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Analytic Continuation of Germs of Holomorphic Mappings between Real Hypersurfaces in $\mathbb{C}^n$

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1. Introduction

The classical theorem of Poincaré states that a biholomorphic map from an open piece of $\partial\mathbb{B}^2$ to $\partial\mathbb{B}^2$ extends to a global biholomorphism of the unit spheres. A general question that arises from this result can be stated as follows.

Problem. Let ${}_\zeta\mathbf{f}\colon \Gamma \to \Gamma'$ be a germ of a holomorphic map, at a point $\zeta \in \Gamma$, between two smooth real-analytic connected hypersurfaces Γ and Γ' in $\mathbb{C}^n$. Under what conditions on Γ and Γ' does f extend analytically along any path on Γ?

We will usually identify the germ ${}_\zeta\mathbf{f}$ with one of its representatives—that is, a map $f\colon U \to \mathbb{C}^n$ defined in a small neighborhood $U \ni \zeta$ and satisfying $f(U \cap \Gamma) \subset \Gamma'$.

Several authors have studied this problem. Alexander [A] generalized Poincaré's theorem to higher dimensions in 1974. A year later, Pinchuk [P1] proved that any germ of a biholomorphic mapping from a connected strictly pseudoconvex real-analytic hypersurface $\Gamma \subset \mathbb{C}^n$ to $\partial\mathbb{B}^n$ extends analytically along any path on Γ as a locally biholomorphic map with the inclusion $f(\Gamma) \subset \partial\mathbb{B}^n$.

Recall that a strictly pseudoconvex real-analytic hypersurface $\Gamma \subset \mathbb{C}^n$ is called *spherical* at a point $p \in \Gamma$ if there exists a germ of a biholomorphic map at p from Γ to $\partial\mathbb{B}^n$. It follows from [P1] that, if a connected strictly pseudoconvex hypersurface is spherical at a point, then it is spherical at any point. Pinchuk's result clearly holds if, in the target space, $\partial\mathbb{B}^n$ is replaced by an arbitrary simply connected compact strictly pseudoconvex spherical hypersurface Γ'. Indeed, if Γ' is spherical then a germ of a biholomorphic mapping $g\colon \Gamma' \to \partial\mathbb{B}^n$ extends along any path on Γ'. Since Γ' is simply connected, g extends to a global mapping from Γ' to $\partial\mathbb{B}^n$. But then Γ' is biholomorphically equivalent to $\partial\mathbb{B}^n$. If Γ' is not simply connected, the result is no longer true. In fact, Burns and Shnider [BS] constructed some examples of compact and spherical but not simply connected hypersurfaces in $\mathbb{C}^n$. For any such hypersurface $\Gamma' \subset \mathbb{C}^n$, there exists a germ of a biholomorphic mapping $f\colon \partial\mathbb{B}^n \to \Gamma'$ that does not extend holomorphically along some paths on $\partial\mathbb{B}^n$.

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In 1978, Pinchuk [P2] proved that, if Γ is connected real-analytic strictly pseudoconvex and $\Gamma' \subset \mathbb{C}^n$ is nonspherical compact and strictly pseudoconvex, then any germ of a biholomorphic map $f : \Gamma \to \Gamma'$ continues analytically along any path on Γ as a locally biholomorphic mapping with the inclusion $f(\Gamma) \subset \Gamma'$. Note that Γ' is not assumed to be simply connected.

If we do not require strict pseudoconvexity of Γ' then f may not extend holomorphically to certain points on Γ, as the following example shows.

EXAMPLE. Let $\Gamma' = \{z' \in \mathbb{C}^2 : |z_1'|^2 + |z_2'|^4 = 1\}$. Then $f(z_1, z_2) = (z_1, \sqrt{z_2})$ maps $\partial\mathbb{B}^2$ to Γ', but f can not be extended as a holomorphic mapping to a neighborhood of $(1, 0) \in \partial\mathbb{B}^2$.

Nonetheless, it is possible to generalize Pinchuk's results for non–strictly pseudoconvex hypersurfaces in the preimage. In this case, of course, we can extend the germ of a mapping only holomorphically, not locally biholomorphically. The goal of this paper is to present the following theorem.

THEOREM 1.1. *Let Γ be a connected, essentially finite, smooth, real-analytic hypersurface in $\mathbb{C}^n$, and let $\zeta \in \Gamma$. Let Γ' be a compact strictly pseudoconvex real-algebraic hypersurface in $\mathbb{C}^n$. Let f be a germ of a holomorphic mapping from Γ to Γ' defined at ζ. Then f extends holomorphically along any path on Γ with the inclusion $f(\Gamma) \subset \Gamma'$.*

By a real-algebraic hypersurface we mean a hypersurface in $\mathbb{C}^n$ globally defined by $P(z, \bar{z}) = 0$, where $P(z, \bar{z})$ is a real polynomial. Precise definition of essential finiteness will be given in Section 2. We do not claim that f extends to a global holomorphic mapping from Γ to Γ'. Without further topological assumptions on Γ it could happen that analytic continuation along different paths with the same endpoint z^0 will give different holomorphic mappings in a neighborhood of z^0. However, if Γ is assumed to be simply connected, then (by the Monodromy theorem) f does extend to a global mapping. Note that we do not require compactness or pseudoconvexity of Γ, and f is not assumed a priori to be biholomorphic.

COROLLARY 1.2. *Suppose that Γ is an essentially finite real-analytic hypersurface in $\mathbb{C}^n$. If there exists a germ of a nonconstant holomorphic mapping from Γ to a compact strictly pseudoconvex real-algebraic hypersurface $\Gamma' \subset \mathbb{C}^n$, then Γ is pseudoconvex at any point. Moreover, the set of points on Γ where the Levi form is degenerate has real dimension at most $2n - 3$.*

COROLLARY 1.3. *Suppose that D is a bounded domain in $\mathbb{C}^n$ with a smooth real-analytic connected and simply connected boundary ∂D. Suppose f is a nonconstant holomorphic mapping defined in some open set U such that $U \cap \partial D$ is not empty and connected and $f(U \cap \partial D) \subset \partial D'$, where D' is a compact strictly pseudoconvex real-algebraic domain in $\mathbb{C}^n$ and $\partial D'$ is its boundary. Then f extends to a proper holomorphic mapping from D to D'.*

The proof of Theorem 1.1 is based on the technique of Segre varieties and the reflection principle; the actual proof will be carried out in Section 6. We will first show that, by choosing a point arbitrarily close to ζ (denote it again by ζ) and a defining function of Γ near ζ, we may assume Γ to be strictly pseudoconvex in some neighborhood $U_\zeta \ni \zeta$. Let Γ_s denote the set of strictly pseudoconvex points of Γ and let $\Sigma \subset \Gamma$ denote the set of points where the Levi form of Γ is degenerate. Then, using the results in [P1] and [P2], we can show that f extends analytically along any path in the connected component of Γ_s that contains ζ. The difficult part of the proof is showing that f extends along Γ past Σ; in Sections 3, 4, and 5 we will build the necessary tools for such extension. Background material is presented in Section 2.

Under the additional assumptions that Γ is real-algebraic and compact, the conclusion of Theorem 1.1 was obtained in [HJ]. The proof of this special case is easier, since by Webster's theorem [W] the germ f is automatically algebraic, which immediately gives its globalization.

