Analytic Continuation of Germs of Holomorphic Mappings between Real Hypersurfaces in $\mathbb{C}^n$

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1. Introduction

The classical theorem of Poincaré states that a biholomorphic map from an open piece of $\partial \mathbb{B}^2$ to $\partial \mathbb{B}^2$ extends to a global biholomorphism of the unit spheres. A general question that arises from this result can be stated as follows.

**Problem.** Let $\xi : \Gamma \to \Gamma'$ be a germ of a holomorphic map, at a point $\xi \in \Gamma$, between two smooth real-analytic connected hypersurfaces $\Gamma$ and $\Gamma'$ in $\mathbb{C}^n$. Under what conditions on $\Gamma$ and $\Gamma'$ does $f$ extend analytically along any path on $\Gamma'$?

We will usually identify the germ $f$ with one of its representatives—that is, a map $f : U \to \mathbb{C}^n$ defined in a small neighborhood $U \ni \xi$ and satisfying $f(U \cap \Gamma) \subset \Gamma'$.

Several authors have studied this problem. Alexander [A] generalized Poincaré’s theorem to higher dimensions in 1974. A year later, Pinchuk [P1] proved that any germ of a biholomorphic mapping from a connected strictly pseudoconvex real-analytic hypersurface $\Gamma \subset \mathbb{C}^n$ to $\partial \mathbb{B}^n$ extends analytically along any path on $\Gamma$ as a locally biholomorphic map with the inclusion $f(\Gamma) \subset \partial \mathbb{B}^n$.

Recall that a strictly pseudoconvex real-analytic hypersurface $\Gamma \subset \mathbb{C}^n$ is called *spherical* at a point $p \in \Gamma$ if there exists a germ of a biholomorphic map at $p$ from $\Gamma$ to $\partial \mathbb{B}^n$. It follows from [P1] that, if a connected strictly pseudoconvex hypersurface is spherical at a point, then it is spherical at any point. Pinchuk’s result clearly holds if, in the target space, $\partial \mathbb{B}^n$ is replaced by an arbitrary simply connected compact strictly pseudoconvex spherical hypersurface $\Gamma'$. Indeed, if $\Gamma'$ is spherical then a germ of a biholomorphic mapping $g : \Gamma' \to \partial \mathbb{B}^n$ extends along any path on $\Gamma'$. Since $\Gamma'$ is simply connected, $g$ extends to a global mapping from $\Gamma'$ to $\partial \mathbb{B}^n$. But then $\Gamma'$ is biholomorphically equivalent to $\partial \mathbb{B}^n$. If $\Gamma'$ is not simply connected, the result is no longer true. In fact, Burns and Shnider [BS] constructed some examples of compact and spherical but not simply connected hypersurfaces in $\mathbb{C}^n$. For any such hypersurface $\Gamma' \subset \mathbb{C}^n$, there exists a germ of a biholomorphic mapping $f : \partial \mathbb{B}^n \to \Gamma'$ that does not extend holomorphically along some paths on $\partial \mathbb{B}^n$.
In 1978, Pinchuk [P2] proved that, if $\Gamma$ is connected real-analytic strictly pseudoconvex and $\Gamma'$ is nonspherical compact and strictly pseudoconvex, then any germ of a biholomorphic map $f : \Gamma \to \Gamma'$ continues analytically along any path on $\Gamma$ as a locally biholomorphic mapping with the inclusion $f(\Gamma) \subset \Gamma'$. Note that $\Gamma'$ is not assumed to be simply connected.

If we do not require strict pseudoconvexity of $\Gamma'$ then $f$ may not extend holomorphically to certain points on $\Gamma$, as the following example shows.

**Example.** Let $\Gamma' = \{z' \in \mathbb{C}^2 : |z'_1|^2 + |z'_2|^4 = 1\}$. Then $f(z_1, z_2) = (z_1, \sqrt{z'_2})$ maps $\partial B^2$ to $\Gamma'$, but $f$ cannot be extended as a holomorphic mapping to a neighborhood of $(1, 0) \in \partial B^2$.

Nonetheless, it is possible to generalize Pinchuk’s results for non–strictly pseudoconvex hypersurfaces in the preimage. In this case, of course, we can extend the germ of a mapping only holomorphically, not locally biholomorphically. The goal of this paper is to present the following theorem.

**Theorem 1.1.** Let $\Gamma$ be a connected, essentially finite, smooth, real-analytic hypersurface in $\mathbb{C}^n$, and let $\xi \in \Gamma$. Let $\Gamma'$ be a compact strictly pseudoconvex real-algebraic hypersurface in $\mathbb{C}^n$. Let $f$ be a germ of a holomorphic mapping from $\Gamma$ to $\Gamma'$ defined at $\xi$. Then $f$ extends holomorphically along any path on $\Gamma$ with the inclusion $f(\Gamma) \subset \Gamma'$.

By a real-algebraic hypersurface we mean a hypersurface in $\mathbb{C}^n$ globally defined by $P(z, \bar{z}) = 0$, where $P(z, \bar{z})$ is a real polynomial. Precise definition of essential finiteness will be given in Section 2. We do not claim that $f$ extends to a global holomorphic mapping from $\Gamma$ to $\Gamma'$. Without further topological assumptions on $\Gamma$ it could happen that analytic continuation along different paths with the same endpoint $z^0$ will give different holomorphic mappings in a neighborhood of $z^0$. However, if $\Gamma$ is assumed to be simply connected, then (by the Monodromy theorem) $f$ does extend to a global mapping. Note that we do not require compactness or pseudoconvexity of $\Gamma$, and $f$ is not assumed a priori to be biholomorphic.

**Corollary 1.2.** Suppose that $\Gamma$ is an essentially finite real-analytic hypersurface in $\mathbb{C}^n$. If there exists a germ of a nonconstant holomorphic mapping from $\Gamma$ to a compact strictly pseudoconvex real-algebraic hypersurface $\Gamma' \subset \mathbb{C}^n$, then $\Gamma$ is pseudoconvex at any point. Moreover, the set of points on $\Gamma$ where the Levi form is degenerate has real dimension at most $2n - 3$.

**Corollary 1.3.** Suppose that $D$ is a bounded domain in $\mathbb{C}^n$ with a smooth real-analytic connected and simply connected boundary $\partial D$. Suppose $f$ is a non-constant holomorphic mapping defined in some open set $U$ such that $U \cap \partial D$ is not empty and connected and $f(U \cap \partial D) \subset \partial D'$, where $D'$ is a compact strictly pseudoconvex real-algebraic domain in $\mathbb{C}^n$ and $\partial D'$ is its boundary. Then $f$ extends to a proper holomorphic mapping from $D$ to $D'$.

The proof of Theorem 1.1 is based on the technique of Segre varieties and the reflection principle; the actual proof will be carried out in Section 6. We will first
show that, by choosing a point arbitrarily close to $\xi$ (denote it again by $\xi$) and a defining function of $\Gamma$ near $\xi$, we may assume $\Gamma$ to be strictly pseudoconvex in some neighborhood $U_{\xi} \ni \xi$. Let $\Gamma_{\xi}$ denote the set of strictly pseudoconvex points of $\Gamma$ and let $\Sigma \subset \Gamma$ denote the set of points where the Levi form of $\Gamma$ is degenerate. Then, using the results in [P1] and [P2], we can show that $f$ extends analytically along any path in the connected component of $\Gamma_{\xi}$ that contains $\xi$. The difficult part of the proof is showing that $f$ extends along $\Gamma$ past $\partial \Sigma$ in Sections 3, 4, and 5 we will build the necessary tools for such extension. Background material is presented in Section 2.

Under the additional assumptions that $\Gamma$ is real-algebraic and compact, the conclusion of Theorem 1.1 was obtained in [HJ]. The proof of this special case is easier, since by Webster’s theorem [W] the germ $f$ is automatically algebraic, which immediately gives its globalization.

