C^j_m \tag{5.5}$$

is thus well defined as an intersection multiplicity chain. By the generalized Poincaré–Lelong formula (cf. [4])

$$V_{\text{loc}} \wedge Z_{p_j} = \frac{1}{2\pi} dd^c \left(\log \left| r\left(z, z', \overline{p}_j\right)\right| \cdot [V_{\text{loc}}]\right) \tag{5.6}$$

in the sense of currents for all j.

Lemma 5.3. *Fix $p \in cl_g(0') \cap \gamma \cap U_1$. Let $\{\mathcal{C}_{p_j}\}$ be the sequence as in Lemma 5.2. Then there exists a subsequence $\{\mathcal{C}_{p_{j_k}}\}$ converging to an analytic set $\mathcal{C}_p \subset U_1 \times (U' \setminus \overline{\mathcal{D}}')$. Moreover, $\overline{\mathcal{C}}_p \subset U_1 \times U'$ is also analytic.*

Proof. By Lemma 5.2 $\mathcal{C}_p$ is not empty. We now show that the sequence $\{\mathcal{C}_{p_j}\}$ has locally uniformly bounded volume in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$.

Fix $K \Subset U_1 \times (U' \setminus \overline{\mathcal{D}}')$. Choose a test function ψ in $U_1 \times (U' \setminus \overline{\mathcal{D}}')$, $0 \le \psi \le 1$ such that $\psi \equiv 1$ on K. Let ω be the standard fundamental form on $\mathbb{C}^4$. Then

$$\begin{aligned} \text{Vol}\left(\left(V_{\text{loc}} \wedge Z_{p_j}\right) \cap K\right) &= \int_{(V_{\text{loc}} \wedge Z_{p_j}) \cap K} \omega \\ &\le \int_{(V_{\text{loc}} \wedge Z_{p_j})} \psi\omega = \frac{1}{2\pi} \int_{V_{\text{loc}}} \log \left| r\left(z, z', \overline{p}_j\right)\right| dd^c(\psi\omega) \,. \end{aligned}$$

Since $r(z, z', \overline{w})$ is antiholomorphic in w, it follows that $\log |r(z, z', \overline{p}_j)| \to \log |r(z, z', \overline{p})|$ in L^1_{loc} and hence

$$\frac{1}{2\pi} \int_{V_{\text{loc}}} \log \left| r\left(z, z', \overline{p}_j\right)\right| dd^c(\psi\omega) \lesssim \frac{1}{2\pi} \int_{V_{\text{loc}}} \log \left| r\left(z, z', \overline{p}\right)\right| dd^c(\psi\omega) := C(K, \psi) < \infty$$

by the dominated convergence theorem. As noted earlier $\mathcal{C}_{p_j} = C^j_m$ for some m and hence the assertion follows.

By Bishop's theorem (cf. [4]), $\{\mathcal{C}_{p_j}\}$ has a subsequence that converges to an analytic set $\mathcal{C}_p \subset U_1 \times (U' \setminus \overline{\mathcal{D}}')$ locally uniformly. $\mathcal{C}_p$ has pure dimension 1 and will be reducible in general. By Proposition 4.3, it is known that all the limit points of $\mathcal{C}_{p_j} \subset (Q_{p_j} \cap M \cap U_1) \times \{f(p_j)\}$ and hence in the limit $\overline{\mathcal{C}}_p \setminus \mathcal{C}_p \subset (Q_p \cap M \cap U_1) \times \{0'\}$. But $(Q_p \cap M \cap U_1) \times \{0'\}$ is a smooth, real analytic arc that is also pluripolar. Sibony's result combined with Bishop's theorem applied exactly as before shows that $\overline{\mathcal{C}}_p \subset U_1 \times U'$ is analytic of pure dimension 1. □

Note that $(p, 0') \in \overline{\mathcal{C}}_p$. The association of each $p \in cl_g(0') \cap \gamma \cap U_1$ with an analytic set is thus complete. Moreover, by (5.2) we have

$$\overline{\mathcal{C}}_p \subset \overline{V_{\text{loc}} \cap \big(U_1 \times (Q'_{0'} \setminus \{0'\})\big)} \tag{5.7}$$

for all such p.

Lemma 5.4. *Let p_1, p_2 be the points in $cl_g(0') \cap \gamma \cap U_1$ such that $p_1 \notin I_{p_2}$. Then* $\dim(\overline{\mathcal{C}}_{p_1} \cap \overline{\mathcal{C}}_{p_2}) < 1$.

