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On boundary regularity of proper holomorphic mappings

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Abstract. We show that a proper holomorphic mapping $f : D \to D'$ from a domain $D \Subset \mathbb{C}^n$ with real-analytic boundary to a domain $D' \Subset \mathbb{C}^n$ with real-algebraic boundary extends holomorphically to a neighborhood of $\overline{D}$.

1. Introduction and the main result

The problem of boundary regularity of proper holomorphic mappings between bounded domains in $\mathbb{C}^n$, $n \geq 2$, has been studied for a long time. This problem seems to be completely solved for strictly pseudoconvex domains. Positive answers have been also obtained for pseudoconvex domains of finite type with C^∞ boundary. For a survey on the subject until 1989 see [F].

The question remains open for non pseudoconvex domains even in the case when both domains have smooth real-analytic boundary. The goal of this paper is to present the following result.

Theorem 1. *Let D, D' be bounded domains in $\mathbb{C}^n$, $n \geq 2$, let ∂D, the boundary of D, be smooth real-analytic and $\partial D'$ be smooth real-algebraic. Let $f : D \to D'$ be a proper holomorphic mapping. Then f extends holomorphically to a neighborhood of $\overline{D}$.*

Analogous theorems for bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth real-analytic boundaries were proved in [DF2] and [BR]. For arbitrary bounded domains with smooth real-analytic boundaries in $\mathbb{C}^2$ the result was proved in [DP1].

By a real-algebraic boundary we mean a real hypersurface in $\mathbb{C}^n$ globally defined by a polynomial equation $P(z, \overline{z}) = 0$. Our proof of Theorem 1 is based on the idea of analytic continuation of holomorphic mappings along hypersurfaces. We first show that f extends to some open set in ∂D and then continue f holomorphically along ∂D. Note that we do not require pseudoconvexity of D or D', and we do not assume *apriori* any regularity of f on the boundary.

The following corollary generalizes a well-known theorem of H. Alexander [A] stating that any proper holomorphic self-map of a unit ball is biholomorphic.

Corollary 1. *Let f be a proper holomorphic self-map of a bounded domain $D \subset \mathbb{C}^n$, $n > 1$ with smooth real-algebraic boundary. Then f is biholomorphic.*

Proof. By the result of [HP] f is biholomorphic if f extends smoothly to ∂D. By Theorem 1 f extends holomorphically to a neighborhood of $\overline{D}$. $\square$

In Sect. 2 we give basic definitions of Segre varieties and holomorphic correspondences. In Section 3 we prove analytic continuation of germs of holomorphic mappings along Segre varieties. The proof of Theorem 1 is contained in Section 4.

2. Notation and preliminaries

Let Γ be an arbitrary smooth real-analytic hypersurface with a defining function $\rho(z, \overline{z})$ and let $z^0 \in \Gamma$. In a suitable neighborhood $U \ni z^0$ to every point $w \in U$ we can associate its so-called Segre variety defined as

$$Q_w = \{z \in U : \rho(z, \overline{w}) = 0\} . \tag{1}$$

We can find neighborhoods $U_1 \Subset U_2$ of z^0, where

$$U_2 = {}'U_2 \times {}''U_2 \subset \mathbb{C}^{n-1}_{'z} \times \mathbb{C}_{z_n}, \tag{2}$$

such that for any $w \in U_1$, Q_w is a closed smooth complex-analytic hypersurface in U_2. Here $z = ({}'z, z_n)$. Furthermore, a Segre variety can be written as a graph of a holomorphic function,

$$Q_w = \left\{({}'z, z_n) \in ({}'U_2 \times {}''U_2) : \; z_n = h({}'z, \overline{w})\right\} , \tag{3}$$

where $h(\cdot, \overline{w})$ is holomorphic in ${}'U_2$. U_1 and U_2 are usually called a *standard* pair of neighborhoods of z^0. A detailed discussion of Segre varieties can be found in [DW], [DF2] or [DP1].