2. Notation and Background Material

Let Γ be a smooth real-analytic hypersurface in $\mathbb{C}^n$ with a defining function $\rho(z, \bar{z})$. For a fixed point $z^0 \in \Gamma$, choose the coordinate system so that $\frac{\partial \rho}{\partial z_n}(z^0) \neq 0$. Let $U = \{z : |z_j - z_j^0| < \sigma,\ j = 1, \dots, n\}$ be a polydisk centered at z^0. Choose σ sufficiently small that (a) $\rho(z, \bar{z})$ has a well-defined complexification $\rho(z, \bar{w})$ that is holomorphic in z and antiholomorphic in w for $(z, w) \in U \times U$ and (b) $\frac{\partial \rho}{\partial z_n}(z, \bar{w}) \neq 0$ for $(z, w) \in U \times U$.

Definition 2.1. Let $w \in U$. The analytic variety $Q_w := \{z \in U : \rho(z, \bar{w}) = 0\}$ is called the *Segre variety* of w with respect to the hypersurface Γ.

Another analytic variety associated with the hypersurface Γ and a point $w \in U$ is the set

$$I_w := \{\zeta \in U : Q_\zeta = Q_w\}.$$

Let $z_j = x_j + i y_j$, $'z = (z_1, \dots, z_{n-1})$, and $z = ('z, z_n)$. We next list some important properties of Q_w and I_w (see e.g. [DF2; DW] for proofs).

Properties of Segre Varieties.

(a) $z \in Q_w \iff w \in Q_z$.
(b) $z \in Q_z \iff z \in \Gamma$.
(c) $z \in I_z$.
(d) If $z \in \Gamma$ then I_z is a complex subvariety of Γ.
(e) $I_w = \bigcap \{Q_z : z \in Q_w\}$.
(f) Q_w is independent of the choice of the defining function.
(g) Let $z^0 \in \Gamma$ and $\frac{\partial \rho}{\partial z_n}(z^0) \neq 0$. Then there exists a pair of neighborhoods U_1 and $U_2 = {}'U_2 \times {}^nU_2 \subset \mathbb{C}^{n-1}_{'z} \times \mathbb{C}_{z_n}$ of z^0 with $U_1 \Subset U_2$ and such that, for any $w \in U_1$, Q_w is a closed smooth complex-analytic hypersurface in U_2 that can be written as a graph of a holomorphic function,

$$Q_w = \{('z, z_n) \in ({}'U_2 \times {}^nU_2) : z_n = h('z, \bar{w})\},$$

where $h(\cdot, \bar{w})$ is holomorphic in ${}'U_2$.

(h) The Segre map $\lambda\colon w \to Q_w$ is locally one-to-one near strictly pseudoconvex points of Γ.

Following [DP], we will call the neighborhoods U_1 and U_2 just defined a *standard pair* of neighborhoods of the point z_0.

Recall that a smooth real-analytic hypersurface $\Gamma \subset \mathbb{C}^n$ is called *essentially finite* at $z \in \Gamma$ if $I_z = \{w \in U_z : Q_w = Q_z\} = \{z\}$, where U_z is a sufficiently small neighborhood of z. The hypersurface Γ is said to be essentially finite if it is essentially finite at any point. Here are some useful properties of essentially finite hypersurfaces.

(i) Any real-analytic hypersurface of finite type is essentially finite. This follows from property (d) of Segre varieties.

(ii) If Γ contains a complex hypersurface passing through $z^0 \in \Gamma$, then it is not essentially finite at z^0.

(iii) If Γ is essentially finite at $z^0 \in \Gamma$, then the Segre map $\lambda\colon z \to Q_z$ is finite-to-one near z^0, as $\dim I_z = 0$ for z sufficiently close to z^0.

Suppose that Γ and Γ' are real-analytic hypersurfaces in $\mathbb{C}^n$ and that (U_1, U_2) and (U_1', U_2') are standard pairs of neighborhoods for $z_0 \in \Gamma$ and $z_0' \in \Gamma'$, respectively. Let $f : U_2 \to U_2'$ be a holomorphic map, with $f(U_1) \subset U_1'$ and $f(\Gamma \cap U_2) \subset (\Gamma' \cap U_2')$. Then the following invariance property holds:

$$f(Q_w \cap U_2) \subset Q'_{f(w)} \cap U_2' \quad \text{for all } w \in U_1.$$

Throughout this paper we follow the convention of using the (right) prime to denote the objects in the target domain. For instance, $Q'_{w'}$ will mean the Segre variety of w' with respect to the hypersurface Γ'.

Since every real hypersurface Γ in $\mathbb{C}^n$ is orientable, there exists a neighborhood U containing Γ such that Γ divides U into two connected components, which we denote by U^- and U^+. Let

$$\delta(z) = \begin{cases} \operatorname{dist}(z, \Gamma) & \text{if } z \in U^+ \cup \Gamma, \\ -\operatorname{dist}(z, \Gamma) & \text{if } z \in U^-. \end{cases}$$

If U is sufficiently small, then δ is a defining function of Γ and $\delta \in C^\omega(U)$. Any other defining function has the form $\rho(z) = \alpha(z)\delta(z)$, where $\alpha(z)$ is of constant sign in U. If $\alpha > 0$ then ρ defines the same orientation on Γ as δ; if $\alpha < 0$, the orientation is opposite.

Suppose the orientation of Γ is fixed by ρ. Then we say that Γ is pseudoconvex (resp. strictly pseudoconvex) at a point $a \in \Gamma$ if the Levi form of ρ is nonnegative (resp. positive) on the complex tangent plane $T_z^c(\Gamma)$ for all $z \in \Gamma$ sufficiently close to a. Clearly, this definition depends only on the orientation. We will assume that the orientations of the hypersurfaces are always suitably chosen. In particular, if Γ is a compact connected real hypersurface then it is the boundary of some bounded domain $D \subset\subset \mathbb{C}^n$, and we assume that $\rho < 0$ in D.

Finally, we will need the following definition.

DEFINITION 2.2. A holomorphic correspondence between two domains D and D' in $\mathbb{C}^n$ is a complex-analytic set $A \subset D \times D'$ that satisfies: (i) A is of pure complex dimension n; and (ii) the natural projection $\pi : A \to D$ is proper.

We will also treat A as the graph of a multivalued mapping defined by $\hat{f} := \pi' \circ \pi^{-1}$, where π' is the natural projection of A to D'.

3. Extension along Segre Varieties

Let $\Gamma \subset \mathbb{C}^n$ be a connected smooth real-analytic hypersurface with $a \in \Gamma$, and let U_1 and U_2 be a standard pair of neighborhoods of a.

Recall that a nonempty connected complex submanifold Λ of a complex manifold M is called an *analytically constructible leaf* if $\bar{\Lambda}$ and $\bar{\Lambda} \setminus \Lambda$ are closed complex analytic subsets of M. A locally finite union of analytically constructible leaves is called an analytically constructible set; for details, see [L]. In this section we will prove the following proposition.