Proof. It may a priori happen that for $p_1 \neq p_2 \in cl_g(0') \cap \gamma \cap U_1$, $Q_{p_1} = Q_{p_2}$. But the set of such points is at most countable since $\lambda : U_1 \to \mathcal{S}$ is proper. Thus the content of this proposition lies in the assertion that for most points in $cl_g(0') \cap \gamma \cap U_1$, the associated analytic sets are also distinct.

Arguing by contradiction assume that $\mathcal{C} \subset \overline{\mathcal{C}}_{p_1} \cap \overline{\mathcal{C}}_{p_2}$ is an irreducible component of dimension 1. Since $\overline{\mathcal{C}}_{p_i}$ lies over Q_{p_i} for $i = 1, 2$ it follows that $\pi(\mathcal{C}) \subset Q_{p_1} \cap Q_{p_2}$. By hypothesis, $Q_{p_1} \cap Q_{p_2}$ is discrete and the irreducibility of $\mathcal{C}$ implies that $\pi(\mathcal{C})$ is a point, say $w \in Q_{p_1} \cap Q_{p_2}$. By (5.7) it follows that $\mathcal{C} \subset \{w\} \times Q'_{0'}$ and thus

$$\mathcal{C} = \{w\} \times Q'_{0'} \,. \tag{5.8}$$

There are two cases to consider. First, if $w \in M$, then (5.8) would force V_{loc} to have limit points on $(M \cap U_1) \times (U' \setminus \overline{\mathcal{D}}')$ and this contradicts Proposition 4.3. Second, if $w \in U_1 \setminus \overline{\mathcal{D}}$, then (5.8) shows that

$$f(Q_w \cap \mathcal{D}) \supset {}_{s_{w'}} Q'_{w'}$$

for all $w' \in Q'_{0'} \setminus \{0'\}$. This contradicts the injectivity of $\lambda' : U' \to \mathcal{S}'$. □

Lemma 5.5. *$\overline{V_{\text{loc}} \cap \big(U_1 \times (Q'_{0'} \setminus \{0'\})\big)}$ is analytic in $U_1 \times U'$ of pure dimension one.*

Proof. By (5.7) $V_{\text{loc}} \cap \big(U_1 \times (Q'_{0'} \setminus \{0'\})\big)$ is a non-empty analytic set of pure dimension one. Proposition 4.3 says that all of its limit points are contained in $(M \cap U_1) \times \{0'\}$. Let $(w, 0') \in ((M \setminus \gamma) \cap U_1) \times \{0'\}$ be a limit point for V_{loc}. The same argument as in case (a) of Proposition 4.3 shows that $f(w) = 0'$. Since $g : \mathcal{D}' \to \mathcal{D}$ has finite multiplicity, it follows that there are only finitely many $w \in M \setminus \gamma$ such that $(w, 0')$ is a limit point for $V_{\text{loc}} \cap \big(U_1 \times (Q'_{0'} \setminus \{0'\})\big)$. Each of these is removable by the Remmert–Stein theorem. The remaining limit points are contained in $(\gamma \cap U_1) \times \{0'\}$. Sibony's result combined with Bishop's theorem as before show that they are also removable. □

Proof of Proposition 3.2. The first step is to show that 0 is an isolated point in $cl_g(0') \cap \gamma \cap U_1$. If not, let $\alpha > 0$ be the Hausdorff dimension of the continuum $\mathcal{E} := cl_g(0') \cap \gamma \cap U_1$. Each $p \in \mathcal{E}$

is associated with a pure 1 dimensional analytic set $\overline{C}_p \subset U_1 \times U'$. By (5.7) it follows that

$$\bigcup_{p\in\mathcal{E}} \overline{C}_p \subset \overline{V_{\mathrm{loc}} \cap \left(U_1 \times (Q'_{0'} \setminus \{0'\})\right)}\,. \tag{5.9}$$

By Lemma 5.4, the Hausdorff dimension of the left side in (5.9) is at least $\alpha + 2$, while the right side has Hausdorff dimension 2 by Lemma 5.5. Thus $\alpha = 0$ and that is a contradiction. Hence $0 \in \mathcal{E}$ is isolated.