A real-analytic hypersurface Γ is called *essentially finite* at a point $z^0 \in \Gamma$ if the *Segre* map $\lambda : z \to Q_z$ is finite-to-one in some neighborhood of z^0, or equivalently, if the set

$$I_w = \{z \in U_1 : Q_z = Q_w\} \tag{4}$$

is finite for every w near z^0. Γ is said to be essentially finite if it is essentially finite at every point. Throughout the paper we assume that a standard pair of neighborhoods $U_1 \Subset U_2$ of any point on Γ is always chosen in such a way that I_w is finite for any $w \in U_1$. For further discussion of essential finiteness of real-analytic hypersurfaces see [BJT] or [DF2].

Definition 1. *A holomorphic correspondence between two domains D and D' in $\mathbb{C}^n$ is a complex-analytic set $A \subset D \times D'$ which satisfies: (i) $\dim_C A \equiv n$ and (ii) the natural projection $\pi : A \to D$ is proper.*

We use the right prime to denote the objects in the target domain. The set A can also be treated as a graph of the multiple valued mapping defined by $F := \pi' \circ \pi^{-1}$, where $\pi' : A \to D'$ is the natural projection.

Definition 2. *Let U and U' be open sets in $\mathbb{C}^n$ and let $f : U \to U'$ be a holomorphic mapping. We say that f extends as a holomorphic correspondence to an open set $V \supset U$, if there exist an open set $V' \subset \mathbb{C}^n$ and a holomorphic correspondence $A \subset V \times V'$ such that $\Gamma_f \subset A$, where Γ_f is the graph of the mapping f.*

Remark 1. If f extends to V as a correspondence, then V' can always be chosen to be $\mathbb{C}^n$.

3. Extension as a correspondence

The next proposition is the main tool in propagation of analyticity of holomorphic mappings along real-analytic hypersurfaces.

Proposition 1. *Let $\Gamma \subset \mathbb{C}^n$ be a smooth real-analytic essentially finite hypersurface and $a \in \Gamma$. Let $U_1 \Subset U_2$ be a standard pair of neighborhoods of a. Let $f : U_a \to \mathbb{C}^n$ be a biholomorphic mapping, $f(U_a \cap \Gamma) \subset \Gamma'$, where U_a is an arbitrarily small neighborhood of a, $U_a \subset U_1$ and $\Gamma' \Subset \mathbb{C}^n$ is a compact smooth real-algebraic hypersurface. Then for any $b \in (Q_a \cap U_1) \setminus \Lambda$, where $\Lambda \subset (Q_a \cap U_1)$ is an analytic set of dimension at most $n-2$, there exists a neighborhood W of a connected component of $Q_b \cap \Gamma \cap U_1$ containing a, such that $f|_{U_a \cap W}$ extends as a holomorphic correspondence to W.*

Let us clarify the statement of Proposition 1. Consider a point $b \in Q_a \cap U_1$. From the properties of Segre varieties (see e.g. [DW]), $a \in Q_b$. In general, $Q_b \cap \Gamma \cap U_1$ may contain several connected components. We choose the component of $Q_b \cap \Gamma \cap U_1$ which contains a. Then the proposition claims that f extends as a holomorphic correspondence to a neighborhood of that component for almost any $b \in Q_a \cap U_1$.

Proof. The idea of the proof is similar to that of [S], Proposition 5.1. For completeness we give a proof here, emphasizing the changes that should be applied when Γ' is not strictly pseudoconvex. The proof of the proposition consists of several steps:

Step 1: Construct a correspondence F in a neighborhood U_b of a point $b \in Q_a \setminus \Lambda$.

Step 2: Construct a multiple valued mapping F^* in a neighborhood W of a connected component of $Q_b \cap \Gamma$.

Step 3: Show that F^* contains f as a branch in some neighborhood of a.

Step 4: Show that W in Step 2 can be chosen in such a way that the multiple valued mapping F^* is a holomorphic correspondence.

Step 1. Let us choose a thin neighborhood V of the set $Q_a \cap U_1$ and shrink the neighborhood U_a so that for any $w \in V$ the set $Q_w \cap U_a$ is nonempty and connected. Define

$$A = \{(w, w') \in V \times \mathbb{C}^n : f(Q_w \cap U_a) \subset Q'_{w'}\}. \tag{5}$$

It is shown in [S], Proposition 3.1 that A is an analytic set in $V \times \mathbb{C}^n$. Further, from the algebraicity of Γ' it follows that the equations defining A are algebraic in w' (for details see [S], Proposition 3.1; similar argument is used in Step 2 of the proof of this proposition). Thus A extends to an analytic set in $V \times \mathbb{P}^n$ which we denote for simplicity by A.