2. Notation and Background Material

Let $\Gamma$ be a smooth real-analytic hypersurface in $\mathbb{C}^n$ with a defining function $\rho(z, \bar{z})$. For a fixed point $z^0 \in \Gamma$, choose the coordinate system so that $\frac{\partial \rho}{\partial z_j}(z^0) \neq 0$. Let $U = \{z : |z - z^0_j| < \sigma, j = 1, \ldots, n\}$ be a polydisk centered at $z^0$. Choose $\sigma$ sufficiently small that (a) $\rho(z, \bar{z})$ has a well-defined complexification $\rho(z, \bar{w})$ that is holomorphic in $z$ and antiholomorphic in $w$ for $(z, w) \in U \times U$ and (b) $\frac{\partial \rho}{\partial z_j}(z, \bar{w}) \neq 0$ for $(z, w) \in U \times U$.

**Definition 2.1.** Let $w \in U$. The analytic variety $Q_w := \{z \in U : \rho(z, \bar{w}) = 0\}$ is called the *Segre variety* of $w$ with respect to the hypersurface $\Gamma$.

Another analytic variety associated with the hypersurface $\Gamma$ and a point $w \in U$ is the set $I_w := \{\xi \in U : Q_\xi = Q_w\}$.

Let $z_j = x_j + iy_j$, $\bar{z}_j = (z_1, \ldots, z_{n-1})$, and $z = (\bar{z}, z_n)$. We next list some important properties of $Q_w$ and $I_w$ (see e.g. [DF2; DW] for proofs).

**Properties of Segre Varieties.**

(a) $z \in Q_w$ $\iff$ $w \in Q_z$.

(b) $z \in Q_z$ $\iff$ $z \in \Gamma$.

(c) $z \in I_z$.

(d) If $z \in \Gamma$ then $I_z$ is a complex subvariety of $\Gamma$.

(e) $I_w = \cap\{Q_z : z \in Q_w\}$.

(f) $Q_w$ is independent of the choice of the defining function.

(g) Let $z^0 \in \Gamma$ and $\frac{\partial \rho}{\partial z_j}(z^0) \neq 0$. Then there exists a pair of neighborhoods $U_1$ and $U_2 = \{U_2 \times U_2 \subset \mathbb{C}^{n-1} \times \mathbb{C} \} \ni z^0$ with $U_1 \subset U_2$ and such that, for any $w \in U_1$, $Q_w$ is a closed smooth complex-analytic hypersurface in $U_2$ that can be written as a graph of a holomorphic function,

$$Q_w = \{(z, z_n) \in (U_2 \times U_2) : z_n = h(z, \bar{w})\},$$

where $h(\cdot, \bar{w})$ is holomorphic in $U_2$. 

(h) The Segre map \( \lambda : w \to Q_w \) is locally one-to-one near strictly pseudoconvex points of \( \Gamma \).

Following [DP], we will call the neighborhoods \( U_1 \) and \( U_2 \) just defined a **standard pair** of neighborhoods of the point \( z_0 \).

Recall that a smooth real-analytic hypersurface \( \Gamma \subset \mathbb{C}^n \) is called **essentially finite** at \( z \in \Gamma \) if \( I_z = \{ w \in U_z : Q_w = \tilde{Q}_z \} = \{ z \} \), where \( U_z \) is a sufficiently small neighborhood of \( z \). The hypersurface \( \Gamma \) is said to be essentially finite if it is essentially finite at any point. Here are some useful properties of essentially finite hypersurfaces.

(i) Any real-analytic hypersurface of finite type is essentially finite. This follows from property (d) of Segre varieties.

(ii) If \( \Gamma \) contains a complex hypersurface passing through \( z \), then it is not essentially finite at \( z \).

(iii) If \( \Gamma \) is essentially finite at \( z \), then the Segre map \( \lambda : z \to Q_z \) is finite-to-one near \( z \), as \( \dim I_z = 0 \) for \( z \) sufficiently close to \( z_0 \).

Suppose that \( \Gamma \) and \( \Gamma' \) are real-analytic hypersurfaces in \( \mathbb{C}^n \) and that \( (U_1, U_2) \) and \( (U'_1, U'_2) \) are standard pairs of neighborhoods for \( z_0 \in \Gamma \) and \( z'_0 \in \Gamma' \), respectively. Let \( f : U_2 \to U'_2 \) be a holomorphic map, with \( f(U_1) \subset U'_1 \) and \( f(\Gamma \cap U_2) \subset (\Gamma' \cap U'_2) \). Then the following invariance property holds:

\[
\lambda(Q_w \cap U_2) \subset \lambda(Q_{f(w)} \cap U'_2) \quad \text{for all } w \in U_1.
\]

Throughout this paper we follow the convention of using the (right) prime to denote the objects in the target domain. For instance, \( Q_{w'} \) will mean the Segre variety of \( w' \) with respect to the hypersurface \( \Gamma' \).

Since every real hypersurface \( \Gamma \) in \( \mathbb{C}^n \) is orientable, there exists a neighborhood \( U \) containing \( \Gamma \) such that \( \Gamma \) divides \( U \) into two connected components, which we denote by \( U^- \) and \( U^+ \). Let

\[
\delta(z) = \begin{cases} 
\text{dist}(z, \Gamma) & \text{if } z \in U^+ \cup \Gamma, \\
-\text{dist}(z, \Gamma) & \text{if } z \in U^-.
\end{cases}
\]

If \( U \) is sufficiently small, then \( \delta \) is a defining function of \( \Gamma \) and \( \delta \in C^\infty(U) \). Any other defining function has the form \( \rho(z) = \alpha(z)\delta(z) \), where \( \alpha(z) \) is of constant sign in \( U \). If \( \alpha > 0 \) then \( \rho \) defines the same orientation on \( \Gamma \) as \( \delta \); if \( \alpha < 0 \), the orientation is opposite.

Suppose the orientation of \( \Gamma \) is fixed by \( \rho \). Then we say that \( \Gamma \) is pseudoconvex (resp. strictly pseudoconvex) at a point \( a \in \Gamma \) if the Levi form of \( \rho \) is nonnegative (resp. positive) on the complex tangent plane \( T_a^c(\Gamma) \) for all \( z \in \Gamma \) sufficiently close to \( a \). Clearly, this definition depends only on the orientation. We will assume that the orientations of the hypersurfaces are always suitably chosen. In particular, if \( \Gamma \) is a compact connected real hypersurface then it is the boundary of some bounded domain \( D \subset \subset \mathbb{C}^n \), and we assume that \( \rho < 0 \) in \( D \).

Finally, we will need the following definition.

**Definition 2.2.** A holomorphic correspondence between two domains \( D \) and \( D' \) in \( \mathbb{C}^n \) is a complex-analytic set \( A \subset D \times D' \) that satisfies: (i) \( A \) is of pure complex dimension \( n \); and (ii) the natural projection \( \pi : A \to D \) is proper.
We will also treat $A$ as the graph of a multivalued mapping defined by $\hat{f} := \pi' \circ \pi^{-1}$, where $\pi'$ is the natural projection of $A$ to $D'$.

3. Extension along Segre Varieties

Let $\Gamma \subset \mathbb{C}^n$ be a connected smooth real-analytic hypersurface with $a \in \Gamma$, and let $U_1$ and $U_2$ be a standard pair of neighborhoods of $a$.

Recall that a nonempty connected complex submanifold $\Lambda$ of a complex manifold $M$ is called an analytically constructible leaf if $\Lambda$ and $\Lambda \setminus \Lambda$ are closed complex analytic subsets of $M$. A locally finite union of analytically constructible leaves is called an analytically constructible set; for details, see [L]. In this section we will prove the following proposition.

**Proposition 3.1.** Let $f$ be a germ of a biholomorphic map from $\Gamma$ to a compact strictly pseudoconvex real-algebraic hypersurface $\Gamma' \subset \mathbb{C}^n$ defined at $a \in \Gamma$. Then there exist a neighborhood $V$ of $Q_w \cap U_1$ in $\mathbb{C}^n$ and an analytically constructible set $\Lambda \subset V$ with $\dim_C \Lambda \leq n - 1$ such that $f$ extends analytically along any path $\theta \subset V \setminus \Lambda$.