By shrinking the neighborhoods U_1, U_2, U' if needed, it is possible to peel off a local correspondence $g' : U' \cap \mathcal{D}' \to U_1 \cap \mathcal{D}$ from the global inverse $g : \mathcal{D}' \to \mathcal{D}$. Note that $cl_{g'}(0') = 0$ and $cl_{g'}(M' \cap U') \Subset M \cap U_1$ after shrinking U' even further perhaps. The analytic set $V_{\mathrm{loc}} \subset U_1 \times (U' \setminus \overline{\mathcal{D}})$ defined in Section 4 contains the graph of the extension of g' across points near $0'$. Also, V_{loc} does not have limit points on $\partial U_1 \times (U' \setminus \overline{\mathcal{D}}')$ because $cl_{g'}(M' \cap U') \Subset M \cap U_1$. Thus

$$\pi' : V_{\mathrm{loc}} \to U' \setminus \overline{\mathcal{D}}'$$

is proper. The canonical pseudo-polynomials defining this cover are monic with coefficients that are holomorphic in $U' \setminus \overline{\mathcal{D}}'$. All of them clearly extend to U', after perhaps shrinking U' and this shows that g' extends as a correspondence. Theorem 4.1 in [22] shows that f also extends as a correspondence near $(0, 0')$ and this is enough to conclude that f extends holomorphically across the origin by Theorem 7.4 in [9]. □

Recall the strategy of the proof of Theorem 1.1 outlined in Section 3. Let ζ_0 be as before and $cl_f(\zeta_0) \cap M'^{+}_s \neq \emptyset$. Then by Proposition 3.2, f extends holomorphically to a neighborhood of ζ_0 and therefore the argument used in case (a) can be applied to obtain a contradiction. Thus the global analytic set V as in Definition 3.1 has no limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times M'^{+}_s$.

6. Removability of $\mathcal{L} \times T'^{+}_2$, $\mathcal{L} \times (M'_e \cup T'^{+}_1 \cup T'^{+}_0)$ and the extendability of f

We will now focus on the global analytic set $V \subset (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ defined in Section 3. Note that the neighborhoods U_1, U' are chosen as described in the general situation. The results of Section 5 show that V has no limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times M'^{+}_s$. The goal of this section is to study the limit points of V on $\mathcal{L} \times (M'_e \cup T'^{+}_2 \cup T'^{+}_1 \cup T'^{+}_0)$ and to show that they are all removable singularities. We begin with the limit points on $\mathcal{L} \times T'^{+}_2$.

To start with, note that $\mathcal{L} = \bigcup_{z \in \gamma \cap U_1} (Q_z \cap U_1)$ is defined by a single real analytic equation. Indeed, γ can be 'straightened' by a change of variables so that in the new coordinate system it becomes the x_1-axis. This may destroy all previous normalizations of the defining function $r(z, \bar{z})$. But nevertheless, it is clear that

$$\mathcal{L} = \{r(z, x_1) = 0\}\,.$$

By the theorems of Cartan–Bruhat (see Proposition 14 and 18 in [16]), it follows that $\mathcal{L}_{\mathrm{sng}}$ is contained in a real analytic set of dimension at most two and is defined in U_1 by finitely many real analytic equations.

Proof of Proposition 3.3. Suppose that $p \in \mathcal{L}_{\mathrm{reg}}$ and choose neighborhoods $p \in U_p$, $p' \in U'_{p'}$ so that $\mathcal{L} \times T'^{+}_2$ is smooth, real analytic in $U_p \times U'_{p'}$. Since $\mathcal{L}$ is locally foliated by open pieces of Segre varieties, it follows that the CR dimension of $\mathcal{L} \times T'^{+}_2$ is 1 near (p, p'). Note that

$(\overline{V} \setminus V) \cap (U_p \times U'_{p'}) \subset \mathcal{L} \times T_2'^{+}$. As dim $V \equiv 2$, it follows by Theorem 20.5 in [4] that V has analytic continuation, say $V^{\text{ext}} \subset U_p \times U'_{p'}$ that is a closed analytic set, after shrinking these neighborhoods. Let $\pi : V^{\text{ext}} \to U_p$ and $\pi' : V^{\text{ext}} \to U'_{p'}$ be the projections and

$$S = \left\{(z, z') \in V^{\text{ext}} : \dim_{(z,z')} \left(\pi'\right)^{-1}\left(z'\right) \geq 1\right\}$$

just as in Proposition 4.1. The condition

$$f\left(Q_w \cap \mathcal{D}\right) \supset {}_{^s w'}Q'_{w'}$$

for $(w, w') \in V$ forces

$$\pi' : V \cap \left(U_p \times U'_{p'}\right) \to U'_{p'}$$

to be locally proper. Thus $\pi'(V \cap (U_p \times U'_{p'}))$ contains an open subset of $U'_{p'}$. By the same reasoning used in Proposition 4.1, it follows that dim $S \leq 1$.