PROPOSITION 3.1. *Let f be a germ of a biholomorphic map from Γ to a compact strictly pseudoconvex real-algebraic hypersurface $\Gamma' \subset \mathbb{C}^n$ defined at $a \in \Gamma$. Then there exist a neighborhood V of $Q_a \cap U_1$ in $\mathbb{C}^n$ and an analytically constructible set $\Lambda \subset V$ with* $\dim_C \Lambda \le n-1$ *such that f extends analytically along any path $\theta \subset V \setminus \Lambda$.*

Proof. Without loss of generality we may assume that $a = 0$. Let U be a neighborhood of the origin where f is biholomorphic and $U = {}'U \times {}^nU$ (here, ${}'z \in {}'U$). We assume that U is smaller than U_1. Choose U and V so that, for any w in V, $Q_w \cap U$ is connected. Observe that if V is small enough then $Q_w \cap U \neq \emptyset$ for any w in V, as $w \in Q_0$ implies $0 \in Q_w$. Following the ideas in [DF2; DP], define

$$A = \{(w, w') \in V \times \mathbb{C}^n : f(Q_w \cap U) \subset Q'_{w'}\}. \tag{3.1}$$

We would like to have $Q_w \cap U$ connected for any $w \in V$ to avoid ambiguity in the condition $f(Q_w \cap U) \subset Q'_{w'}$, since different components of $Q_w \cap U$ could be mapped a priori to different Segre varieties. We will also use this in further constructions.

Let $P'(z', \bar{z}')$ be a defining polynomial of Γ'. Let $z \in U$ and $z' = f(z)$. The condition $f(Q_w \cap U) \subset Q'_{w'}$ can be expressed as

$$P'(f(z), \bar{w}') = 0 \quad \text{for any } \; z \in Q_w \cap U.$$

Therefore by property (g) of Segre varieties, (3.1) is equivalent to

$$A = \{(w, w') \in V \times \mathbb{C}^n : P'(f({}'z, h({}'z, \bar{w})), \bar{w}') = 0 \ \forall {}'z \in {}'U)\}. \tag{3.2}$$

Thus (3.2) is defined by an infinite system of holomorphic equations in $\bar{w}$ and $\bar{w}'$ that are polynomials in $\bar{w}'$. By the Noetherian property of the ring of holomorphic functions, we can choose finitely many points ${}'z^1, \dots, {}'z^m$ so that (3.2) can be written as a finite system:

$$\sum_{|J| \le d} \alpha_J^k(w) w'^J = 0, \tag{3.3}$$

where $k = 1, \dots, m$ and d is the degree of P' in w'. We define the closure of A in $V \times \mathbb{P}^n$ in the following way. Let $\tilde{t} = (t_0, t_1, \dots, t_n)$ be homogeneous coordinates in $\mathbb{P}^n$, and let $w'_j = t_j/t_0$ and $t = (t_1, \dots, t_n)$. Then

$$t_0^d \sum_{|J|\le d} \alpha_J^k(w)\left(\frac{t}{t_0}\right)^J = 0, \quad k = 1, \dots, m, \tag{3.4}$$

is a system of equations homogeneous in $\tilde{t}$ that defines an analytic variety in $V \times \mathbb{P}^n$. Denote this variety again by A. Clearly, its restriction to $V \times (\mathbb{P}^n \setminus H_0) = V \times (\mathbb{C}^n)$ coincides with the set defined by (3.2). Here $H_0 = \{t_0 = 0\}$ is the "the hyperplane at infinity".

Let $U' = f(U)$. Let us show that $A \cap (U \times U') = \Gamma_f$. Suppose

$$(w, w') \in A \cap (U \times U').$$

Then $f(Q_w \cap U) \subset Q'_{w'}$. Since $f(Q_w \cap U) \subset Q'_{f(w)}$ and $\dim_C f(Q_w \cap U) = n-1$, we have $Q'_{w'} = Q'_{f(w)}$ and therefore $w' \in I_{f(w)}$. Since Γ' is strictly pseudoconvex, we may assume that U is chosen so small that the Segre map λ' is one-to-one in $U' = f(U)$ and

$$I_{f(w)} \cap f(U) = \{f(w)\}.$$

Thus, $w' = f(w)$.

Consider the irreducible component of A that coincides with Γ_f in $U \times U'$; for simplicity, denote this component again by A. Then $\dim_C A = n$. Let $\pi : A \to V$ and $\pi' : A \to \mathbb{P}^n$ be the natural projections. Notice that projection π is proper because $\mathbb{P}^n$ is compact.

By Remmert's theorem, the image of an analytic set under a proper map is an analytic set. Hence $\pi(A)$ is analytic and, moreover, $U \subset \pi(A)$. Therefore, $\pi(A) = V$. Let

$$\begin{aligned}
\Lambda_1 &:= \pi(\pi'^{-1}(H_0) \cap A), \\
\Lambda_2 &:= \pi\{(w, w') \in A : \pi \text{ is not biholomorphic near } (w, w')\}, \\
\Lambda &:= \Lambda_1 \cup \Lambda_2.
\end{aligned}$$

For any path $\theta : [0, 1] \to V \setminus \Lambda$ such that $\theta(0) = a \in (U \setminus \Lambda)$, there exists a unique lifting $\hat{\theta} \subset \pi^{-1}(\theta) \subset A$ with the starting point $(a, f(a))$. This lifting defines the analytic continuation of f along θ. To finish the proposition we need only prove the following lemma.

LEMMA 3.2. *Λ is an analytically constructible set in V, and* $\dim_C \Lambda < n$.

Proof. Λ_1 is a proper analytic subset of V because $\pi'^{-1}(H_0)$ is a proper analytic subset of A and π is proper. Thus, $\dim_C \Lambda_1 < n$.

The set $\{(w, w') \in A : \pi$ is not biholomorphic near $(w, w')\}$ is the union of two sets: $S := \{(w, w') \in A^{\text{reg}} : \pi$ is not biholomorphic near $(w, w')\}$ and A^{sng}, where A^{reg} and A^{sng} are the regular and the singular parts (respectively) of the variety A. For $(w^0, w'^0) \in A^{\text{sng}}$, π is not biholomorphic in any neighborhood of (w^0, w'^0) because A is not a complex manifold near (w^0, w'^0), by the definition of A^{sng}, and hence cannot be biholomorphically equivalent to an open set in $\mathbb{C}^n$. According to [L, Thm. 1, p. 265], S is an analytically constructible set in $V \times \mathbb{P}^n$. Since π is proper on $\bar{S}$, by the Chevallay–Remmert theorem π is an analytically constructible set in V. Thus $\Lambda_2 = \pi(S) \cup \pi(A^{\text{sng}})$ is analytically constructible in V. Clearly, $\dim_C \Lambda_2 < n$.

This proves Proposition 3.1. □

Note that any analytically constructible set Λ of a complex dimension less than n does not divide V. Therefore, for any $b \in V \setminus \Lambda$ there exists a path along which f extends to some neighborhood of b.

For the proof of Theorem 1.1, we need to consider an additional set:

$$\Lambda_3 := \pi\big(\{(w, w') \in A : \dim_C(\pi^{-1}(w) \cap A) \geq 1\}\big). \tag{3.5}$$

PROPOSITION 3.3. *Λ_3 is an analytic set and* $\dim_C \Lambda_3 \leq n - 2$.