**Proof.** Without loss of generality we may assume that $a = 0$. Let $U$ be a neighborhood of the origin where $f$ is biholomorphic and $U = U_U \times U (\text{here}, \cdot \in U)$. We assume that $U$ is smaller than $U_1$. Choose $U$ and $V$ so that, for any $w$ in $V$, $Q_w \cap U$ is connected. Observe that if $V$ is small enough then $Q_w \cap U \neq \emptyset$ for any $w$ in $V$, as $w \in Q_0$ implies $0 \in Q_w$. Following the ideas in [DF2; DP], define

$$A = \{(w, w') \in V \times \mathbb{C}^n : f(Q_w \cap U) \subset Q_{w'}\}. \tag{3.1}$$

We would like to have $Q_w \cap U$ connected for any $w \in V$ to avoid ambiguity in the condition $f(Q_w \cap U) \subset Q_{w'}$, since different components of $Q_w \cap U$ could be mapped a priori to different Segre varieties. We will also use this in further constructions.

Let $P(\cdot, \cdot')$ be a defining polynomial of $\Gamma'$. Let $z \in U$ and $z' = f(z)$. The condition $f(Q_w \cap U) \subset Q_{w'}$ can be expressed as

$$P'(f(z), \tilde{w}') = 0 \quad \text{for any } z \in Q_w \cap U.$$

Therefore by property (g) of Segre varieties, (3.1) is equivalent to

$$A = \{(w, w') \in V \times \mathbb{C}^n : P'(f(z, \tilde{w}(z)), \tilde{w}') = 0 \quad \forall \tilde{z} \in U\}. \tag{3.2}$$

Thus (3.2) is defined by an infinite system of holomorphic equations in $\tilde{w}$ and $\tilde{w}'$. By the Noetherian property of the ring of holomorphic functions, we can choose finitely many points $\tilde{z}^1, \ldots, \tilde{z}^m$ so that (3.2) can be written as a finite system:

$$\sum_{|J| \leq d} a_J^*(w) w^{*J} = 0, \tag{3.3}$$

where $k = 1, \ldots, m$ and $d$ is the degree of $P'$ in $w'$. We define the closure of $A$ in $V \times \mathbb{P}^n$ in the following way. Let $\tilde{t} = (t_0, t_1, \ldots, t_n)$ be homogeneous coordinates in $\mathbb{P}^n$, and let $w_j' = t_j/t_0$ and $t = (t_1, \ldots, t_n)$. Then
According to [L, Thm. 1, p. 265],

\[ f(U) = \mathbb{V}(f(U)) \]

is a system of equations homogeneous in \( f \) that defines an analytic variety in \( V \times \mathbb{P}^n \).

Denote this variety again by \( A \). Clearly, its restriction to \( V \times (\mathbb{P}^n \setminus H_0) = V \times (\mathbb{C}^n) \) coincides with the set defined by (3.2). Here \( H_0 = \{ t_0 = 0 \} \) is the "hyperplane at infinity".

Let \( U' = f(U) \). Let us show that \( A \cap (U \times U') = \Gamma_f \). Suppose

\[ (w, w') \in A \cap (U \times U') \]

Then \( f(Q_w \cap U) \subset Q_{w'} \). Since \( f(Q_w \cap U) \subset Q_{f(w)} \) and \( \dim C f(Q_w \cap U) = n - 1 \), we have \( Q_{w'} = Q_{f(w)} \) and therefore \( w' \in I_{f(w)} \). Since \( \Gamma_f \) is strictly pseudoconvex, we may assume that \( U \) is chosen so small that the Segre map \( \lambda \) is one-to-one in \( U' = f(U) \) and

\[ \Gamma_f \cap f(U) = \{ f(w) \} \]

Thus, \( w' = f(w) \).

Consider the irreducible component of \( A \) that coincides with \( \Gamma_f \) in \( U \times U' \); for simplicity, denote this component again by \( A \). Let \( \pi : A \to V \) and \( \pi' : A \to \mathbb{P}^n \) be the natural projections. Notice that projection \( \pi \) is proper because \( \mathbb{P}^n \) is compact.

By Remmert’s theorem, the image of an analytic set under a proper map is an analytic set. Hence \( \pi(A) \) is analytic and, moreover, \( U \subseteq \pi(A) \). Therefore, \( \pi(A) = V \).

Let

\[ \Lambda_1 := \pi(\pi'^{-1}(H_0) \cap A) \]
\[ \Lambda_2 := \pi \{(w, w') \in A : \pi \text{ is not biholomorphic near } (w, w')\} \]
\[ \Lambda := \Lambda_1 \cup \Lambda_2 \]

For any path \( \theta : [0, 1] \to V \setminus \Lambda \) such that \( \theta(0) = a \in (U \setminus \Lambda) \), there exists a unique lifting \( \tilde{\theta} \subset \pi^{-1}(\theta) \subset A \) with the starting point \( (a, f(a)) \). This lifting defines the analytic continuation of \( f \) along \( \tilde{\theta} \). To finish the proposition we need only prove the following lemma.

**Lemma 3.2.** \( \Lambda \) is an analytically constructible set in \( V \), and \( \dim C \Lambda < n \).

**Proof.** \( \Lambda_1 \) is a proper analytic subset of \( V \) because \( \pi'^{-1}(H_0) \) is a proper analytic subset of \( A \) and \( \pi \) is proper. Thus, \( \dim C \Lambda_1 < n \).

The set \( \{(w, w') \in A : \pi \text{ is not biholomorphic near } (w, w')\} \) is the union of two sets: \( S := \{(w, w') \in \mathcal{A}^{\text{reg}} : \pi \text{ is not biholomorphic near } (w, w')\} \) and \( \mathcal{A}^{\text{sing}} \), where \( \mathcal{A}^{\text{reg}} \) and \( \mathcal{A}^{\text{sing}} \) are the regular and the singular parts (respectively) of the variety \( A \). For \( (w^0, w^0) \in \mathcal{A}^{\text{reg}} \), \( \pi \) is not biholomorphic in any neighborhood of \( (w^0, w^0) \) because \( A \) is not a complex manifold near \( (w^0, w^0) \), by the definition of \( \mathcal{A}^{\text{reg}} \), and hence cannot be biholomorphically equivalent to an open set in \( \mathbb{C}^n \). According to [L, Thm. 1, p. 265], \( S \) is an analytically constructible set in \( V \times \mathbb{P}^n \). Since \( \pi \) is proper on \( \hat{S} \), by the Chevallay–Remmert theorem \( \pi \) is an analytically
constructible set in $V$. Thus $\Lambda_2 = \pi(S) \cup \pi(A^\text{sing})$ is analytically constructible in $V$. Clearly, $\dim_C \Lambda_2 < n$.

This proves Proposition 3.1.

Note that any analytically constructible set $\Lambda$ of a complex dimension less than $n$ does not divide $V$. Therefore, for any $b \in V \setminus \Lambda$ there exists a path along which $f$ extends to some neighborhood of $b$.

For the proof of Theorem 1.1, we need to consider an additional set:

$$\Lambda_3 := \pi \left( \{(w, w') \in A : \dim_C(\pi^{-1}(w) \cap A) \geq 1\} \right).$$

Proposition 3.3. $\Lambda_3$ is an analytic set and $\dim_C \Lambda_3 \leq n - 2$.