Let $\Omega \subset (U_a \cap V)$ be a small open set containing a. By the invariance property of Segre varieties under biholomorphic mappings, for $w \in \Omega$

$$f(Q_w \cap U_a) \subset Q'_{f(w)}. \tag{6}$$

Therefore $\Gamma_{f|_\Omega} \subset A$ and $A \neq \varnothing$. Let $(w, w') \in A \cap (\Omega \times f(\Omega))$ be an arbitrary point. Then $f(Q_w \cap U_a) \subset Q'_{w'}$. In view of (6) we conclude that

$$w' \in I'_{f(w)}. \tag{7}$$

By [DF1], any compact real-analytic hypersurface in $\mathbb{C}^n$ is of finite type (in the sense of D'Angelo), in particular it is essentially finite. Therefore by shrinking Ω if necessary, we may assume that $I'_{f(w)}$ is a finite set in $f(\Omega)$. This implies that $\dim_C A \cap (\Omega \times f(\Omega)) = n$.

We consider only the irreducible component of A that contains $\Gamma_{f|_\Omega}$. Denote this component again by A.

Let $\pi : A \to V$ and $\pi' : A \to \mathbb{P}^n$ be the natural projections. Define

$$\widetilde{\Lambda} = \{w \in V : \dim(\pi^{-1}(w) \cap A) \geq 1\}. \tag{8}$$

By Cartan-Remmert's theorem (see e.g. [L]) $\widetilde{\Lambda}$ is an analytic set, and it was shown in [S], Proposition 3.3, that $\dim_C \widetilde{\Lambda} \leq n-2$. We set $\Lambda = Q_a \cap \widetilde{\Lambda}$. From [S], Proposition 3.1 and Lemma 5.4, for any $b \in (Q_a \cap U_1) \setminus \Lambda$ one can find a simply connected set $V_1 \subset V \setminus \widetilde{\Lambda}$ with $a, b \in V_1$ such that after possibly a linear fractional transformation of the target coordinates w', which is holomorphic on Γ', the set $A \cap (V_1 \times \mathbb{C}^n)$ is a holomorphic correspondence extending $f|_{V_1 \cap U_a}$.

Consider the restriction of the extended correspondence to some neighborhood $U_b \ni b$, $U_b \subset V_1$, and let $F : U_b \to \mathbb{C}^n$ be a corresponding multiple valued mapping, that is $F = \pi' \circ \pi^{-1}|_{U_b}$. We mention some important properties of F. Let $z' \in F(U_b)$. Then for any $z \in F^{-1}(z')$, $f(Q_z \cap U_a) \subset Q'_{z'}$. Since f is biholomorphic in U_a,

$$Q_{z^1} \cap U_a = Q_{z^2} \cap U_a, \ \ \forall\, z^1, z^2 \in F^{-1}(z'). \tag{9}$$

Therefore since Γ is essentially finite, $F^{-1}(z')$ is finite for any $z' \in F(U_b)$. It follows that

$$\dim F(Q_z \cap U_b) = 2n-2, \ \text{ if } z \in U_1 \text{ and } Q_z \cap U_b \neq \varnothing. \tag{10}$$

This in particular implies that if $w' \in F(w)$, then

$$F(w) \subset I'_{w'}. \tag{11}$$

Step 2. Let W be a neighborhood of the connected component of $Q_b \cap \Gamma \cap U_1$ that contains a. We choose W and shrink U_b so that for all $z \in W$, $Q_z \cap U_b$ is nonempty and connected. Let

$$\Sigma = \{z \in U_b : \pi \text{ is not locally biholomorphic near } \pi^{-1}(z)\} \tag{12}$$

be the singular locus of F and let

$$E = \{z \in W : (Q_z \cap U_b) \subset \Sigma\}. \tag{13}$$

Define

$$A^* = \left\{(w, w') \in (W \setminus E) \times \mathbb{C}^n : \ F(Q_w \cap U_b) \subset Q'_{w'}\right\}. \tag{14}$$

