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Boundary regularity of correspondences in $\mathbb{C}^n$

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Abstract. Let M, M' be smooth, real analytic hypersurfaces of finite type in $\mathbb{C}^n$ and $\hat{f}$ a holomorphic correspondence (not necessarily proper) that is defined on one side of M, extends continuously up to M and maps M to M'. It is shown that $\hat{f}$ must extend across M as a locally proper holomorphic correspondence. This is a version for correspondences of the Diederich–Pinchuk extension result for CR maps.

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1. Introduction and statement of results

1.1 *Boundary regularity*

Let U, U' be domains in $\mathbb{C}^n$ and let $M \subset U$, $M' \subset U'$ be relatively closed, connected, smooth, real analytic hypersurfaces of finite type (in the sense of D'Angelo). A recent result of Diederich and Pinchuk [DP3] shows that a continuous CR mapping $f: M \to M'$ is holomorphic in a neighbourhood of M. The purpose of this note is to show that their methods can be adapted to prove the following version of their result for correspondences. We assume additionally that M (resp. M') divides the domain U (resp. U') into two connected components U^+ and U^- (resp. $U'^{\pm}$).

Theorem 1.1. *Let $\hat{f}: U^- \to U'$ be a holomorphic correspondence that extends continuously up to M and maps M to M', i.e., $\hat{f}(M) \subset M'$. Then $\hat{f}$ extends as a locally proper holomorphic correspondence across M.*

We recall that if $\mathcal{D} \subset \mathbb{C}^p$ and $\mathcal{D}' \subset \mathbb{C}^m$ are bounded domains, a holomorphic correspondence $\hat{f}: \mathcal{D} \to \mathcal{D}'$ is a complex analytic set $A \subset \mathcal{D} \times \mathcal{D}'$ of pure dimension p such that $\overline{A} \cap (\mathcal{D} \times \partial\mathcal{D}') = \emptyset$, where $\partial D'$ is the boundary of D'. In this situation, the natural projection $\pi: A \to \mathcal{D}$ is proper, surjective and a finite-to-one branched covering. If in addition the other projection $\pi': A \to \mathcal{D}'$ is proper, the correspondence is called proper. The analytic set A can be regarded as the graph of the multiple valued mapping $\hat{f} := \pi' \circ \pi^{-1}: \mathcal{D} \to \mathcal{D}'$. We also use the notation $A = \mathrm{Graph}(\hat{f})$.

The branching locus σ of the projection π is a codimension one analytic set in $\mathcal{D}$. Near each point in $\mathcal{D} \setminus \sigma$, there are finitely many well-defined holomorphic inverses of π^{-1}. The symmetric functions of these inverses are globally well-defined holomorphic functions on

$\mathcal{D}$. To say that $\hat{f}$ is continuous up to $\partial\mathcal{D}$ simply means that the symmetric functions extend continuously up to $\partial\mathcal{D}$. Thus in Theorem 1.1 the various branches of $\hat{f}$ are continuous up to M and each branch maps points on M to those on M'.

We say that $\hat{f}$ in Theorem 1.1 extends as a *holomorphic correspondence* across M if there exist open neighbourhoods $\tilde{U}$ of M and $\tilde{U}'$ of M', and an analytic set $\tilde{A} \subset \tilde{U} \times \tilde{U}'$ of pure dimension n such that (i) Graph($\hat{f}$) intersected with $(\tilde{U} \cap U^-) \times (\tilde{U}' \cap U')$ is contained in $\tilde{A}$ and (ii) the projection $\tilde{\pi}\colon \tilde{A} \to \tilde{U}$ is proper. Without condition (ii), $\hat{f}$ is said to extend as an *analytic set*. Finally, the extension of $\hat{f}$ is a proper holomorphic correspondence if in addition to (i) and (ii), $\tilde{\pi}'\colon \tilde{A} \to \tilde{U}'$ is also proper.

COROLLARY 1.1

Let D and D' be bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth real-analytic boundary. Let $\hat{f}\colon D \to D'$ be a holomorphic correspondence. Then $\hat{f}$ extends as a locally proper holomorphic correspondence to a neighbourhood of the closure of D.