Proof. $\{(w, w') \in A : \dim_C(\pi^{-1}(w) \cap A) \geq 1\}$ is an analytic subset of $A$ by Cartan–Remmert’s theorem (see e.g. [L]). Therefore, its image $\Lambda_3$ (under a proper mapping $\pi$) is also an analytic set. Suppose $\dim_C \Lambda_3 = n - 1$. Then there exists some locally analytic set $Z \subset \Lambda_3$ such that $\dim_C Z = n - 1$ and, for any $w$ in $Z$, $\dim_C(\pi^{-1}(w)) = 1$. By [L, Cor. 2, p. 266], $\dim_C(\pi^{-1}(Z)) = n$. This yields $A = \pi^{-1}(Z)$, since $A$ is irreducible. But $\pi(A) = V$ and so we obtain a contradiction, proving the claim.

4. Connecting Points on $\Gamma$ by Segre Varieties

Recall that a real submanifold $M \subset \mathbb{C}^n$ of real dimension $k \geq n$ is called generic if, for any $z \in M$, $\dim_C T_z(M) = k - n$ (here $T_z(M)$ is the complex tangent plane to $M$ at the point $z$). Following Trepreau and Tumanov, we call a hypersurface minimal if it does not contain germs of complex hypersurfaces. Although (for the proof of Theorem 1.1) we need only essential finiteness of $\Gamma$ in the following proposition, we would like to prove it in full generality.

Proposition 4.1. Let $\Gamma \subset \mathbb{C}^n$ be a minimal smooth real-analytic hypersurface. Let $M \subset \Gamma$ be a generic submanifold of dimension $2n - 2$, and let $p \in M$. Let $U$ be a neighborhood of $p$ such that $U \cap (\Gamma \setminus M)$ consists of two connected components, which we denote by $\Gamma^-$ and $\Gamma^+$. Then $Q_p \cap U$ contains an open subset $\omega$ such that, for any point $b \in \omega$, there exists a closed path $\gamma$ satisfying (i) $\gamma \subset (Q_b \cap \Gamma^+) \cup \{p\}$ and (ii) $\gamma \cap M = \{p\}$.

Proof. We will prove this proposition in two steps: for $n = 2$ and $n > 2$.

Step 1. Suppose that $n = 2$. Then $M$ is totally real. After an appropriate change of coordinates we may assume that $p = 0$ and, in a small neighborhood $U$ of the origin, $\Gamma$ is given by the defining function

$$\rho(z, \bar{z}) = z_2 + \bar{z}_2 + \sum_{k,l} \rho_{kl}(y_2)z^k_1\bar{z}^l_1$$

and $M$ is given by

$$\begin{cases} x_1 = 0, \\ \rho(z, \bar{z}) = 0. \end{cases}$$

Assume that $\Gamma^+ = \{z \in \Gamma \cap U : x_1 > 0\}$. 


To simplify computations, we introduce special ("normal") coordinates, which first appeared in [CM] as an intermediate step in their construction of normal forms of strictly pseudoconvex hypersurfaces. The form of the defining function that we use here is valid for arbitrary real-analytic hypersurfaces. It was shown in [CM] that—if we subject $0$ to a holomorphic transformation $\begin{align*}
abla \nabla & \begin{pmatrix} z_1 \\
abla z_2 + g(z_1, z_2) \end{pmatrix},
abla \nabla \end{align*}$

where $g(z_1, z_2)$ is some holomorphic function satisfying $g(0, z_2) = 0$—then the defining function of $0$ in new coordinates (to simplify the notation we omit the asterisks) takes the form

$$
\rho(z, \tilde{z}) = z_2 + \sum_{k,l > 0} \rho_{kl}(y_2) z_1^k \tilde{z}_1^l. 
$$

(4.3)

It is clear that $M$ in these coordinates is also given by (4.2).

In dimension 2, a finite-type condition is equivalent to minimality. Thus we may assume that $0$ is of finite type and that there exists $m = \min_{k,l > 0} \{ (k + l) : \rho_{kl}(0) \neq 0 \}, \quad m < \infty.$

Then $Q_0 = \{ \rho(z, 0) = 0 \} = \{ z_2 = 0 \}$. For $b \in Q_0$ where $b = (b, 0)$, we have

$$
Q_b = \left\{ z \in U : z_2 + \sum_{k,l > 0} \rho_{kl}(\frac{z_2}{\bar{z}_2}) z_1^k \bar{z}_1^l = 0 \right\}.
$$

By solving this equation for $z_2$ near the origin, we obtain

$$
z_2 = \eta z_1^q + \alpha(z_1),
$$

(4.4)

where $\eta$ depends holomorphically on $\bar{b}_1$, $\eta \neq \text{const}$; $\alpha(z_1) = o(z_1^q)$ with $q = \min\{k : \rho_{kl}(0) \neq 0 \}$ and $1 \leq q < m$.

The set $Q_b \cap \Gamma$ is given by the system

$$
\begin{align*}
\begin{cases}
\quad z_2 + \tilde{z}_2 + \sum_{k,l > 0} \rho_{kl}(z_2) z_1^k \tilde{z}_1^l = 0, \\
\quad z_2 = \eta z_1^q + \alpha(z_1).
\end{cases}
\end{align*}
$$

(4.5)

By plugging the second equation into the first, we obtain

$$
2 \text{Re}(\eta z_1^q + \alpha(z_1)) + \sum_{k,l > 0} \rho_{kl}(\text{Im}(\eta z_1^q + \alpha(z_1))) z_1^k \tilde{z}_1^l = 0.
$$

(4.6)

Choose $\omega \subset Q_0$ such that, for $b \in \omega$, $\text{Re} \eta \neq 0$ and $\text{Im} \eta \neq 0$.

If $q = 1$ then, by the implicit function theorem, equation (4.6) can be rewritten in the form $x_1 = cy_1 + \tilde{a}(y_1)$, where $\tilde{a}(y_1) = o(y_1)$ and $c \neq 0$. For $b \in \omega$, $\gamma = \Gamma \cap Q_b$ is then given by

$$
\begin{align*}
\begin{cases}
\quad x_1 = cy_1 + \tilde{a}(y_1), \\
\quad z_2 = \eta z_1 + \alpha(z_1).
\end{cases}
\end{align*}
$$

Hence $\gamma$ intersects both $\Gamma^+$ and $\Gamma^-$, and $\gamma \cap M = \{0\}$ for small $y_1$. 


If $q > 1$ then $q < m$ and (4.6) admits the form
\[ \Re((\eta z_1^n) + o(|z|^m)) = 0. \] (4.7)
Let $z_1 = re^{i\theta}$. Then (4.7) is equivalent to
\[ \Re \eta \cos q\theta - \Im \eta \sin q\theta + r\tilde{\alpha}(r, \theta) = 0, \]
where $\tilde{\alpha}$ is a real-analytic function in a neighborhood of the line $[0] \times \mathbb{R} \subset \mathbb{R}^2$. Let $\Psi(r, \theta) = \Re \eta \cos q\theta - \Im \eta \sin q\theta + r\tilde{\alpha}(r, \theta)$. Choose $\vartheta_0$ such that $(\Re \eta) \cos q\vartheta_0 - (\Im \eta) q\vartheta_0 = 0$ and $(\Re \eta) \sin q\vartheta_0 + (\Im \eta) \cos q\vartheta_0 \neq 0$. Then
\[ \frac{\partial \Psi}{\partial \vartheta}(0, \vartheta_0) \neq 0. \]
By the implicit function theorem, the equation $\Psi(r, \theta) = 0$ can be rewritten as $\vartheta = \beta(r)$ near the point $(0, \vartheta_0)$, where $\beta(r)$ is some analytic function near the origin.

Thus, $Q_b \cap \Gamma$ contains the curve given by
\[ (z_1, z_2) = \left( re^{i\beta(r)}, r^q e^{iq\beta(r)} + a(re^{i\beta(r)}) \right), \quad r \geq 0. \] (4.8)
Additionally, $\vartheta_0$ can be chosen to satisfy $\cos \vartheta_0 > 0$. Then $x_1 > 0$ as $r \to 0$, so the curve (4.8) is contained in $\Gamma^+$. These computations are valid for any point $b \in \omega$. Hence, the proposition is proved for $n = 2$.