Let $(w, w') \in A^*$. Consider an open simply connected set $\Omega \subset (U_b \setminus \Sigma)$ such that $Q_w \cap \Omega \neq \varnothing$. Then the branches of F are correctly defined in Ω, and $F(Q_w \cap U_b) \subset Q'_{w'}$ is equivalent to

$$\tilde{F}(Q_w \cap \Omega) \subset Q'_{w'} \tag{15}$$

for all branches $\tilde{F}$ of F. Such neighborhood Ω exists for any $w \in (W \setminus E)$. The inclusion (15) can be written as a system of holomorphic equations as follows. Let $P(z', \overline{z}')$ be the defining polynomial of Γ'. Then (15) can be expressed as

$$P'(\tilde{F}(z), \overline{w}') = 0, \text{ for any } z \in Q_w \cap \Omega. \tag{16}$$

Choosing Ω as in (2) and using (3) we obtain

$$P'\left(\tilde{F}('z, h('z, \overline{w})), \overline{w}\right) = 0, \; \forall\, 'z \in {'\Omega}, \tag{17}$$

which is an infinite system of equations holomorphic in $\overline{w}$ and algebraic in $\overline{w}'$. Thus locally (14) is given by a system of equations holomorphic in some neighborhood of (w, w'). To prove that A^* is a complex analytic set in $(W \setminus E) \times \mathbb{C}^n$ it remains to show that A^* is closed. Suppose $(z^\nu, z^{\nu\prime}) \to (z^0, z^{0\prime})$ as $\nu \to \infty$, $(z^\nu, z^{\nu\prime}) \in A^*$ and $z^0 \in (W \setminus E)$. Then $F(Q_{z^\nu} \cap U_b) \subset Q'_{z^{\nu\prime}}$. Since $Q_{z^\nu} \to Q_{z^0}$ and $Q'_{z^{\nu\prime}} \to Q'_{z^{0\prime}}$, we deduce that $F(Q_{z^0} \cap U_b) \subset Q'_{z^{0\prime}}$, $(z^0, z^{0\prime}) \in A^*$, and A^* is closed.

In view of essential finiteness of Γ, the set E is finite. Let $p \in E$. Then

$$\overline{A^*} \cap (\{p\} \times \mathbb{C}^n) \subset \{p\} \times \{z' \in U' : F(Q_p) \subset Q'_{z'}\}. \tag{18}$$

Notice that if $w' \in F(Q_p) \subset Q'_{z'}$, then $z' \in Q'_{w'}$. Hence the set $\{z' \in U' : F(Q_p) \subset Q'_{z'}\}$ has dimension at most $2n - 2$. Therefore $\overline{A^*} \cap (E \times \mathbb{C}^n)$ has Hausdorff $2n$-measure zero, and by Bishop's theorem (see e.g. [C]), it is a removable singularity for A^*, and $\overline{A^*}$ is an analytic set in $W \times \mathbb{C}^n$. The multiple valued map F^* is now defined by $F^* = \pi' \circ \pi^{-1}$, where $\pi : A^* \to W$ and $\pi' : A^* \to \mathbb{C}^n$ are the natural projections.

Step 3. To simplify notations we denote $\overline{A^*}$ by A^* and $W \cap U_a \cap V$ by U_a. To show that $\Gamma_{f|_{U_a}} \subset A^*$ it is enough to prove the following lemma.

Lemma 1. $A^* \cap (U_a \times \mathbb{C}^n) = A \cap (U_a \times \mathbb{C}^n)$.

Proof. Let us first show that the following three inclusions are equivalent for $z \in U_a$ and $Q_z \cap U_a \neq \varnothing$:

$$\begin{aligned} &\text{i) } f(Q_z \cap U_a) \subset Q'_{z'} \\ &\text{ii) } F(Q_z \cap U_a) \subset Q'_{z'} \\ &\text{iii) } F(Q_z \cap U_b) \subset Q'_{z'}. \end{aligned} \tag{19}$$

Indeed, suppose $f(Q_z \cap U_a) \subset Q'_{z'}$. Then by the invariance property of the Segre varieties, $z' \in I'_{f(z)}$. Let $w \in Q_z \cap U_a$ and $w' \in F(w)$. It follows from the definition of F that $f(Q_w \cap U_a) \subset Q'_{w'}$, and $z \in Q_w$ implies $f(z) \in Q'_{w'}$. Therefore $w' \in Q'_{f(z)}$. Since $(w, w') \in A$ was arbitrary, $F(Q_z \cap U_a) \subset Q'_{f(z)} = Q'_{z'}$. Thus i) $\Rightarrow$ ii).