Proof. $\{(w, w') \in A : \dim_C(\pi^{-1}(w) \cap A) \geq 1\}$ is an analytic subset of A by Cartan–Remmert's theorem (see e.g. [L]). Therefore, its image Λ_3 (under a proper mapping π) is also an analytic set. Suppose $\dim_C \Lambda_3 = n - 1$. Then there exists some locally analytic set $Z \subset \Lambda_3$ such that $\dim_C Z = n - 1$ and, for any w in Z, $\dim_C(\pi^{-1}(w)) = 1$. By [L, Cor. 2, p. 266], $\dim_C(\pi^{-1}(Z)) = n$. This yields $A = \pi^{-1}(Z)$, since A is irreducible. But $\pi(A) = V$ and so we obtain a contradiction, proving the claim. □

4. Connecting Points on Γ by Segre Varieties

Recall that a real submanifold $M \subset \mathbb{C}^n$ of real dimension $k \geq n$ is called *generic* if, for any $z \in M$, $\dim_C T_z^c(M) = k - n$ (here $T_z^c(M)$ is the complex tangent plane to M at the point z). Following Trepreau and Tumanov, we call a hypersurface *minimal* if it does not contain germs of complex hypersurfaces. Although (for the proof of Theorem 1.1) we need only essential finiteness of Γ in the following proposition, we would like to prove it in full generality.

PROPOSITION 4.1. *Let $\Gamma \subset \mathbb{C}^n$ be a minimal smooth real-analytic hypersurface. Let $M \subset \Gamma$ be a generic submanifold of dimension $2n - 2$, and let $p \in M$. Let U be a neighborhood of p such that $U \cap (\Gamma \setminus M)$ consists of two connected components, which we denote by Γ^- and Γ^+. Then $Q_p \cap U$ contains an open subset ω such that, for any point $b \in \omega$, there exists a closed path γ satisfying* (i) $\gamma \subset (Q_b \cap \Gamma^+) \cup \{p\}$ *and* (ii) $\gamma \cap M = \{p\}$.

Proof. We will prove this proposition in two steps: for $n = 2$ and $n > 2$.

Step 1. Suppose that $n = 2$. Then M is totally real. After an appropriate change of coordinates we may assume that $p = 0$ and, in a small neighborhood U of the origin, Γ is given by the defining function

$$\rho(z, \bar{z}) = z_2 + \bar{z}_2 + \sum_{k,l} \rho_{kl}(y_2) z_1^k \bar{z}_1^l \tag{4.1}$$

and M is given by

$$\begin{cases} x_1 = 0, \\ \rho(z, \bar{z}) = 0. \end{cases} \tag{4.2}$$

Assume that $\Gamma^+ = \{z \in \Gamma \cap U : x_1 > 0\}$.

To simplify computations, we introduce special ("normal") coordinates, which first appeared in [CM] as an intermediate step in their construction of normal forms of strictly pseudoconvex hypersurfaces. The form of the defining function that we use here is valid for arbitrary real-analytic hypersurfaces. It was shown in [CM] that—if we subject Γ to a holomorphic transformation

$$\begin{cases} z_1^* = z_1, \\ z_2^* = z_2 + g(z_1, z_2), \end{cases}$$

where $g(z_1, z_2)$ is some holomorphic function satisfying $g(0, z_2) \equiv 0$—then the defining function of Γ in new coordinates (to simplify the notation we omit the asterisks) takes the form

$$\rho(z, \bar z) = z_2 + \bar z_2 + \sum_{k,l>0} \rho_{kl}(y_2) z_1^k \bar z_1^l. \tag{4.3}$$

It is clear that M in these coordinates is also given by (4.2).

In dimension 2, a finite-type condition is equivalent to minimality. Thus we may assume that Γ is of finite type and that there exists

$$m = \min_{k,l>0}\{(k+l) : \rho_{kl}(0) \neq 0\}, \quad m < \infty.$$

Then $Q_0 = \{\rho(z, 0) = 0\} = \{z_2 = 0\}$. For $b \in Q_0$ where $b = (b_1, 0)$, we have

$$Q_b = \left\{ z \in U : z_2 + \sum_{k,l>0} \rho_{kl}\left(\frac{z_2}{2i}\right) z_1^k \bar b_1^l = 0 \right\}.$$

By solving this equation for z_2 near the origin, we obtain

$$z_2 = \eta z_1^q + \alpha(z_1), \tag{4.4}$$

where η depends holomorphically on $\bar b_1$, $\eta \neq \text{const}$; $\alpha(z_1) = o(z_1^q)$ with $q = \min\{k : \rho_{kl}(0) \neq 0\}$ and $1 \le q < m$.

The set $Q_b \cap \Gamma$ is given by the system

$$\begin{cases} z_2 + \bar z_2 + \sum_{k,l>0} \rho_{kl}(y_2) z_1^k \bar z_1^l = 0, \\ z_2 = \eta z_1^q + \alpha(z_1). \end{cases} \tag{4.5}$$

By plugging the second equation into the first, we obtain

$$2\,\mathrm{Re}(\eta z_1^q + \alpha(z_1)) + \sum_{k,l>0} \rho_{kl}(\mathrm{Im}(\eta z_1^q + \alpha(z_1))) z_1^k \bar z_1^l = 0. \tag{4.6}$$

Choose $\omega \subset Q_0$ such that, for $b \in \omega$, $\mathrm{Re}\,\eta \neq 0$ and $\mathrm{Im}\,\eta \neq 0$.

If $q = 1$ then, by the implicit function theorem, equation (4.6) can be rewritten in the form $x_1 = c y_1 + \tilde\alpha(y_1)$, where $\tilde\alpha(y_1) = o(y_1)$ and $c \neq 0$. For $b \in \omega$, $\gamma = \Gamma \cap Q_b$ is then given by

$$\begin{cases} x_1 = c y_1 + \tilde\alpha(y_1), \\ z_2 = \eta z_1 + \alpha(z_1). \end{cases}$$

Hence γ intersects both Γ^+ and Γ^-, and $\gamma \cap M = \{0\}$ for small y_1.

If $q > 1$ then $q < m$ and (4.6) admits the form

$$\operatorname{Re}(\eta z_1^q) + o(|z_1|^q) = 0. \tag{4.7}$$

Let $z_1 = re^{i\vartheta}$. Then (4.7) is equivalent to

$$\operatorname{Re}\eta \cos q\vartheta - \operatorname{Im}\eta \sin q\vartheta + r\tilde{\alpha}(r, \vartheta) = 0,$$

where $\tilde{\alpha}$ is a real-analytic function in a neighborhood of the line $\{0\} \times \mathbb{R} \subset \mathbb{R}^2_{r,\vartheta}$. Let $\Psi(r, \vartheta) = \operatorname{Re}\eta \cos q\vartheta - \operatorname{Im}\eta \sin q\vartheta + r\tilde{\alpha}(r, \vartheta)$. Choose ϑ_0 such that $(\operatorname{Re}\eta)\cos q\vartheta_0 - (\operatorname{Im}\eta) q\vartheta_0 = 0$ and $(\operatorname{Re}\eta)\sin q\vartheta_0 + (\operatorname{Im}\eta)\cos q\vartheta_0 \neq 0$. Then

$$\frac{\partial\Psi}{\partial\vartheta}(0, \vartheta_0) \neq 0.$$

By the implicit function theorem, the equation $\Psi(r, \vartheta) = 0$ can be rewritten as $\vartheta = \beta(r)$ near the point $(0, \vartheta_0)$, where $\beta(r)$ is some analytic function near the origin.