The corollary follows immediately from Theorem 1.1 and [BS] where the continuity of $\hat{f}$ is proved. This generalizes a well-known result of [BR] and [DF] where the extension past the boundary of D is proved for holomorphic mappings.

1.2 *Preservation of strata*

Let M_s^+ (resp. M_s^-) be the set of strongly pseudoconvex (resp. pseudoconcave) points on M. The set of points where the Levi form $\mathcal{L}_\rho$ has eigenvalues of both signs on $T^{\mathbb{C}}(M)$ and no zero eigenvalue will be denoted by $M^{\pm}$ and finally M^0 will denote those points where $\mathcal{L}_\rho$ has at least one zero eigenvalue on $T^{\mathbb{C}}(M)$. M^0 is a closed real analytic subset of M of real dimension at most $2n-2$. Then

$$M = M_s^+ \cup M_s^- \cup M^{\pm} \cup M^0.$$

Further, let M^+ (resp. M^-) be the pseudoconvex (resp. pseudoconcave) part of M, which equals the relative interior of $\overline{M_s^+}$ (resp. $\overline{M_s^-}$). For non-negative integers i, j such that $i + j = n - 1$, let $M_{i,j}$ denote those points at which $\mathcal{L}_\rho$ has exactly i positive and j negative eigenvalues on $T^{\mathbb{C}}(M)$. Each (non-empty) $M_{i,j}$ is relatively open in M and semi-analytic whose relative boundary is contained in M^0. With this notation, $M_{0,n-1} = M_s^-$ and $M_{n-1,0} = M_s^+$. Moreover, $M^{\pm}$ is the union of all (non-empty) $M_{i,j}$ where both i, j are at least 1 and $i + j = n - 1$. Note that points in $M_s^-, M^{\pm}$ are in the envelope of holomorphy of U^-. Following [B], there is a semi-analytic stratification for M^0 given by

$$M^0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \tag{1.1}$$

where Γ_4 is a closed, real analytic set of dimension at most $2n-4$ and $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is also a closed, real analytic set of dimension at most $2n-3$. Further, $\Gamma_1, \Gamma_2, \Gamma_3$ are either empty or smooth, real analytic manifolds; Γ_2, Γ_3 have dimension $2n-3$, and Γ_1 has dimension $2n-2$. Finally, Γ_2 and Γ_3 are CR manifolds of complex dimension $n-2$ and $n-3$ respectively. The set of points, denoted by Γ_h^1 in Γ_1 where the complex tangent space to Γ_1 has dimension $n-1$ is semi-analytic and has real dimension at most $2n-3$, as otherwise there would exist a germ of a complex manifold in M contradicting the finite type hypothesis. Then $\Gamma_1 \setminus \Gamma_h^1$ is a real analytic manifold of dimension $2n-2$ and has CR dimension $n-2$. Using the same letters to denote the various strata of M^0, there exists a refinement of (1.1), so that $\Gamma_1, \Gamma_2, \Gamma_3$ are all smooth, real analytic manifolds of dimensions $2n-2$, $2n-3$, $2n-3$ respectively, while the corresponding CR dimensions are $n-2$, $n-2$, and $n-3$. Finally, Γ_4 is a closed, real analytic set of dimension at most $2n-4$.

Theorem 1.2. *With the hypothesis of Theorem* 1.1, *the extended correspondence* $\hat{f}\colon M \to M'$ *satisfies the additional properties*: $\hat{f}(M^+) \subset M'^+$, $\hat{f}(M^+ \cap M^0) \subset M'^+ \cap M'^0$ *and* $\hat{f}(M^-) \subset M'^-$, $\hat{f}(M^- \cap M^0) \subset M'^- \cap M'^0$. *Moreover*, $\hat{f}(M^+ \cap \Gamma_j) \subset M'^+ \cap \Gamma'_j$ *and* $\hat{f}(M^- \cap \Gamma_j) \subset M'^- \cap \Gamma'_j$ *for* $j = 1, 3, 4$. *Finally*, $\hat{f}$ *maps the relative interior of* $\overline{M}^{\pm}$ *to the relative interior of* $\overline{M}'^{\pm}$.