Suppose ii) holds. From (7) we have $z' \in I'_{f(z)}$. Let $w \in Q_z \cap U_b$ and $w' \in F(w)$. Then by the definition of F, $f(Q_w \cap U_a) \subset Q'_{w'}$, in particular, $f(z) \in Q'_{w'}$ as $z \in Q_w \cap U_a$. But then $w' \in Q'_{f(z)}$. Since (w, w') was arbitrary, $F(Q_z \cap U_b) \subset Q'_{f(z)} = Q'_{z'}$, and ii) $\Rightarrow$ iii).

Finally, suppose $F(Q_z \cap U_b) \subset Q'_{z'}$. Let $w \in Q_z \cap U_b$ and $w' \in F(w)$. Then $w' \in Q'_{z'}$. On the other hand, by the definition of F, $f(Q_w \cap U_a) \subset Q'_{w'}$, in particular, $f(z) \in Q'_{w'}$ and therefore $w' \in Q'_{f(z)}$. Thus $w' \in Q'_{z'} \cap Q'_{f(z)}$. Since $\dim F(Q_z \cap U_b) = 2n-2$, we conclude that $z' \in I'_{f(z)}$. This proves that iii) $\Rightarrow$ i).

Now the assertion of the lemma easily follows. Let $(z, z') \in A^* \cap (U_a \times \mathbb{C}^n)$. Then $F(Q_z \cap U_b) \subset Q'_{z'}$ and therefore by (19), $f(Q_z \cap U_a) \subset Q'_{z'}$. But this means $(z, z') \in A$. Conversely, if $(z, z') \in A \cap (U_a \times \mathbb{C}^n)$, then $f(Q_z \cap U_a) \subset Q'_{z'}$ and therefore by (19), $F(Q_z \cap U_b) \subset Q'_{z'}$. But then $(z, z') \in A^*$. Lemma 1 is proved. $\square$

Step 4. Consider the irreducible component of A^* which coincides with A in $U_a \times \mathbb{C}^n$. For simplicity denote it by A^*. To finish the proof of Proposition 1 we need to show that we can choose W and a neighborhood U' of Γ' such that $A^* \cap (W \times U')$ is a holomorphic correspondence. Let U' be a neighborhood of Γ' such that the Segre map $\lambda' : z' \to Q'_{z'}$ is finite-to-one in U'. Let $\pi : A^* \to W$ and $\pi' : A^* \to U'$ be the natural projections.

We first show that for any $z \in \Gamma \cap W$

$$F^*(z) = \pi' \circ \pi^{-1}(z) \subset \Gamma'. \tag{20}$$

Indeed, consider the set $\pi^{-1}(\Gamma \cap W) \subset A^*$. This is a real-analytic subset of A^* of dimension $2n-1$. Let $S = {\pi'}^{-1}(\Gamma')$. If $(z, z') \in A^* \cap (U_a \times \mathbb{C}^n)$, and $z \in \Gamma$, then by Lemma 1 and (7), $z' \in I_{f(z)}$. By [DW], for any $z' \in \Gamma'$, $I'_{z'} \subset \Gamma'$, and therefore

$$\pi^{-1}(\Gamma \cap U_a) \subset S. \tag{21}$$

Hence the whole irreducible component of $\pi^{-1}(\Gamma \cap W)$ containing $\pi^{-1}(a)$ is contained in S. From (11) and the fact that $I'_{z'} \subset \Gamma'$ for any $z' \in \Gamma'$, the assertion follows.

Now let us show that W can be chosen so small that

$$A^* \cap (W \times \partial U') = \varnothing. \tag{22}$$

If not, then there exists a sequence $(z^j, z'^j) \in A^*$ such that $z^j \to z^0 \in \Gamma \cap W$ and $z'^j \to z'^0 \in \partial U'$ as $j \to \infty$. Then $(z^0, z'^0) \in A^*$ and $z'^0 \notin \Gamma'$ (recall that Γ' is compact and $U' \supset \Gamma'$). But this contradicts (20).