Thus, $Q_b \cap \Gamma$ contains the curve given by

$$(z_1, z_2) = \left(re^{i\beta(r)}, r^q e^{iq\beta(r)} + \alpha(re^{i\beta(r)})\right), \quad r \geq 0. \tag{4.8}$$

Additionally, ϑ_0 can be chosen to satisfy $\cos\vartheta_0 > 0$. Then $x_1 > 0$ as $r \to 0$, so the curve (4.8) is contained in Γ^+.

These computations are valid for any point $b \in \omega$. Hence, the proposition is proved for $n = 2$.

Step 2. Suppose $n > 2$. Choose the coordinate system so that $p = 0$; similar to (4.3), the defining function of Γ is given by

$$\rho(z, \bar{z}) = 2x_n + \sum_{|K|,|L|>0} \rho_{KL}(y_n)' z^{K\prime} \bar{z}^L.$$

Then $Q_0 = \{z : z_n = 0\}$. Consider the family of 2-dim complex planes L_b such that $b \in L_b$ for $b = ('b, 0)$ and $\{z_1 = \cdots = z_{n-1} = 0\} \subset L_b$. Since Γ is minimal, in any arbitrary neighborhood of the origin there exists an open set $\omega \subset Q_0$ such that, for any $b \in \omega$, $\Gamma \cap L_b$ is a real surface of real dimension 3 that is of finite type in $\mathbb{C}^2 = L_b$. It is easy to see that $m(\Gamma \cap L_b, 0)$, the type of $\Gamma \cap L_b$ at point $0 \in \Gamma \cap L_b$, is an upper semicontinuous function of b. Therefore, we can find an open subset of ω with $m(\Gamma \cap L_b, 0) = \text{const}$. Denote this subset again by ω.

If we repeat step 1 for $\Gamma \cap L_{b_0}$ then we can find a point $b_0 \in \omega$ such that Q_{b_0} contains the path required by the proposition. Let $\eta = \eta(b)$ and $q = q(b)$ be the functions from (4.4) satisfying $\operatorname{Re}(\eta(b_0)) \neq 0$, $\operatorname{Im}(\eta(b_0)) \neq 0$, and $q(b_0) < m(\Gamma \cap L_{b_0}, 0)$. It is clear that if $b \in \omega$ is sufficiently close to b_0 then $\operatorname{Re}(\eta(b)) \neq 0$, $\operatorname{Im}(\eta(b)) \neq 0$, and $q(b) < m(\Gamma \cap L_b, 0)$. The last inequality holds because $q(b)$, the order of contact of $Q_b \cap L_b$ with $\Gamma \cap L_b$, is an upper semicontinuous function. Therefore, for all such b we can apply the argument of step 1 for $\Gamma \cap L_b$. $\square$

REMARKS. 1. Analogously, it can be shown that the same set ω also satisfies Proposition 4.1 with Γ^+ replaced by Γ^-.

2. It follows from the construction of the set ω that, for any $b \in \omega$ and any small neighborhood U_0 of the origin, the Segre variety Q_b intersects both connected components of $U_0 \setminus \Gamma$. To see this, notice that if $q = 1$ in (4.4) then Q_b intersects Γ transversally. If $q > 1$, then from (4.3) and (4.4) we obtain

$$\rho(z, \bar{z})\big|_{Q_b \cap U_0} = \mathrm{Re}(\eta z_1^q) + o(|z_1|^q), \quad (z_1, z_2) \in U_0,$$

and the assertion follows. Note that this implies $\dim Q_b \cap \Gamma = 2n - 3$ at the origin.

3. Proposition 4.1 is false if M is a complex hypersurface (in this case, Γ is not minimal). Indeed, let $p = 0$ and

$$\Gamma = \{z \in \mathbb{C}^n : x_n + y_n \phi('z, '\bar{z}) = 0\},$$

where ϕ is real-analytic and $\phi(0) = 0$. Let $M = \{z : z_n = 0\}$. Then $M \subset \Gamma$ and Γ is not minimal. For any point $z \in Q_0 = M$, we have $Q_z = Q_0$ near the origin and the path γ does not exist.

5. Extension across Generic Submanifolds

The next proposition is the key result for the proof of Theorem 1.1.

PROPOSITION 5.1. *Let Γ be an essentially finite, smooth, real-analytic hypersurface, and let Γ' be a compact, real-algebraic, strictly pseudoconvex hypersurface. Let M be a generic submanifold of dimension $2n - 2$, and let $p \in M$. Let U be a neighborhood of p. Denote by Γ^- and Γ^+ the connected components of $U \cap (\Gamma \setminus M)$. Suppose that f is a holomorphic mapping defined in a neighborhood of Γ^+, $f(\Gamma^+) \subset \Gamma'$, and suppose that J_f, the Jacobian of the mapping f, is not identically zero. Then f extends holomorphically to a neighborhood of p.*

Proof. Let U_1, U_2 be a standard pair of neighborhoods of p. Since Γ is essentially finite, we may assume that the Segre map λ is finite-to-one in U_1 and that $I_p \cap U_1 = \{p\}$. By Proposition 4.1, there exists an open set $\omega \subset (Q_p \cap U_1)$ such that, for any point $b \in \omega$, $Q_b \cap \Gamma$ contains a path γ in Γ^+ with the end point at p. The choice of $b \in \omega$, γ, and a point $a \in \gamma \cap U_1$ will form a triple, which we will denote by (b, γ, a). We can choose a so close to p that, possibly after a small perturbation, U_1, U_2 will also be a standard pair of neighborhoods for a.

We can choose (b, γ, a) such that $J_f(a) \neq 0$. Indeed, by Remark 2 following Proposition 4.1, $\dim(Q_z \cap \Gamma) = 2n - 3$. Since Γ is essentially finite, there exists a neighborhood U_b of the point b such that

$$\#\{z \in U_b : J_f\big|_{Q_z \cap \Gamma^+} = 0\} < \infty.$$

Moving b if necessary, we may assume that $J_f\big|_{Q_b \cap \Gamma^+}$ is not identically zero.

Let U_a be a neighborhood of a, so that f is biholomorphic in U_a. By Proposition 3.1, f extends analytically along any path in $V \setminus \Lambda$, where V is a neighborhood of $Q_a \cap U_1$ and $\Lambda \subset V$ is an analytically constructible set of complex dimension at most $n - 1$. There are two cases to be considered: either

(1) $Q_p \cap V$ is not contained in Λ; or
(2) $Q_p \cap V$ is contained in Λ.

Case 1. In this situation we can slightly perturb the triple (b, γ, a) so that $b \in (\omega \cap V)$, $b \notin \Lambda$, and f is biholomorphic in U_a. Notice that slight changes of (b, γ, a) do not change Λ. Since $V \setminus \Lambda$ is connected, we can find a continuous path $\theta \subset V$, with no self-intersections, connecting a and b and such that $\theta \cap \Lambda = \emptyset$. Choose a simply connected neighborhood U_θ of θ so that $U_\theta \subset V$ and $U_\theta \cap \Lambda = \emptyset$. Then, by the Monodromy theorem, $f\big|_{U_a}$ extends holomorphically to U_θ.

Denote by F the extension of $f\big|_{U_a}$ to U_θ obtained by Proposition 3.1. Choose a small neighborhood $U_b \subset U_\theta$ of the point b such that, for any z in some small neighborhood U_γ of γ, $Q_z \cap U_b$ is nonempty and connected. (Since $\gamma \subset Q_b$, we have $Q_z \ni b$ for all $z \in \gamma$.) Thus, F is holomorphic in U_b. Consider the set

$$A^* = \{(w, w') \in U_\gamma \times \mathbb{C}^n : F(Q_w \cap U_b) \subset Q'_{w'}\}. \tag{5.1}$$

As in Proposition 3.1, A^* is a closed complex-analytic subset of $U_\gamma \times \mathbb{C}^n$.