Step 2. Suppose $n > 2$. Choose the coordinate system so that $p = 0$; similar to (4.3), the defining function of $\Gamma$ is given by
\[ \rho(z, \bar{z}) = 2x_n + \sum_{|K|, |L| > 0} \rho_{KL}(y)_0 z^K \bar{z}^L. \]
Then $Q_0 = \{ z : z_n = 0 \}$. Consider the family of 2-dim complex planes $L_b$ such that $b \in L_b$ for $b = (b, 0)$ and $\{ z_1 = \cdots = z_{n-1} = 0 \} \subset L_b$. Since $\Gamma$ is minimal, in any arbitrary neighborhood of the origin there exists an open set $\omega \subset \mathcal{Q}_b$ such that, for any $b \in \omega$, $\Gamma \cap L_b$ is a real surface of real dimension 3 that is of finite type in $\mathbb{C}^2 = L_b$. It is easy to see that $m(\Gamma \cap L_b, 0)$, the type of $\Gamma \cap L_b$ at point $0 \in \Gamma \cap L_b$, is an upper semicontinuous function of $b$. Therefore, we can find an open subset of $\omega$ with $m(\Gamma \cap L_b, 0) = \text{const}$. Denote this subset again by $\omega$.

If we repeat step 1 for $\Gamma \cap L_{b_0}$ then we can find a point $b_0 \in \omega$ such that $Q_{b_0}$ contains the path required by the proposition. Let $\eta = \eta(b)$ and $q = q(b)$ be the functions from (4.4) satisfying $\Re(\eta(b_0)) \neq 0$, $\Im(\eta(b_0)) \neq 0$, and $q(b_0) < m(\Gamma \cap L_{b_0}, 0)$. It is clear that if $b \in \omega$ is sufficiently close to $b_0$ then $\Re(\eta(b)) \neq 0$, $\Im(\eta(b)) \neq 0$, and $q(b) < m(\Gamma \cap L_b, 0)$. The last inequality holds because $q(b)$, the order of contact of $Q_b \cap L_b$ with $\Gamma \cap L_b$, is an upper semicontinuous function. Therefore, for all such $b$ we can apply the argument of step 1 for $\Gamma \cap L_b$. \[ \square \]

Remarks. 1. Analogously, it can be shown that the same set $\omega$ also satisfies Proposition 4.1 with $\Gamma^+$ replaced by $\Gamma^-$. 
2. It follows from the construction of the set $\omega$ that, for any $b \in \omega$ and any small neighborhood $U_0$ of the origin, the Segre variety $Q_b$ intersects both connected components of $U_0 \setminus \Gamma$. To see this, notice that if $q = 1$ in (4.4) then $Q_b$ intersects $\Gamma$ transversally. If $q > 1$, then from (4.3) and (4.4) we obtain

$$\rho(z, \bar{z})|_{Q_b \cap U_0} = \text{Re}(qz_1^q) + o(|z|^q), \quad (z_1, z_2) \in U_0,$$

and the assertion follows. Note that this implies $\dim Q_b \cap \Gamma = 2n - 3$ at the origin.

3. Proposition 4.1 is false if $M$ is a complex hypersurface (in this case, $\Gamma$ is not minimal). Indeed, let $p \in U$ and $\gamma : t \mapsto (t, t, t)$, $t \in \mathbb{C}$, be a path in $U_1$ such that $\gamma(0) = p$. Then $M \subset \Gamma$ and $\Gamma$ is not minimal. For any point $z \in Q_0 = M$, we have $Q_z = Q_0$ near the origin and the path $\gamma$ does not exist.

5. Extension across Generic Submanifolds

The next proposition is the key result for the proof of Theorem 1.1.

**Proposition 5.1.** Let $\Gamma$ be an essentially finite, smooth, real-analytic hypersurface, and let $\Gamma'$ be a compact, real-algebraic, strictly pseudoconvex hypersurface. Let $M$ be a generic submanifold of dimension $2n - 2$, and let $p \in M$. Let $U$ be a neighborhood of $p$. Denote by $\Gamma^-$ and $\Gamma^+$ the connected components of $U \cap (\Gamma \setminus M)$. Suppose that $f$ is a holomorphic mapping defined in a neighborhood of $\Gamma^+$, $f(\Gamma^+) \subset \Gamma'$, and suppose that $J_f$, the Jacobian of the mapping $f$, is not identically zero. Then $f$ extends holomorphically to a neighborhood of $p$.

**Proof.** Let $U_1, U_2$ be a standard pair of neighborhoods of $p$. Since $\Gamma$ is essentially finite, we may assume that the Segre map $\lambda$ is finite-to-one in $U_1$ and that $I_p \cap U_1 = \{ p \}$. By Proposition 4.1, there exists an open set $\omega \subset (Q_p \cap U_1)$ such that, for any point $b \in \omega$, $Q_b \cap \Gamma$ contains a path $\gamma$ in $\Gamma^+$ with the end point at $p$. The choice of $b \in \omega$, $\gamma$, and a point $a \in \gamma \cap U_1$ will form a triple, which we will denote by $(b, \gamma, a)$. We can choose $a$ so close to $p$ that, possibly after a small perturbation, $U_1, U_2$ will also be a standard pair of neighborhoods for $a$.

We can choose $(b, \gamma, a)$ such that $J_f(a) \neq 0$. Indeed, by Remark 2 following Proposition 4.1, $\dim(Q_z \cap \Gamma) = 2n - 3$. Since $\Gamma$ is essentially finite, there exists a neighborhood $U_b$ of the point $b$ such that

$$\#\{ z \in U_b : J_f|_{Q_z \cap \Gamma^+} = 0 \} < \infty.$$

Moving $b$ if necessary, we may assume that $J_f|_{Q_b \cap \Gamma^+}$ is not identically zero.

Let $U_a$ be a neighborhood of $a$, so that $f$ is biholomorphic in $U_a$. By Proposition 3.1, $f$ extends analytically along any path in $V \setminus \Lambda$, where $V$ is a neighborhood of $Q_a \cap U_1$ and $\Lambda \subset V$ is an analytically constructible set of complex dimension at most $n - 1$. There are two cases to be considered: either
(1) \( Q_p \cap V \) is not contained in \( \Lambda \); or
(2) \( Q_p \cap V \) is contained in \( \Lambda \).

**Case I.** In this situation we can slightly perturb the triple \((b, \gamma, a)\) so that \( b \in (\omega \cap V) \), \( b \not\in \Lambda \), and \( f \) is biholomorphic in \( U_a \). Notice that slight changes of \((b, \gamma, a)\) do not change \( \Lambda \). Since \( V \setminus \Lambda \) is connected, we can find a continuous path \( \theta \subset V \), with no self-intersections, connecting \( a \) and \( b \) and such that \( \theta \cap \Lambda = \emptyset \). Choose a simply connected neighborhood \( U_\theta \) of \( \theta \) so that \( U_\theta \subset V \) and \( U_\theta \cap \Lambda = \emptyset \). Then, by the Monodromy theorem, \( f|_{U_\theta} \) extends holomorphically to \( U_\theta \).

Denote by \( F \) the extension of \( f|_{U_\theta} \) to \( U_\theta \) obtained by Proposition 3.1. Choose a small neighborhood \( U_b \subset U_\theta \) of the point \( b \) such that, for any \( z \in U_b \) in some small neighborhood \( U_\gamma \) of \( \gamma \), \( Q_z \cap U_b \) is nonempty and connected. (Since \( \gamma \subset Q_b \), we have \( Q_z \ni b \) for all \( z \in \gamma \).) Thus, \( F \) is holomorphic in \( U_b \). Consider the set

\[
A^* = \{(w, w') \in U_\gamma \times \mathbb{C}^n : F(Q_w \cap U_b) \subset Q_w'\}.
\]

(5.1)

As in Proposition 3.1, \( A^* \) is a closed complex-analytic subset of \( U_\gamma \times \mathbb{C}^n \).