Since the change of coordinates w' performed earlier, is holomorphic near Γ', (22) also holds for the original coordinate system. From that equation $\pi : A^* \to W$ is proper, and by Lemma 1 $\dim_C A^* = n$ and $\Gamma_{f|_{U_a}} \subset A^*$. Thus $A^* \cap (W \times U')$ is the desired holomorphic correspondence. □

4. Proof of the main result

Proposition 1 allows us to extend a germ of a holomorphic mapping defined at a point on the hypersurface along certain Segre varieties. This will be the main tool in the proof of Theorem 1. To apply Proposition 1 we need to connect different points on ∂D by Segre varieties. The proposition below provides some information on the existence of such Segre varieties.

Let $\Gamma \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface and let $0 \in \Gamma$ be a strictly pseudoconvex point. By [CM] there exists a biholomorphic change of coordinates near the origin such that in a new coordinate system the defining function of Γ has the form

$$\rho(z, \overline{z}) = 2x_n + \sum_{k=1}^{n-1} |z_k|^2 + \sum_{|K|,|L| \geq 2} \rho_{KL}(y_n)('z)^K('\overline{z})^L. \tag{23}$$

Proposition 2. *Suppose Γ is a smooth real-analytic hypersurface with the defining function given as in* (23)*. Let U_1, U_2 be a standard pair of neighborhoods of the origin. Then there exists $\delta > 0$ such that for any $w \in U_1 \cap \Gamma$, $w = ('w, u_n + iv_n)$, satisfying*

$$|v_n| < \delta|'w|, \ \ |w| < \delta, \tag{24}$$

we can find a point $z \in (U_1 \cap Q_0 \cap Q_w)$.

Proof. Let $w \in \Gamma$ satisfy (24). Then

$$Q_w = \left\{ z_n + \overline{w}_n + \sum_{k=1}^{n-1} z_k \overline{w}_k + \sum_{|K|,|L| \geq 2} \rho_{KL}\left(\frac{z_n - \overline{w}_n}{2i}\right) {'z}^K {'\overline{w}}^L = 0 \right\}. \tag{25}$$

$z \in Q_0$ implies $z_n = 0$, and therefore z satisfies

$$\overline{w}_n + \sum_{k=1}^{n-1} z_k \overline{w}_k + \sum_{|K|,|L| \geq 2} \rho_{KL}\left(-\frac{\overline{w}_n}{2i}\right) {'z}^K {'\overline{w}}^L = 0. \tag{26}$$

Since $w \in \Gamma$,

$$2u_n + |'w|^2 + \sum_{|K|,|L| \geq 2} \rho_{KL}(v_n)('w)^K('\overline{w})^L = 0. \tag{27}$$

From (26) and (27) we obtain

$$iv_n - \frac{1}{2}|'w|^2 + \sum_{k=1}^{n-1} z_k \overline{w}_k + \Phi('z, 'w, v_n) = 0, \tag{28}$$

where $\Phi('z, 'w, v_n)$ is holomorphic in $'z$ and $\Phi('z, 'w, v_n) = o(|'z|^2 + |w|^2)$. Let $'z = t\frac{'w}{|w|}$, $t \in \mathbb{C}$. Then from (28),

$$\frac{iv_n}{|'w|} - \frac{1}{2}|'w| + t + \tilde{\Phi}(t, 'w, v_n) = 0. \tag{29}$$

If $|t| = \epsilon$, $\epsilon > 0$, w satisfies (24) and δ is sufficiently small, then

$$\left| \frac{iv}{|'w|} - \frac{1}{2}|'w| + \tilde{\Phi}(t, 'w, v_n) \right| < \epsilon. \tag{30}$$

Therefore, by Rouché's theorem equation (28) has a solution $'z = t\frac{'w}{|w|}$, where $|t| < \epsilon$. Finally, If ϵ is chosen small enough, then $z \in U_1$. $\square$

Now let $L = \{z \in U_1 : 'z = x_n = 0\}$ and let fix some $z = ('0,\ iy_n) \in L$. Then there exists $\delta > 0$ such that for any $w \in U_1 \cap \Gamma$, $w = ('w, u_n + iv_n)$, satisfying

$$|y_n - v_n| < \delta|'w|,\ |z - w| < \delta, \tag{31}$$

one can find a point $\zeta \in (U_1 \cap Q_z \cap Q_w)$. For that it is sufficient to perform a translation shifting z to the origin and apply Proposition 2. Moreover, δ can be chosen uniformly for all $z \in L$ in a small neighborhood of the origin.