Preservation of Γ_2 is not always possible even for holomorphic mappings as the following example shows: the domain $\Omega = \{(z_1, z_2)\colon |z_1|^2 + |z_2|^4 < 1\}$ is mapped to the unit ball in $\mathbb{C}^2$ by the proper holomorphic mapping $f(z_1, z_2) = (z_1, z_2^2)$. Points of the form $\{(e^{i\theta}, 0)\} \subset \partial\Omega$ are weakly pseudoconvex and in fact form $\Gamma_2 \subset \partial\Omega$, and f maps them to strongly pseudoconvex points.

2. Segre varieties

We will write $z = ('z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ for a point $z \in \mathbb{C}^n$. The word 'analytic' will always mean complex analytic unless stated otherwise. The techniques of Segre varieties will be used and here is a synopsis of the main properties that will be needed. The proofs of these can be found in [DF] and [DW]. As described above, let M be a smooth, real analytic hypersurface of finite type in $\mathbb{C}^n$ that contains the origin. If U is small enough, the complexification $\rho(z, \overline{w})$ of ρ is well-defined by means of a convergent power series in $U \times U$. Note that $\rho(z, \overline{w})$ is holomorphic in z and anti-holomorphic in w. For any $w \in U$, the associated Segre variety is defined as

$$Q_w = \{z \in U : \rho(z, \overline{w}) = 0\}.$$

By the implicit function theorem, it is possible to choose neighbourhoods $U_1 \subset\subset U_2$ of the origin such that for any $w \in U_1$, Q_w is a closed, complex hypersurface in U_2 and

$$Q_w = \{z = ('z, z_n) \in U_2 : z_n = h('z, \overline{w})\},$$

where $h('z, \overline{w})$ is holomorphic in $'z$ and anti-holomorphic in w. Such neighbourhoods will be called a standard pair of neighbourhoods and they can be chosen to be polydiscs centered at the origin. It can be shown that Q_w is independent of the choice of ρ. For $\zeta \in Q_w$, the germ Q_w at ζ will be denoted by ${}_\zeta Q_w$. Let $\mathcal{S} := \{Q_w\colon w \in U_1\}$ be the set of all Segre varieties, and let $\lambda\colon w \mapsto Q_w$ be the so-called Segre map. Then $\mathcal{S}$ admits the structure of a finite dimensional analytic set. It can be shown that the analytic set

$$I_w := \lambda^{-1}(\lambda(w)) = \{z : Q_z = Q_w\}$$

is contained in M if $w \in M$. Consequently, the finite type assumption on M forces I_w to be a discrete set of points. Thus λ is proper in a small neighbourhood of each point of M. For $w \in U_1^+$, the symmetric point ${}^s w$ is defined to be the unique point of intersection of the complex normal to M through w and Q_w. The component of $Q_w \cap U_2^-$ that contains the symmetric point is denoted by Q_w^c.

Finally, for all objects and notions considered above, we simply add a prime to define their corresponding analogs in the target space.

3. Localization and extension across an open dense subset of *M*

In the proof of Theorem 1.1 in order to show extension of $\hat{f}$ as a holomorphic correspondence, it is enough to consider the problem in an arbitrarily small neighbourhood of any point $p \in M$. The reason is the following. Firstly, since the projection $\pi\colon \mathrm{Graph}(\hat{f}) \to U^-$ is proper, the closure of $\mathrm{Graph}(\hat{f})$ has empty intersection with $U^- \times \partial U'$. Therefore, by [C] § 20.1, to prove the continuation of $\hat{f}$ across M as an analytic set, it is enough to do that in a neighbourhood of any point in M. Secondly, once the extension of $\hat{f}$ as a holomorphic correspondence in a neighbourhood of any point $p \in M$ is established, then globally there exists a holomorphic correspondence defined in a neighbourhood $\tilde{U}$ of M which extends $\hat{f}$. To see that simply observe that if $F \subset \tilde{U} \times \tilde{U}'$ is an analytic set extending $\hat{f}$, then by choosing smaller $\tilde{U}$ we may ensure that the projection to the first component is proper, as otherwise there would exist a point z on M such that $\hat{F}(z)$ has positive dimension (here $\hat{F}$ is the map associated with the set F). This however contradicts local extension of $\hat{f}$ near z as a holomorphic correspondence.