Suppose that $(a, a') \in (\overline{V} \setminus V) \setminus S \subset \mathcal{L} \times T_2'^{+}$. It is then possible to choose neighborhoods $U_a, U'_{a'}$ so that

$$\pi' : V^{\text{ext}} \cap \left(U_a \times U'_{a'}\right) \to U'_{a'} \tag{6.1}$$

is proper. Since $(\overline{V} \setminus V) \subset \mathcal{L} \times T_2'^{+}$, (6.1) shows that $\pi'(V \cap (U_a \times U'_{a'}))$ contains a one sided neighborhood of a', say $\Omega' \subset U' \setminus \overline{\mathcal{D}}'$. Clearly, $\partial\Omega'$ contains points from $M_s'^{+}$ and this contradicts the fact that V has no limit points on $(U_1 \setminus \overline{\mathcal{D}}) \times M_s'^{+}$. Thus $\overline{V} \setminus V \subset S$. But the three dimensional Hausdorff measure of S is zero and by Shiffman's theorem (cf. [4]), it follows that $\overline{V}$ itself is analytic in $U_p \times U'_{p'}$. Therefore $V^{\text{ext}} = \overline{V}$.

This argument works if $p \in \mathcal{L}_{\text{reg}}$. As observed above, $\mathcal{L}_{\text{sng}}$ is contained in a real analytic set of dimension at most 2 that is defined by finitely many equations. Thus it is possible to proceed by downward induction to conclude that $\mathcal{L} \times T_2'^{+}$ is removable. □

As a consequence $\overline{V}$ is analytic in $\left((U_1 \setminus \overline{\mathcal{D}}) \times U'\right) \setminus \left(\mathcal{L} \times (M'_e \cup T_1'^{+} \cup T_0'^{+})\right)$.

Proof of Proposition 3.4. Consider the global analytic set $V \subset (U_1 \setminus \overline{\mathcal{D}}) \times (U' \setminus \overline{\mathcal{D}}')$ of Definition 3.1. We will show that even the bigger set $(U_1 \setminus \overline{\mathcal{D}}) \times (M'_e \cup T_1'^{+} \cup T_0'^{+})$ is removable for $\overline{V}$. As observed in Section 3,

$$\pi : \overline{V} \setminus \left(\mathcal{L} \times \left(M'_e \cup T_1'^{+} \cup T_0'^{+}\right)\right) \to U_1 \setminus \overline{\mathcal{D}}$$

is proper. Note that $M'_e \cup T_1'^{+} \cup T_0'^{+}$ is a locally finite union of real analytic arcs and points and is thus a locally pluripolar set. But such sets are also globally pluripolar by Josefson's theorem. Hence it is possible to choose a plurisubharmonic function on $\mathbb{C}^4$, say ϕ, such that $(U_1 \setminus \overline{\mathcal{D}}) \times (M'_e \cup T_1'^{+} \cup T_0'^{+}) \subset \{\phi = -\infty\}$. Also, choose $a \in M \setminus \gamma$ across which f is known to extend. Fix a small ball $B \Subset (U_1 \setminus \overline{\mathcal{D}}) \setminus \mathcal{L}$ close to a so that f is well defined in B. Then $\overline{V} \setminus (\mathcal{L} \times (M'_e \cup T_1'^{+} \cup T_0'^{+}))$ has no limit points on $B \times M'$. Indeed, suppose $(w_0, w'_0) \in B \times M'$ is such a limit point, and let $(w_j, w'_j) \in \overline{V} \setminus (\mathcal{L} \times (M'_e \cup T_1'^{+} \cup T_0'^{+}))$ be a sequence converging to (w_0, w'_0). Then

$$f\left(Q_{w_j} \cap \mathcal{D}\right) \supset {}_{^s w'_j}Q'_{w'_j}$$

holds for all j. Choose $\zeta_j \in Q_{w_j} \cap \mathcal{D}$ so that $f(\zeta_j) = {}^s w'_j$. After passing to a subsequence, $\zeta_j \to \zeta_0 \in M \cap U_2$ and ${}^s w'_j \to {}^s w'_0 = w'_0$. Since $B \cap \mathcal{L} = \emptyset$ and $\zeta_0 \in Q_{w_0}$, it follows that $\zeta_0 \notin \gamma$. A contradiction can now be obtained exactly as in case (a) of Section 3. The non-empty analytic set $\left(\overline{V} \setminus (\mathcal{L} \times (M'_e \cup T_1'^+ \cup T_0'^+))\right) \setminus \{\phi = -\infty\}$ thus satisfies all the hypotheses of Bishop's lemma (cf. Section 18.2 in [4]). Hence $\overline{V}$ is analytic in $(U_1 \setminus \overline{\mathcal{D}}) \times U'$. The projection