Remark 2. If $\zeta \in (U_1 \cap Q_z \cap Q_w)$, then there exists a path in $Q_\zeta \cap \Gamma \cap U_1$ which connects z and w. Indeed, it is well known that if Γ is strictly pseudoconvex, ζ is close to Γ, and $\rho(\zeta, \overline{\zeta}) > 0$, then $Q_\zeta \cap \{z \in U_1 : \rho(z, \overline{z}) < 0\}$ is connected, and Γ and Q_ζ intersect in general position.

For further reference define

$$\Omega_{z,\delta} = \left\{ w \in U_1 \cap \Gamma : |y_n - v_n| < \delta|'w|,\ |z - w| < \delta \right\}, \tag{32}$$

where $z \in L$ and $\delta > 0$.

Proof of Theorem 1. We use the following notation: $\Gamma = \partial D$, $\Gamma' = \partial D'$. Let Γ^+ be the strictly pseudoconvex part of Γ, Γ^- be the set of points on Γ

where the Levi form of the defining function of Γ has at least one negative eigenvalue, and let $\Gamma^0 = \Gamma \setminus (\Gamma^+ \cup \Gamma^-)$.

In view of the result of [DF2] and [BR] it is enough to consider the case when both domains are not pseudoconvex. It is well-known that any holomorphic function in D extends past Γ^-. Let us show that f extends holomorphically to a neighborhood of a dense open subset of $M = \Gamma^0 \setminus \overset{\circ}{\overline{\Gamma^+}}$, where $\overset{\circ}{\overline{\Gamma^+}}$ is the interior of the set $\overline{\Gamma^+}$. Observe that M is a semi-analytic set which admits stratification $M = \cup_k M_k$, where M_k is a locally finite union of smooth real-analytic manifolds of dimension k, $k = 0, 1, \ldots, 2n-2$. For details see [BM].

Suppose $0 \in M$, U is a neighborhood of the origin, $M \cap U$ is a smooth connected generic submanifold of Γ and $\dim(M \cap U) = 2n-2$. Then M divides $\Gamma \cap U$ into two connected components which we denote by M^+ and M^-. Let M^- be the component which is contained in Γ^-, and we may assume that f is holomorphic on M^-. Let $b \in Q_0 \cap U$ and S_b be the connected component of $Q_b \cap \Gamma \cap U$ that contains 0.

Lemma 2. *We can choose points $b \in Q_0 \cap U$ and $a \in S_b$ such that*
(i) $a \in M^-$
(ii) $J_f(a) \neq 0$
(iii) if Λ is the set from Proposition 1, then $b \in (Q_a \cap U) \setminus \Lambda$

Proof. Proposition 4.1 of [S] can be reformulated as follows. There exists an open set $\omega \subset Q_0 \cap U$ such that for any $b \in \omega$, $M^- \cap S_b$ is a nonempty real-analytic subset of M^- of dimension $2n-3$. Thus we can choose $b \in \omega$ such that $J_f \not\equiv 0$ on $M^- \cap S_b$, where $J_f(z)$ is the Jacobian of the mapping f. Let $a \in M^- \cap S_b$ with $J_f(a) \neq 0$. Notice that $b \in \omega \cap Q_a$ and $\dim_C Q_a \cap \omega = n-2$. If $b \in \Lambda$, but $\omega \cap Q_a \not\subset \Lambda$, we replace b by another point from $(\omega \cap Q_a) \setminus \Lambda$. If $(\omega \cap Q_a) \subset \Lambda$ we can slightly change the point $a \in S_b \cap M^-$ such that $J_f(a) \neq 0$ still holds but $\omega \cap Q_a \not\subset \Lambda$, as the set $\widetilde{\Lambda}$ defined in Proposition 1 does not depend on a. After that we can find a new $b \in (Q_a \cap \omega) \setminus \Lambda$. $\square$

By Proposition 1, for a small neighborhood $U_a \ni a$, $f|_{U_a}$ extends as a holomorphic correspondence to a neighborhood of S_b. Therefore there exists a neighborhood U_0 of the origin such that $f|_{U_0 \cap D}$ extends to U_0 as a holomorphic correspondence. By [DP2], if a proper holomorphic map $f : D \to D'$ extends as a holomorphic correspondence to some neighborhood of a point $0 \in \partial D$, then f in fact extends as a holomorphic mapping.