Lemma 5.2. *There exists a small neighborhood Ω of a such that*

$$A^* \cap (\Omega \times \Omega') = \Gamma_{f|\Omega}, \tag{5.2}$$

where $\Omega' = f(\Omega)$.

Proof. Choose some small neighborhood Ω containing a and a point z in Ω. Let $w \in Q_z \cap U_b$ be an arbitrary point, and let $w' = F(w)$. It follows from the definition of F that $f(Q_w \cap U_a) \subset Q'_{w'}$ and $z \in Q_w$. This implies that $f(z) \in Q'_{w'} = Q'_{F(w)}$. But then $F(w) \in Q'_{f(z)}$. Since $w \in Q_z$ was arbitrary, we deduce that $F(Q_z \cap U_b) \subset Q'_{f(z)}$. This means that $(z, z') \in A^*$ if $z' \in I_{f(z)}$; in particular, $A^* \cap (\Omega \times \Omega') \neq \emptyset$, since $(z, f(z)) \in A^*$. If Ω is chosen small enough, then $\Omega' \cap I_{f(z)} = \{f(z)\}$ and we conclude that $A^* \cap (\Omega \times \Omega') = \Gamma_{f|_\Omega}$. □

Consider the irreducible component of A^* that coincides with Γ_f in $\Omega \times \Omega'$. For simplicity, denote this component again by A^*. Then $\dim_C A^* = n$. Let $z^j \to p$ as $j \to \infty$, $z^j \in \gamma$. By passing to a subsequence if necessary, we may assume that there exists $p' \in \Gamma'$ such that $p' = \lim_{j\to\infty} f(z^j)$. Since the graph of $f\big|_{U_\gamma \cap \Gamma^+}$ is contained in A^*, we have $(z^j, f(z^j)) \in A^*$ and thus $(p, p') \in A^*$. Let $\pi : A^* \to U_\gamma$ and $\pi' : A^* \to \mathbb{C}^n$ be the natural projections.

Lemma 5.3. *There exist neighborhoods $U_p \ni p$ and $U'_{p'} \ni p'$ such that $\hat{f} := \pi' \circ \pi^{-1}(z)$ is a holomorphic mapping in U_p that extends f. Here $\pi^{-1} : U_p \to A^* \cap (U_p \times U'_{p'})$.*

Proof. Choose $U'_{p'} \ni p'$ so small that the Segre map λ' is one-to-one in $U'_{p'}$, and let U_p be a small neighborhood of p such that $U_p \subset \pi(\pi'^{-1}(U'_{p'}))$. Let us show that $\pi : A^* \cap (U_p \times U'_{p'}) \to U_p$ is one-to-one. If not, then we can find $z \in U_p$ and $z'^1, z'^2 \in U'_{p'}$ ($z'^1 \neq z'^2$) such that

$$(z, z'^1), (z, z'^2) \in A^* \cap (U_p \times U'_{p'}). \tag{5.3}$$

Then $F(Q_z \cap U_b) \subset Q'_{z'^j}$ for $j = 1, 2$. It follows from the definition of F that, for any $w \in U_b$, we have

$$f(Q_w \cap U_a) \subset Q'_{F(w)}.$$

Since $\lambda\colon z \to Q_z$ is finite-to-one in U_b, there exist only finitely many points in U_b that have the same Segre variety as w. Thus,

$$\#\{F^{-1}(F(w))\} < \infty \quad \text{for any } w \in U_b.$$

This shows that $\dim_C F(Q_z \cap U_b) = n - 1$. But then, since λ' is one-to-one in $U'_{p'}$, there exists at most one point $z' \in U'_{p'}$ such that $F(Q_z \cap U_b) \subset Q'_{z'}$. This contradicts (5.3) and therefore π is one-to-one.

By [Ch, Sec. 3.3, Prop. 3], $\pi : A^* \cap (U_p \times U'_{p'}) \to U_p$ is a biholomorphic mapping and hence $\hat{f} := \pi' \circ \pi^{-1}(z)$ is holomorphic in U_p and extends f. By analyticity, we also have $\hat{f}(\Gamma \cap U_p) \subset \Gamma'$. □

Case 2. $Q_p \subset \Lambda$. In this situation, f may not extend holomorphically to a neighborhood U_b of $b \in Q_p$ because $\omega \subset \Lambda$. However, one can show that f extends as a holomorphic correspondence. By such extension we mean a complex-analytic set of pure dimension n, defined in $U_\theta \times \mathbb{C}^n$, with proper projection onto the first component that contains $\Gamma_{f|_{U_a}}$.

LEMMA 5.4. *There exists a triple (b^*, γ^*, a^*) such that $b^* \in (\omega \cap V)$, $\gamma^* \subset \Gamma^+ \cap Q_{b^*}$, $a^* \in \gamma^* \cap U_a$, and $f\big|_{U_a}$ extends to a neighborhood of b^* as a holomorphic correspondence along some path $\theta \subset V$, possibly after a biholomorphic change of variables in the target space.*

Proof. We use the notation of Proposition 3.1. First we can exclude the case when $Q_p \cap V \subset \Lambda_1$. Indeed, after a biholomorphic change of coordinates in $\mathbb{P}^n$, we may assume that (in new coordinates) Γ' remains compact in $\mathbb{C}^n \subset \mathbb{P}^n$ and that $\pi'(\pi^{-1}(Q_p))$ is not entirely contained in $H_0 \subset \mathbb{P}^n$. Thus, b^* can be chosen so that $b^* \notin \Lambda_1$. If Q_p is not contained in Λ_2, then b^* can be chosen so that $b^* \notin \Lambda_2$ and we are in the conditions of case 1. Otherwise, since (by Proposition 3.3) $\dim \Lambda_3 < n-1$ and hence $Q_p \cap V$ is not contained in Λ_3, we can find a point b^* in $(\omega \cap V) \setminus (\Lambda_1 \cup \Lambda_3)$, that is, $b^* \in \Lambda_2 \setminus \Lambda_3$. Furthermore, since Λ is analytically constructible, we may choose $b^* \in \Lambda^{\text{reg}}$. Let $\gamma^* \subset Q_{b^*} \cap \Gamma^+$ be close to γ. Choose a^* so that $a^* \in U_a \cap \gamma^*$. Analogously to case 1, there exists a path $\theta \subset V$ (without self-intersections) connecting a^* and b^*, and $\theta \cap \Lambda = \{b^*\}$. Let U_θ be a simply connected neighborhood of θ such that $U_\theta \subset V$ and

$$\Lambda \cap U_\theta = (\Lambda \setminus (\Lambda_1 \cup \Lambda_3)) \cap U_\theta = Q_p \cap U_\theta.$$

Let A be the analytic set from Proposition 3.1, defined in $V \times \mathbb{P}^n$. Consider the irreducible component of $A \cap (U_\theta \times \mathbb{P}^n)$ that contains $\Gamma_{f|_{U_{a^*}}}$. Denote this component again as A. Then A is the desired extension of $f\big|_{U_{a^*}}$ as a correspondence because $A \cap (U_\theta \times H_0) = \emptyset$, since $U_\theta \cap (\Lambda_1 \cup \Lambda_3) = \emptyset$ and $\pi : A \to U_\theta$ is proper. □