$$\pi : \overline{V} \to U_1 \setminus \overline{\mathcal{D}}$$

is still proper and thus $\overline{V}$ defines a correspondence in $(U_1 \setminus \overline{\mathcal{D}}) \times U'$. The canonical defining equations of this correspondence have coefficients that are holomorphic in $U_1 \setminus \overline{\mathcal{D}}$. Since $0 \notin \hat{\mathcal{D}}$, by Trepreau's theorem all the coefficients extend to U_1, perhaps after shrinking U_1 . Thus $\overline{V}$ extends as a correspondence to $U_1 \times U'$ and this provides a multivalued extension of f. Theorem 7.4 in [9] now shows that f extends holomorphically across the origin. $\square$

7. Proof of Theorem 1.2

By Theorem 1.1, f extends across each point of M. Let us begin by observing the following: Let $z_0 \in M^+$ and consider $f(z_0)$. Suppose that $\{p'_j\} \subset M'$ is a sequence of points converging to $f(z_0)$ such that each p'_j is a strongly pseudoconcave point. By the invariance property of Segre varieties (see Theorems 4.1 and 5.1 in [9]) it is possible to choose small neighborhoods $z_0 \in U$ and $f(z_0) \in U'$ so that the global inverse correspondence $g : \mathcal{D}' \to \mathcal{D}$ extends as a correspondence, say $\tilde{g}$ and

$$\tilde{g} : U' \to U$$

is proper. Let $\sigma' \subset U'$ be the branching locus of $\tilde{g}$. Fix $p'_{j_0} \in U'$ for some large j_0. By shifting p'_{j_0} slightly we may assume that it is still strongly pseudoconcave but $p'_{j_0} \notin \sigma' \cup f(T \cap U)$, where T is the set of Levi flat points on M . Let g_1 be a branch of $\tilde{g}$ that is well defined near p'_{j_0} as a locally biholomorphic map. Then $g_1(p'_{j_0})$ is clearly strongly pseudoconvex and this contradicts the invariance of the Levi form. This shows that $f(z_0)$ cannot belong to the border between the pseudoconvex and pseudoconcave points on M' nor can M' be pseudoconcave near it. Thus $f(z_0) \in M'^+$.

Case (a): If $z_0 \in M_s^+$, then combining the observation above with Theorem 1.1 in [5] shows that $f(z_0) \in M_s'^+$. Thus $f(M_s^+) \subset M_s'^+$.

Case (b): Let $z_0 \in T_2^+$. We know that M' must be pseudoconvex near $f(z_0)$. Suppose that $f(z_0) \in M_s'^+$. Let r, r' be the defining functions of M, M' near $z_0, f(z_0)$, respectively. By the Hopf lemma, $r' \circ f$ is a defining function for M near z_0 and hence, for z close to z_0, the Levi determinant Λ transforms as

$$\Lambda_{r' \circ f}(z) = |J_f(z)|^2 \Lambda_{r'}(f(z)) ,$$

where J_f is the Jacobian determinant of f. Since $f(z_0) \in M_s'^+$, it follows that $\Lambda_{r'}(f(z)) \neq 0$. Thus the zero sets of $\Lambda_{r' \circ f}(z)$ and J_f coincide near z_0. But $\Lambda_{r' \circ f}(z)$ vanishes precisely on T_2^+ and hence J_f is zero on a maximally totally real manifold. Thus $J_f \equiv 0$ near z_0 and this contradicts the properness of f.

If $f(z_0) \in T_1'^+ \cup T_0'^+$, then we may argue in the following manner. Choose small neighborhoods $z_0 \in U$, $f(z_0) \in U'$ so that

$$f : U \to U'$$

is proper. Then $E := f^{-1}((T_1'^{+} \cup T_0'^{+}) \cap U') \cap U$ has real dimension 1 and hence there exists $\tilde{z} \in (T_2^{+} \cap U) \setminus E$. Clearly then $f(\tilde{z}) \in M_s'^{+}$. This is not possible by the discussion above. Thus $f(T_2^{+}) \subset T_2'^{+}$.