**Lemma 5.2.** There exists a small neighborhood \( \Omega \) of a such that

\[
A^* \cap (\Omega \times \Omega') = \Gamma_{f|_{A^*}},
\]

(5.2)

where \( \Omega' = f(\Omega) \).

**Proof.** Choose some small neighborhood \( \Omega \) containing \( a \) and a point \( z \in \Omega \). Let \( w \in Q_z \cap U_b \) be an arbitrary point, and let \( w' = F(w) \). It follows from the definition of \( F \) that \( f(Q_w \cap U_a) \subset Q_w' \) and \( z \in Q_w \). This implies that \( f(z) \in Q_{w'} \). But then \( F(w) \in Q_{f(z)}' \). Since \( w \in Q_z \) was arbitrary, we deduce that \( F(Q_z \cap U_b) \subset Q_{f(z)}' \). This means that \((z, z') \in A^* \) if \( z' \in f(I_{U_\gamma}) \); in particular, \( A^* \cap (\Omega \times \Omega') \neq \emptyset \), since \((z, f(z)) \in A^* \). If \( \Omega \) is chosen small enough, then \( \Omega' \cap f(I_{U_\gamma}) = (f(z)) \) and we conclude that \( A^* \cap (\Omega \times \Omega') = \Gamma_{f|_{A^*}} \).

Consider the irreducible component of \( A^* \) that coincides with \( \Gamma_f \) in \( \Omega \times \Omega' \). For simplicity, denote this component again by \( A^* \). Then \( \dim C A^* = n \). Let \( z^j \to p \) as \( j \to \infty \). By passing to a subsequence if necessary, we may assume that there exists \( p' \in \Gamma_f \) such that \( p' = \lim_{j \to \infty} f(z^j) \).

Since the graph of \( f|_{I_{U_\gamma} \cap \Gamma_f} \) is contained in \( A^* \), we have \((z^j, f(z^j)) \in A^* \) and thus \((p, p') \in A^* \). Let \( \pi : A^* \to U_\gamma \) and \( \pi' : A^* \to \mathbb{C}^n \) be the natural projections.

**Lemma 5.3.** There exist neighborhoods \( U_p \ni p \) and \( U_{p'} \ni p' \) such that \( \hat{f} := \pi' \circ \pi^{-1}(z) \) is a holomorphic mapping in \( U_p \) that extends \( f \). Here \( \pi^{-1} : U_p \to A^* \cap (U_p \times U_{p'}) \).

**Proof.** Choose \( U_{p'} \ni p' \) so small that the Segre map \( \lambda' \) is one-to-one in \( U_{p'} \), and let \( U_p \) be a small neighborhood of \( p \) such that \( U_p \subset \pi(\pi^{-1}(U_{p'})) \). Let us show that \( \pi : A^* \cap (U_p \times U_{p'}) \to U_p \) is one-to-one. If not, then we can find \( z \in U_p \) and \( z^{1'}, z^{2'} \in U_{p'} \) \((z^{1'} \neq z^{2'})\) such that

\[
(z, z^{1'}), (z, z^{2'}) \in A^* \cap (U_p \times U_{p'}).
\]

(5.3)
Then \( F(Q_z \cap U_b) \subset Q'_j \) for \( j = 1, 2 \). It follows from the definition of \( F \) that, for any \( w \in U_b \), we have

\[
f(Q_w \cap U_a) \subset Q'_{F(w)}.
\]

Since \( \lambda : z \to Q_z \) is finite-to-one in \( U_b \), there exist only finitely many points in \( U_b \) that have the same Segre variety as \( w \). Thus,

\[
\#\{F^{-1}(F(w))\} < \infty \quad \text{for any } w \in U_b.
\]

This shows that \( \dim C F(Q_z \cap U_b) = n - 1 \). But then, since \( \lambda' \) is one-to-one in \( U'_j \), there exists at most one point \( z' \in U'_j \) such that \( F(Q_z \cap U_b) \subset Q'_j \). This contradicts (5.3) and therefore \( \pi \) is one-to-one.

By [Ch, Sec. 3.3, Prop. 3], \( \pi : A^* \cap (U_p \times U_{p'}) \to U_p \) is a biholomorphic mapping and hence \( \tilde{\pi} := \pi' \circ \pi^{-1}(z) \) is holomorphic in \( U_p \) and extends \( f \). By analyticity, we also have \( \tilde{f}(\Gamma \cap U_p) \subset \Gamma' \).

Case 2. \( Q_p \subset \Lambda \). In this situation, \( f \) may not extend holomorphically to a neighborhood \( U_b \) of \( b \in Q_p \) because \( \omega \subset \Lambda \). However, one can show that \( f \) extends as a holomorphic correspondence. By such extension we mean a complex-analytic set of pure dimension \( n \), defined in \( U_0 \times \mathbb{C}^n \), with proper projection onto the first component that contains \( \Gamma_f|_{U_0} \).

**Lemma 5.4.** There exists a triple \((b^*, \gamma^*, a^*)\) such that \( b^* \in (\omega \cap V) \), \( \gamma^* \subset \Gamma^+ \cap Q_{b^*} \), \( a^* \subset \gamma^* \cap U_a \), and \( f|_{U_a} \) extends to a neighborhood of \( b^* \) as a holomorphic correspondence along some path \( \theta \subset V \), possibly after a biholomorphic change of variables in the target space.

**Proof.** We use the notation of Proposition 3.1. First we can exclude the case when \( Q_p \cap V \subset \Lambda_1 \). Indeed, after a biholomorphic change of coordinates in \( \mathbb{P}^n \), we may assume that (in new coordinates) \( \Gamma' \) remains compact in \( \mathbb{C}^n \subset \mathbb{P}^n \) and that \( \pi'(\pi^{-1}(Q_p)) \) is not entirely contained in \( H_0 \subset \mathbb{P}^n \). Thus, \( b^* \) can be chosen so that \( b^* \notin \Lambda_1 \). If \( Q_p \) is not contained in \( \Lambda_2 \), then \( b^* \) can be chosen so that \( b^* \notin \Lambda_2 \) and we are in the conditions of case 1. Otherwise, since (by Proposition 3.3) \( \dim A_3 < n - 1 \) and hence \( Q_p \cap V \) is not contained in \( \Lambda_3 \), we can find a point \( b^* \) in \((\omega \cap V) \setminus (A_1 \cup A_3) \), that is, \( b^* \in A_2 \setminus A_3 \). Furthermore, since \( \Lambda \) is analytically constructible, we may choose \( b^* \in A_{\text{reg}} \) close to \( \gamma \). Choose \( a^* \) so that \( a^* \in U_0 \cap \gamma^* \). Analogously to case 1, there exists a path \( \theta \subset V \) (without self-intersections) connecting \( a^* \) and \( b^* \), and \( \theta \cap \Lambda = \{b^*\} \). Let \( U_0 \) be a simply connected neighborhood of \( \theta \) such that \( U_0 \subset V \) and

\[
\Lambda \cap U_0 = (\Lambda \setminus (A_1 \cup A_3)) \cap U_0 = Q_p \cap U_0.
\]

Let \( A \) be the analytic set from Proposition 3.1, defined in \( V \times \mathbb{P}^n \). Consider the irreducible component of \( A \cap (U_0 \times \mathbb{P}^n) \) that contains \( \Gamma_f|_{U_0} \). Denote this component again as \( A \). Then \( A \) is the desired extension of \( f|_{U_a} \) as a correspondence because \( A \cap (U_0 \times H_0) = \emptyset \), since \( U_0 \cap (A_1 \cup A_3) = \emptyset \) and \( \pi : A \to U_0 \) is proper.