To simplify the notation we will drop the asterisks from (b^*, γ^*, a^*). Let F: $U_\theta \to \mathbb{C}^n$ be a multivalued mapping corresponding to A; that is,

$$F(w) = \{w' : (w, w') \in A\}.$$

Let U_γ be a sufficiently small neighborhood of γ, where $\lambda : z \to Q_z$ is finite-to-one. Analogously, let U' be a small neighborhood of Γ', where the Segre map λ' is finite-to-one. Choose a small neighborhood U_b of b ($U_b \subset U_\theta$) such that, for all $z \in U_\gamma$, $Q_z \cap U_b$ is nonempty and connected. Define

$$A^* = \{(w, w') \in (U_\gamma \setminus \{p\}) \times U' : F(Q_w \cap U_b) \subset Q'_{w'}\}.$$

LEMMA 5.5. *A^* is a closed complex-analytic subset of $(U_\gamma \setminus \{p\}) \times U'$ that contains the graph of $f\big|_{U_a}$.*

Proof. For any $(w, w') \in A^*$, the condition

$$F(Q_w \cap U_b) \subset Q'_{w'}$$

can be expressed as follows. Take an open, simply connected set $\Omega \subset (U_b \setminus Q_p)$ such that $Q_w \cap \Omega \neq \emptyset$. Since $\Omega \cap \Lambda = \emptyset$, the branches of F are correctly defined in Ω. Then (5.4) is equivalent to $\tilde{f}(Q_w \cap \Omega) \subset Q'_{w'}$ for all branches $\tilde{f}$ of F. Notice that such an open set Ω can be found for any $w \in U_\gamma \setminus \{p\}$. The inclusion $\tilde{f}(Q_w \cap \Omega) \subset Q'_{w'}$ can be written as a system of holomorphic equations; therefore, A^* is complex-analytic. A^* is also closed because if $(w^j, w'^j) \to (w^0, w'^0)$ as $j \to \infty$ with $(w^j, w'^j) \in A^*$ and $(w_0, w'_0) \in (U_\gamma \setminus \{p\}) \times U'$, then $Q_{w^j} \to Q_{w^0}$ and $Q'_{w'^j} \to Q'_{w'^0}$ as $j \to \infty$. As a result, $F(Q_{w^0}) \subset Q'_{w'^0}$ and $(w^0, w'^0) \in A^*$. By repeating the argument in Lemma 5.2 we can show that A^* contains the graph of $f\big|_{U_a}$. □

Denote again by A^* the irreducible component of A^* that contains $\Gamma_{f|_{U_a}}$. Thus, $\dim_C A^* = n$. Let

$$S = (\{p\} \times U') \subset U_\gamma \times U'.$$

Then S is a removable singularity for A^*; that is, $\overline{A^*}$ is a complex analytic variety in $U_\gamma \times U'$. Indeed, let $(z^j, z'^j) \in A^*$ and $(z^j, z'^j) \to (z^0, z'^0) \in S$ as $j \to \infty$. Then $z^j \to p$, $F(Q_{z^j}) \subset Q'_{z'^j}$, and so $F(Q_p) \subset Q'_{z'^0}$. It follows that

$$\overline{A^*} \cap S \subset \{p\} \times \{z' \in U' : F(Q_p \cap U_b) \subset Q'_{z'}\}.$$

Because

$$\{z' \in U' : F(Q_p \cap U_b) \subset Q'_{z'}\} \subset Q'_{w'}$$

and $\dim_c Q'_{w'} = n - 1$, it follows that $\overline{A^*} \cap S$ has Hausdorff $2n$-measure zero. Since S is a pluripolar set, Bishop's theorem (see e.g. [Ch]) can be applied to conclude that S is a removable singularity for A^*.

Denote $\overline{A^*}$ again by A^*. Note that A^* is an analytic variety in $U_\gamma \times U'$. The rest of the proof is identical to case 1: we show that there exist neighborhoods $U_p \ni p$ and $U'_{p'} \ni p'$ such that π^{-1} is single-valued and, as a result, f extends holomorphically to a neighborhood of p if we set

$$f(z) = \pi' \circ \pi^{-1}(z).$$

This proves Proposition 5.1. □

6. Proof of the Main Result

Let $\rho(z,\bar z)$ be a defining function of Γ in a neighborhood of $\zeta \in \Gamma$. Let U_ζ be a small neighborhood of ζ and let $f\colon U_\zeta \to \mathbb{C}^n$ be a nonconstant holomorphic mapping such that $f(U_\zeta \cap \Gamma) \subset \Gamma'$, where Γ' is a compact strictly pseudoconvex real algebraic hypersurface with the defining function $P'(z',\bar z')$.

PROPOSITION 6.1. *There exists a point $\xi \in U_\zeta \cap \Gamma$ such that all eigenvalues of the Levi form $H_\rho(\xi, v)$, $v \in T^c_\zeta(\Gamma)$, are of the same sign.*

Proof. Since Γ' is strictly pseudoconvex, $f(U_\zeta)$ is not contained in Γ'. Consider the set $f^{-1}(\Gamma')$. This is a real-analytic set in U_ζ, and $\Gamma \subset f^{-1}(\Gamma')$. Since the set of regular points of a real-analytic set is dense, there exists a point $\xi \in U_\zeta \cap \Gamma$ such that $f^{-1}(\Gamma') \cap U_\xi = \Gamma \cap U_\xi$ in some small neighborhood $U_\xi \ni \xi$. Moreover, since Γ is essentially finite, ξ and U_ξ can be chosen such that $H_\rho(z, v)$ is nondegenerate on $T^c_z(\Gamma)$ for any $z \in U_\xi$. Replacing ρ by $-\rho$ (if necessary), we obtain

$$f(\{z \in U_\xi : \rho(z,\bar z) < 0\}) \subset \{P'(z',\bar z') < 0\}. \tag{6.1}$$

Indeed, if there are two points $a, b \in \{z \in U_\xi : \rho(z,\bar z) < 0\}$ that are mapped by f to different sides of Γ', then we can connect a and b by a path γ not intersecting Γ. But $f(\gamma)$ will clearly intersect Γ', which contradicts the fact that $f^{-1}(\Gamma') = \Gamma$ in U_ξ.

Consider the function $P' \circ f(z)$, which is defined in U_ξ and negative in $\{z \in U_\xi : \rho(z,\bar z) < 0\}$ because of (6.1). Since Γ' is strictly pseudoconvex, we can choose P' to be plurisubharmonic in a neighborhood of Γ'. Then $P' \circ f$ is also plurisubharmonic. By the Hopf lemma, $d(P' \circ f) \neq 0$ on $\Gamma \cap U_\xi$; we may thus consider $P' \circ f$ to be a local defining function of Γ in U_ξ. By the invariance property of the Levi form, for any vector $v \in T^c_\xi(\Gamma)$ we have

$$H_{P'\circ f}(\xi, v) = H_{P'}(f(\xi), f_* v) \geq 0. \tag{6.2}$$

Since the Levi form of Γ is nondegenerate, Γ is strictly pseudoconvex at ξ. $\square$

Notice that it follows from (6.2) that $J_f(\xi) \neq 0$. By a suitable choice of the defining function of Γ, by moving ζ to a nearby point (if necessary), and by the choice of U_ζ, we may therefore assume that (6.1) holds for U_ζ. Then Γ is strictly pseudoconvex in U_ζ and f is biholomorphic. Recall that Γ_s denotes the set of strictly pseudoconvex points of Γ. Let us show that f extends along any path in a connected component of Γ_s containing ζ.