Note that M is generic at almost any point where the dimension of M is $2n-2$. If $M \cap U$ is a submanifold and $\dim M < 2n-2$, then we can always find a generic submanifold in Γ of dimension $2n-2$ which contains $M \cap U$. By repeating the argument that we used for extension to generic points of

M we now extend f to a dense open subset of M. Denote by $\Sigma \subset \Gamma$ the set of points of holomorphic extendability of f. Thus we proved the following lemma.

Lemma 3. *$(\Gamma^- \cup M_1) \subset \Sigma$, where M_1 is a dense open subset of M.*

From the above considerations, Σ intersects every connected component of Γ^+. Our next goal is to show that $\Gamma^+ \subset \Sigma$. Let $z^0 \in \Sigma \cap \Gamma^+$ and let $w^0 \in \Gamma^+$ be an arbitrary point in the same component of Γ^+ as z^0. We connect z^0 and w^0 by a path $\tau : [0,1] \to \Gamma^+$.

Lemma 4. *If $t_0 \in (0,1]$ and f extends holomorphically to $\tau(t) \subset \Gamma^+$ for any $0 < t < t_0$, then f extends holomorphically to a neighborhood of $\tau(t_0)$.*

Proof. Without loss of generality assume that $\tau(t_0) = 0$ and τ is real-analytic near t_0. First suppose that τ is tangent to $T_0^c(\Gamma)$, the complex tangent plane to Γ at the origin. After an appropriate change of coordinates we may assume that for any $\delta > 0$ there exists $a \in (\tau \cap \Omega_{0,\delta})$, where $\Omega_{0,\delta}$ is defined as in (32). By Proposition 2, there exists a point b in a neighborhood U of the origin such that $a \in Q_b \cap U$ and $0 \in Q_b$. Furthermore, by Remark 2 for a and b sufficiently close to the origin, $Q_b \cap \Gamma \cap U$ is connected. Again, if necessary we can slightly move points b and a in such a way that $b \in Q_a \setminus \Lambda$ and $J_f(a) \neq 0$. Then Proposition 1 applies, and $f|_{U_a}$ extends as a correspondence to a neighborhood of $Q_b \cap \Gamma \cap U$, and therefore f extends as a correspondence to the origin. By [DP2] f extends as a holomorphic mapping.

Now suppose that the angle between the tangent vector of $\tau(t)$ and $T^c_{\tau(t)}(\Gamma)$ is bounded below from zero as $t \to t_0$. By the result of [CM] there exists a local biholomorphic change of variables such that the defining function of Γ near the origin is given in the form (23) and τ contains the set

$$L = \{z \in U_1 : {}'z = x_n = 0;\ y_n \geq 0\}.$$

By Proposition 2 we can find $\delta > 0$ and $w \in L$, such that

$$\Omega_{w,\delta} \cap \Omega_{0,\delta} \neq \varnothing \tag{33}$$

Let $a \in \Omega_{w,\delta} \cap \Omega_{0,\delta}$. Applying Proposition 1 and [DP2] we first continue f holomorphically from w to a neighborhood of a along the Segre variety connecting these points. Then similarly we extend f holomorphically along the Segre variety connecting point a and the origin. $\square$

It follows from Lemma 4 that $\Gamma^+ \subset \Sigma$. The remaining part of the hypersurface, to which f does not extend holomorphically, is contained in Γ^0. By repeating the arguments which we used for extension across the set M, we can show that f extends holomorphically to a neighborhood of every regular point z of Γ^0. Note that the set of points of Γ where $T_z(\Gamma^0)$ is contained in $T_z^c(\Gamma)$ is a subvariety of dimension at most $2n-3$.

According to [N], the singular part of a real-analytic set defined by a finite system of equations is contained in some real-analytic set of lower dimension. Thus we can use an inductive procedure to extend f to every point on Γ. Theorem 1 is proved. □

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