To simplify the notation we will drop the asterisks from \((b^*, \gamma^*, a^*)\). Let \( F : U_0 \to \mathbb{C}^n \) be a multivalued mapping corresponding to \( A \); that is,
Let $U_{\gamma}$ be a sufficiently small neighborhood of $\gamma$, where $\lambda: z \to Q$ is finite-to-one. Analogously, let $U'$ be a small neighborhood of $\Gamma'$, where the Segre map $\lambda'$ is finite-to-one. Choose a small neighborhood $U_b$ of $b$ ($U_b \subset U_0$) such that, for all $z \in U_\gamma$, $Q_w \cap U_b$ is nonempty and connected. Define

$$A^* = \{(w, w') \in (U_\gamma \setminus \{p\}) \times U': F(Q_w \cap U_b) \subset Q_{w'}\}.$$ 

**Lemma 5.5.** $A^*$ is a closed complex-analytic subset of $(U_\gamma \setminus \{p\}) \times U'$ that contains the graph of $f|_{U_0}$.

**Proof.** For any $(w, w') \in A^*$, the condition

$$F(Q_w \cap U_b) \subset Q_{w'}$$

can be expressed as follows. Take an open, simply connected set $\Omega \subset (U_b \setminus Q_p)$ such that $Q_w \cap \Omega \neq \emptyset$. Since $\Omega \cap A = \emptyset$, the branches of $F$ are correctly defined in $\Omega$. Then (5.4) is equivalent to $\tilde{f}(Q_w \cap \Omega) \subset Q_{w'}$ for all branches $\tilde{f}$ of $F$. Notice that such an open set $\Omega$ can be found for any $w \in U_\gamma \setminus \{p\}$. The inclusion $\tilde{f}(Q_w \cap \Omega) \subset Q_{w'}$ can be written as a system of holomorphic equations; therefore, $A^*$ is complex-analytic. $A^*$ is also closed because if $(w^j, w'^j) \to (w^0, w'^0)$ as $j \to \infty$ with $(w^j, w'^j) \in A^*$ and $(w^0, w'^0) \in (U_\gamma \setminus \{p\}) \times U'$, then $Q_{w^j} \to Q_{w^0}$ and $Q'_{w'^j} \to Q'_{w'^0}$ as $j \to \infty$. As a result, $F(Q_{w^0}) \subset Q'_{w'^0}$ and $(w^0, w'^0) \in A^*$. By repeating the argument in Lemma 5.2 we can show that $A^*$ contains the graph of $f|_{U_0}$.

Denote again by $A^*$ the irreducible component of $A^*$ that contains $\Gamma_f|_{U_0}$. Thus, $\dim C A^* = n$. Let

$$S = \{(p) \times U'\} \subset U_\gamma \times U'.$$

Then $S$ is a removable singularity for $A^*$; that is, $\overline{A^*}$ is a complex analytic variety in $U_\gamma \times U'$. Indeed, let $(z', z'^j) \in A^*$ and $(z', z'^j) \to (z^0, z'^0) \in S$ as $j \to \infty$. Then $z'^j \to p$, $F(Q_{z'^j}) \subset Q'_{z'^0}$, and so $F(Q_p) \subset Q'_{z'^0}$. It follows that

$$\overline{A^*} \cap S \subset \{(p) \times U': F(Q_p \cap U_b) \subset Q'_{z'^0}\}.$$ 

Because

$$\{(z' \in U': F(Q_p \cap U_b) \subset Q'_{z'}) \subset Q_{w'}\}$$

and $\dim C Q_{w'} = n - 1$, it follows that $\overline{A^*} \cap S$ has Hausdorff $2n$-measure zero. Since $S$ is a pluripolar set, Bishop’s theorem (see e.g. [Ch]) can be applied to conclude that $S$ is a removable singularity for $A^*$.

Denote $\overline{A^*}$ again by $A^*$. Note that $A^*$ is an analytic variety in $U_\gamma \times U'$. The rest of the proof is identical to case 1: we show that there exist neighborhoods $U_p \ni p$ and $U_{p'} \ni p'$ such that $\pi^{-1}$ is single-valued and, as a result, $f$ extends holomorphically to a neighborhood of $p$ if we set

$$f(z) = \pi' \circ \pi^{-1}(z).$$

This proves Proposition 5.1. □
6. Proof of the Main Result

Let $\rho(z, \bar{z})$ be a defining function of $\Gamma$ in a neighborhood of $\zeta \in \Gamma$. Let $U_{\zeta}$ be a small neighborhood of $\zeta$ and let $f : U_{\zeta} \to \mathbb{C}^n$ be a nonconstant holomorphic mapping such that $f(U_{\zeta} \cap \Gamma) \subset \Gamma'$, where $\Gamma'$ is a compact strictly pseudoconvex real algebraic hypersurface with the defining function $P'(z', \bar{z}')$.

**Proposition 6.1.** There exists a point $\xi \in U_{\zeta} \cap \Gamma$ such that all eigenvalues of the Levi form $H_{P}(\xi, v)$, $v \in T_{\xi}^{c}(\Gamma)$, are of the same sign.

**Proof.** Since $\Gamma'$ is strictly pseudoconvex, $f(U_{\zeta})$ is not contained in $\Gamma'$. Consider the set $f^{-1}(\Gamma')$. This is a real-analytic set in $U_{\zeta}$, and $\Gamma \subset f^{-1}(\Gamma')$. Since the set of regular points of a real-analytic set is dense, there exists a point $\xi \in U_{\zeta} \cap \Gamma$ such that $f^{-1}(\Gamma') \cap U_{\xi} = \Gamma \cap U_{\xi}$ for some small neighborhood $U_{\xi}$ of $\xi$. Moreover, since $\Gamma$ is essentially finite, $\xi$ and $U_{\xi}$ can be chosen such that $H_{P}(\xi, v)$ is nondegenerate on $T_{\xi}^{c}(\Gamma)$ for any $z \in U_{\xi}$. Replacing $\rho$ by $-\rho$ (if necessary), we obtain

$$f((z \in U_{\xi} : \rho(z, \bar{z}) < 0)) \subset \{P'(z', \bar{z}') < 0\}.$$  \hspace{1cm} (6.1)

Indeed, if there are two points $a, b \in \{z \in U_{\xi} : \rho(z, \bar{z}) < 0\}$ that are mapped by $f$ to different sides of $\Gamma'$, then we can connect $a$ and $b$ by a path $\gamma$ not intersecting $\Gamma$. But $f(\gamma)$ will clearly intersect $\Gamma'$, which contradicts the fact that $f^{-1}(\Gamma') = \Gamma$ in $U_{\xi}$.

Consider the function $P' \circ f(z)$, which is defined in $U_{\xi}$ and negative in $\{z \in U_{\xi} : \rho(z, \bar{z}) < 0\}$ because of (6.1). Since $\Gamma'$ is strictly pseudoconvex, we can choose $P'$ to be plurisubharmonic in a neighborhood of $\Gamma'$. Then $P' \circ f$ is also plurisubharmonic. By the Hopf lemma, $d(P' \circ f) \neq 0$ on $\Gamma \cap U_{\xi}$; we may thus consider $P' \circ f$ to be a local defining function of $\Gamma$ in $U_{\xi}$. By the invariance property of the Levi form, for any vector $v \in T_{\xi}^{c}(\Gamma)$ we have

$$H_{P' \circ f}(\xi, v) = H_{P'}(f(\xi), f_{*}v) \geq 0.$$  \hspace{1cm} (6.2)

Since the Levi form of $\Gamma$ is nondegenerate, $\Gamma$ is strictly pseudoconvex at $\xi$. \hfill $\Box$

Notice that it follows from (6.2) that $J_{f}(\xi) \neq 0$. By a suitable choice of the defining function of $\Gamma'$, by moving $\xi$ to a nearby point (if necessary), and by the choice of $U_{\zeta}$, we may therefore assume that (6.1) holds for $U_{\zeta}$. Then $\Gamma$ is strictly pseudoconvex in $U_{\zeta}$ and $f$ is biholomorphic. Recall that $\Gamma_{s}$ denotes the set of strictly pseudoconvex points of $\Gamma$. Let us show that $f$ extends along any path in a connected component of $\Gamma_{s}$ containing $\xi$.