Any compact strictly pseudoconvex algebraic hypersurface Γ' is either nonspherical or spherical at any point. In the latter case, Γ' is globally biholomorphically equivalent to a unit sphere, by [HJ]. Thus, we may assume that Γ' is either nonspherical or is a unit sphere. By the results in [P1] and [P2], f extends analytically along any path containing in a path-connected component of Γ_s.

As before, let Σ denote the set of points of Γ where the Levi form is degenerate. Note that Σ is a real-analytic set. Let M be the set of regular points $z \in \Sigma$ such that $T_z(\Sigma)$ is not a complex plane, and let $M^* = \Sigma \setminus M$.

LEMMA 6.2. *The set M^* does not divide Γ.*

Proof. M^* is the union of the set Σ^{sng} and the set $M^c := \{z \in \Sigma^{\text{reg}} : T_z(\Sigma) = T_z^c(\Sigma)\}$. Observe that, locally, M^c is a real-analytic set. Indeed, suppose $p \in M^c$ and that locally, near p, Σ is given by

$$\{z \in \Gamma : \phi_j(z, \bar{z}) = 0,\ j = 1, \dots, 2m,\ m < n\},$$

where $\phi_j(z, \bar{z})$ are smooth real-analytic functions and $d\phi_1 \wedge \cdots \wedge d\phi_{2m} \neq 0$. Then M^c is given by the condition $\operatorname{rank}(\partial\phi_j/\partial z_k) = m$ and thus M^c is defined by a finite system of real-analytic equations. Also, $\dim_R M^c \leq 2n - 3$, for if $\dim_R M^c = 2n - 2$ at some regular point $z \in M^c$ then, by the Levi–Civita theorem, M^c near z is a complex hypersurface contained in Γ. This contradicts the essential finiteness of Γ. Since $\dim \Sigma^{\text{sng}} < \dim \Sigma$, we have

$$\dim_R M^* \leq 2n - 3$$

and so M^* does not divide Γ. □

LEMMA 6.3. *f extends along any path in $\Gamma \setminus M^*$.*

Proof. Let $\tau : [0, 1] \to \Gamma$ be an arbitrary simple path with $\tau(0) = \zeta$. Suppose there exists a number $t_0 \in (0, 1]$ such that f extends along $\tau(t)$ for $0 \leq t < t_0$ but does not extend to a neighborhood of $p = \tau(t_0)$. Let U be a small neighborhood of p such that $U \cap \Sigma = U \cap M$. If $\dim_R(M \cap U) < 2n - 2$, then we can find a generic submanifold $\tilde{M}$ such that $\dim_R \tilde{M} = 2n - 2$ and $M \subset \tilde{M}$. We may therefore assume that $\dim_R M = 2n - 2$. The set $M \cap U$ divides $\Gamma \cap U$ into two connected and simply connected components, which we denote by Γ^- and Γ^+. Let $\tau_0 = \tau\big|_{[0,t_0)}$. Denote by f_t the extension of f along τ_0. Then f_t is holomorphic in U_{τ_0}, a small neighborhood of τ_0. There exist $\tau(t_1) \in (\tau_0 \cap U)$ and a neighborhood $U_1 \ni \tau(t_1)$ such that f_t is holomorphic in U_1. Clearly, U_1 intersects at least one of the connected components of $U \setminus M$—say, Γ^+ for definiteness. It follows from Proposition 6.1 that the eigenvalues of $H_\rho(z, v)$ are of the same sign in Γ^+. Hence, by the choice of the defining function, Γ^+ can be assumed to be strictly pseudoconvex and $f_t\big|_{U_1}$ extends to Γ^+ as a locally biholomorphic mapping. Denote this extension by $\tilde{f}$. By Proposition 5.1, $\tilde{f}$ extends holomorphically to some neighborhood U_p of p. If τ_0 also intersects Γ^-, then analogously f_t extends as a locally biholomorphic mapping to Γ^-. For $t < t_0$ and close to t_0, $\tau(t) \in U_p$ and f_t coincides with the extension of $\tilde{f}$ to U_p. In view of Lemma 6.2, these considerations show that f extends along any path in $\Gamma \setminus M^*$. □

The remaining case is $p \in M^*$. It follows from the theorems of Cartan (see e.g. [N, Prop. 15, p. 104]) and Narasimhan [N, Prop. 18, p. 105] that, if a real-analytic set is defined by a finite system of equations, then singular points of this analytic set are contained in some real-analytic set of lower dimension, which is also defined by a finite system of equations. Hence there exists a real-analytic set Σ_1 of real dimension at most $2n-3$ such that $\Sigma^{\text{sng}} \subset \Sigma_1$. It follows that $\Sigma_1 \cup M^c$ is a locally real-analytic set of dimension at most $2n-3$. For any $p \in (\Sigma_1 \cup M^c)^{\text{reg}}$ there exists a small neighborhood U_p such that $U_p \cap (\Sigma_1 \cup M_c)$ is contained in some generic submanifold of Γ, of dimension $2n-2$, and we can show that f extends holomorphically to a neighborhood of p by repeating the argument in Lemma 6.3.

The singular part of $\Sigma_1 \cup M^c$ is now contained in an analytic set of dimension $2n-4$. By induction on dimension, f extends holomorphically to every point in Σ. Theorem 1.1 is proved. □

Proof of Corollary 1.2. Since Γ is essentially finite, the set of points where the Levi form of Γ is degenerate has dimension at most $2n-2$. Let U be an open set, and let $f : U \to \mathbb{C}^n$ be a holomorphic mapping such that $f(U \cap \Gamma) \subset \Gamma'$. Then there is a point in $U \cap \Gamma$ where the Levi form is nondegenerate. By Proposition 6.1, $\Gamma \cap U$ contains strictly pseudoconvex points (up to orientation). If $\dim_R \Sigma < 2n-2$, then Σ does not divide Γ and the latter is globally pseudoconvex. Suppose now that Σ contains a component M of dimension $2n-2$ and that $p \in M$. By Theorem 1.1, f extends holomorphically to a neighborhood $U_p \ni p$ along some path in Γ. Moreover, it follows from Proposition 6.1 that J_f is not identically zero. Since $A := \{z \in U_p : J_f(z) = 0\}$ is an analytic variety and Γ is essentially finite, M is not contained in A; hence there is a point $\xi \in M \cap U_p$ such that $J_f(\xi) \neq 0$ and f is biholomorphic near ξ. But this contradicts the fact that the Levi form of Γ is degenerate at ξ. Thus, $\dim_R \Sigma < 2n-2$. □

Proof of Corollary 1.3. By [DF1], any compact domain with a smooth real-analytic boundary is of finite type; in particular, it is essentially finite. By Theorem 1.1, f extends holomorphically along any path on ∂D and, since ∂D is simply connected, f extends to a global mapping from ∂D to $\partial D'$. By Hartog's theorem, f extends to a holomorphic mapping in $\bar{D}$. Since $f(\partial D) \subset \partial D'$, the extended mapping is proper. □

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