Case (c): Let $z_0 \in T_1^{+} \cup T_0^{+}$ and suppose that $f(z_0) \in T_2'^{+}$. Let $z_0 \in U$, $f(z_0) \in U'$ be small neighborhoods as before so that $f : U \to U'$ is proper. Then $E := f^{-1}(T_2'^{+}) \cap U$ has real dimension 2 and hence it is possible to choose $a \in E \setminus (T_1^{+} \cup T_0^{+})$. Note that $a \in M_s^{+}$. Thus a strongly pseudoconvex point is mapped into $T_2'^{+}$ and this contradicts case (a). Thus $f(T_1^{+} \cup T_0^{+}) \subset M_s'^{+} \cup T_1'^{+} \cup T_0'^{+}$.

Case (d): Let $z_0 \in T_0^{+}$ be an isolated point of T and suppose that $f(z_0) \in M_s'^{+}$. Then $J_f(z_0) = 0$ as otherwise f would locally biholomorphically map z_0, which is a weakly pseudoconvex point to $f(z_0) \in M_s'^{+}$. This contradicts the invariance of the Levi form. Choose small neighborhoods $z_0 \in U$, $f(z_0) \in U'$ so that

$$f : U \to U'$$

is proper. Call $Z_f = \{z \in U : J_f(z) = 0\}$, the branching locus of f. We claim that Z_f intersects both $U \setminus \overline{\mathcal{D}}$ and $U \cap \mathcal{D}$. Indeed, firstly $Z_f \not\subset \overline{U \cap \mathcal{D}}$ as otherwise the continuity principle would force $z_0 \in M \cap \hat{\mathcal{D}}$ which is not possible. Secondly, an open piece of Z_f cannot lie in M due to the finite type condition and hence $Z_f \cap M$ has real dimension at most one. Finally, let us show that $Z_f \cap \mathcal{D} \neq \emptyset$. If not, then observe that by the invariance property of Segre varieties (cf. [9]), f maps $U \cap \mathcal{D}$ to $U' \cap \mathcal{D}'$, $U \setminus \overline{\mathcal{D}}$ to $U' \setminus \overline{\mathcal{D}'}$ and M to M'. That is, f preserves the 'sides' of M. The same is also true for $f^{-1} : U' \to U$. Now choose some branch of f^{-1}, say g_1 that maps a fixed but arbitrary $a' \in U' \cap \mathcal{D}'$ to $a := g_1(a') \in U \cap \mathcal{D}$. Since Z_f does not enter $\mathcal{D}$ by assumption, it is possible to analytically continue g_1 along all paths in $U' \cap \mathcal{D}'$ to get a well defined mapping, still denoted by g_1 and $g_1 : U' \cap \mathcal{D}' \to U \cap \mathcal{D}$. The analytic set $\Gamma_f \subset U \times U'$ extends g_1 as a correspondence and by Theorem 7.4 in [9], it follows that g_1 extends as a holomorphic mapping to all of U', after perhaps shrinking U' slightly and $g_1 : U' \to U$. This shows that f has a well defined holomorphic inverse and hence it must be a biholomorphic mapping. Hence $Z_f = \emptyset$ which is not possible as $z_0 \in Z_f$. Thus Z_f must intersect both sides of M near z_0.

Choose $a \in (Z_f \cap M) \setminus \{z_0\}$ and note that $a \in M_s^{+}$. Thus f is a proper mapping between strongly pseudoconvex hypersurfaces near $(a, f(a))$ that branches at a. This is not possible since the Segre maps of both M, M' are injective and this forces f to be locally biholomorphic. Cases (a) and (b) also rule out the possibility that $f(z_0) \in T_2'^{+} \cup T_1'^{+}$. Thus $f(z_0) \in T_0'^{+}$ is also an isolated point of T'.

Case (e): If $z_0 \in M_e$, then M' cannot be pseudoconvex near $f(z_0)$ as otherwise there would exist a strongly pseudoconcave point close to z_0 that is mapped to a strongly pseudoconvex point close to $f(z_0)$. For the same reason M' cannot be pseudoconcave near $f(z_0)$. Also, $f(z_0)$ cannot belong to the two dimensional strata of the border since it is known to be in the envelope of holomorphy (cf. [6]). Thus $f(M_e) \subset M_e'$.

To conclude, the arguments used in Lemma 3.1 in [9] show that $f(M \cap \hat{\mathcal{D}}) \subset M' \cap \hat{\mathcal{D}}'$.

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Received July 25, 2003

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Communicated by John Erik Fornaess