Any compact strictly pseudoconvex algebraic hypersurface $\Gamma'$ is either nonspherical or spherical at any point. In the latter case, $\Gamma'$ is globally biholomorphically equivalent to a unit sphere, by [HJ]. Thus, we may assume that $\Gamma'$ is either nonspherical or is a unit sphere. By the results in [P1] and [P2], $f$ extends analytically along any path containing in a path-connected component of $\Gamma_{s}$.

As before, let $\Sigma$ denote the set of points of $\Gamma$ where the Levi form is degenerate. Note that $\Sigma$ is a real-analytic set. Let $M$ be the set of regular points $z \in \Sigma$ such that $T_{z}(\Sigma)$ is not a complex plane, and let $M^{*} = \Sigma \setminus M$. 


Lemma 6.2. The set $M^*$ does not divide $\Gamma$.

Proof. $M^*$ is the union of the set $\Sigma^{\text{reg}}$ and the set $M^c := \{z \in \Sigma^{\text{reg}} : T_z(\Sigma) = T_z(\Sigma)\}$. Observe that, locally, $M^c$ is a real-analytic set. Indeed, suppose $p \in M^c$ and that locally, near $p$, $\Sigma$ is given by

$$\{z \in \Gamma : \phi_j(z, \bar{z}) = 0, \ \ j = 1, \ldots, 2m, \ m < n\},$$

where $\phi_j(z, \bar{z})$ are smooth real-analytic functions and $d\phi_1 \wedge \cdots \wedge d\phi_{2m} \neq 0$. Then $M^c$ is given by the condition $\text{rank}(\partial \phi_j/\partial z_k) = m$ and thus $M^c$ is defined by a finite system of real-analytic equations. Also, $\dim_{\mathbb{R}} M^c \leq 2n - 3$, for if $\dim_{\mathbb{R}} M^c = 2n - 2$ at some regular point $z \in M^c$, then, by the Levi–Civita theorem, $M^c$ near $z$ is a complex hypersurface contained in $\Gamma$. This contradicts the essential finiteness of $\Gamma$. Since $\dim \Sigma^{\text{reg}} < \dim \Sigma$, we have

$$\dim_{\mathbb{R}} M^* \leq 2n - 3$$

and so $M^*$ does not divide $\Gamma$.

\[\square\]

Lemma 6.3. $f$ extends along any path in $\Gamma \setminus M^*$.

Proof. Let $\tau : [0, 1] \to \Gamma$ be an arbitrary simple path with $\tau(0) = \zeta$. Suppose there exists a number $t_0 \in (0, 1]$ such that $f$ extends along $\tau(t)$ for $0 \leq t < t_0$ but does not extend to a neighborhood of $p = \tau(t_0)$. Let $U$ be a small neighborhood of $p$ such that $U \cap \Sigma = U \cap M$. If $\dim_{\mathbb{R}} (M \cap U) < 2n - 2$, then we can find a generic submanifold $M$ such that $\dim_{\mathbb{R}} M = 2n - 2$ and $M \subset \tilde{M}$. We may therefore assume that $\dim_{\mathbb{R}} M = 2n - 2$. The set $M \cap U$ divides $\Gamma \cap U$ into two connected and simply connected components, which we denote by $\Gamma^-$ and $\Gamma^+$. Let $\tau_0 = \tau|_{(0, t_0]}$. Denote by $f_t$ the extension of $f$ along $\tau_t$. Then $f_t$ is holomorphic in $U_{t_0}$, a small neighborhood of $\tau_0$. There exist $\tau(t_1) \in (t_0 \cap U)$ and a neighborhood $U_1 \ni \tau(t_1)$ such that $f_t$ is holomorphic in $U_1$. Clearly, $U_1$ intersects at least one of the connected components of $U \setminus M$—say, $\Gamma^+$ for definiteness. It follows from Proposition 6.1 that the eigenvalues of $H_\mu(z, v)$ are of the same sign in $\Gamma^+$. Hence, by the choice of the defining function, $\Gamma^+$ can be assumed to be strictly pseudoconvex and $f_t|_{U_1}$ extends to $\Gamma^+$ as a locally biholomorphic mapping. Denote this extension by $\tilde{f}$. By Proposition 5.1, $\tilde{f}$ extends holomorphically to some neighborhood $U_\rho$ of $p$. If $t_0$ also intersects $\Gamma^-$, then analogously $f_t$ extends as a locally biholomorphic mapping to $\Gamma^-$. For $t < t_0$ and close to $t_0$, $\tau(t) \in U_t$, and $f_t$ coincides with the extension of $f$ to $U_\rho$. In view of Lemma 6.2, these considerations show that $f$ extends along any path in $\Gamma \setminus M^*$.

\[\square\]

The remaining case is $p \in M^*$. It follows from the theorems of Cartan (see e.g. [N, Prop. 15, p. 104]) and Narasimhan [N, Prop. 18, p. 105] that, if a real-analytic set is defined by a finite system of equations, then singular points of this analytic set are contained in some real-analytic set of lower dimension, which is also defined by a finite system of equations. Hence there exists a real-analytic set $\Sigma_1$ of real dimension at most $2n - 3$ such that $\Sigma^{\text{reg}} \subset \Sigma_1$. It follows that $\Sigma_1 \cup M^c$ is a locally real-analytic set of dimension at most $2n - 3$. For any $p \in (\Sigma_1 \cup M^c)^{\text{reg}}$ there
exists a small neighborhood $U_p$ such that $U_p \cap (\Sigma_1 \cup M)$ is contained in some generic submanifold of $\Gamma$, of dimension $2n - 2$, and we can show that $f$ extends holomorphically to a neighborhood of $p$ by repeating the argument in Lemma 6.3.

The singular part of $\Sigma_1 \cup M^\circ$ is now contained in an analytic set of dimension $2n - 4$. By induction on dimension, $f$ extends holomorphically to every point in $\Sigma$. Theorem 1.1 is proved.

**Proof of Corollary 1.2.** Since $\Gamma$ is essentially finite, the set of points where the Levi form of $\Gamma$ is degenerate has dimension at most $2n - 2$. Let $U$ be an open set, and let $f : U \to \mathbb{C}^n$ be a holomorphic mapping such that $f(U \cap \Gamma) \subset \Gamma'$. Then there is a point in $U \cap \Gamma$ where the Levi form is nondegenerate. By Proposition 6.1, $\Gamma \cap U$ contains strictly pseudoconvex points (up to orientation). If $\dim \Sigma < 2n - 2$, then $\Sigma$ does not divide $\Gamma$ and the latter is globally pseudoconvex. Suppose now that $\Sigma$ contains a component $M$ of dimension $2n - 2$ and that $p \in M$. By Theorem 1.1, $f$ extends holomorphically to a neighborhood $U_p \ni p$ along some path in $\Gamma$. Moreover, it follows from Proposition 6.1 that $J_f$ is not identically zero. Since $A := \{z \in U_p : J_f(z) = 0\}$ is an analytic variety and $\Gamma$ is essentially finite, $M$ is not contained in $A$; hence there is a point $\xi \in M \cap U_p$ such that $J_f(\xi) \neq 0$ and $f$ is biholomorphic near $\xi$. But this contradicts the fact that the Levi form of $\Gamma$ is degenerate at $\xi$. Thus, $\dim \Sigma < 2n - 2$.

**Proof of Corollary 1.3.** By [DF1], any compact domain with a smooth real-analytic boundary is of finite type; in particular, it is essentially finite. By Theorem 1.1, $f$ extends holomorphically along any path on $\partial D$ and, since $\partial D$ is simply connected, $f$ extends to a global mapping from $\partial D$ to $\partial D'$. By Hartog’s theorem, $f$ extends to a holomorphic mapping in $\bar{D}$. Since $f(\partial D) \subset \partial D'$, the extended mapping is proper.

**Acknowledgments.** The author would like to express his gratitude to Prof. S. Pinchuk for suggesting this problem and for very valuable discussions throughout the work on this paper.

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