Since the projection $\pi\colon \mathrm{Graph}(\hat{f}) \to U^-$ is proper, $\mathrm{Graph}(\hat{f})$ is contained in the analytic set $A \subset U^- \times U'$, defined by the zero locus of holomorphic functions $P_1(z, z_1'), P_2(z, z_2'), \dots, P_n(z, z_n')$ given by

$$P_j(z, z_j') = {z_j'}^{l} + a_{j1}(z){z_j'}^{l-1} + \cdots + a_{jl}(z), \tag{3.1}$$

where l is the generic number of images in $\hat{f}(z)$, and $1 \le j \le n$ (for details, see [C]). The coefficients $a_{\mu\nu}(z)$ are holomorphic in U^- and extend continuously up to M. This is the definition of continuity of the correspondence $\hat{f}$ up to M which is equivalent to that given in § 1.1.

The discriminant locus is $\{R_j(z) = 0\}$, $1 \le j \le n$, where $R_j(z)$ is a universal polynomial function of $a_{j\mu}(z)$ $(1 \le \mu \le l)$ and hence by the uniqueness theorem, it follows that $\overline{\{R_j(z) = 0\}} \cap M$ is nowhere dense in M, for all j. The set of points S on M which do not belong to $\overline{\{R_j(z) = 0\}} \cap M$ for any j is therefore open and dense in M. Near each point p on S, $\hat{f}$ splits into well-defined holomorphic maps $f_1(z), f_2(z), \dots, f_l(z)$ each of which is continuous up to M.

If $p \in S \cap (M^- \cup M^\pm)$, the functions $a_{\mu\nu}(z)$ extend holomorphically to a neighbourhood of p and hence $\hat{f}$ extends as a holomorphic correspondence across p. It is therefore sufficient to show that $\hat{f}$ extends across an open dense subset of $S \cap M^+$. But this follows from Lemma 3.2 and Corollary 3.3 in [DP3]. We denote by $\Sigma \subset M$ the non-empty open dense subset of M across which $\hat{f}$ extends as a holomorphic correspondence.

4. Extension as an analytic set

Fix $0 \in M$ and let $p_1', p_2', \dots, p_k' \in \hat{f}(0) \cap M'$. The continuity of $\hat{f}$ allows us to choose neighbourhoods $0 \in U_1$ and $p_i' \in U_i'$ and local correspondences $\hat{f}_i\colon U_1^- \to U_i'$ that are irreducible and extend continuously up to M. Moreover, $\hat{f}_i(0) = p_i'$ for all $1 \le i \le k$. It will suffice to focus on one of the $\hat{f}_i'$'s, say $\hat{f}_1'$ and to show that it extends holomorphically across the origin. Abusing notation, we will write $\hat{f}_1' = \hat{f}$, $U_1' = U'$ and $p_1' = 0'$. Thus $\hat{f}\colon U_1^- \to U'$ is an irreducible holomorphic correspondence and $\hat{f}(0) = 0'$. Define

$$V^+ = \{(w, w') \in U_1^+ \times U' : \hat{f}(Q_w^c) \subset Q'_{w'}\}.$$

Then V^+ is non-empty. Indeed, $\hat{f}$ extends across an open dense set near the origin and [V] shows that the invariance property of Segre varieties then holds. Moreover, a similar argument as in [S2] shows that $V^+ \subset U_1^+ \times U'$ is an analytic set of dimension n and exactly the same arguments as in Lemmas 4.2 – 4.4 of [DP3] show that: first, the projection $\pi\colon V^+ \to U^+ := \pi(V^+) \subset U_1^+$ is proper (and hence that $U^+ \subset U_1^+$ is open) and second, the projection $\pi'\colon V^+ \to U'$ is locally proper. Thus, to V^+ is associated a correspondence $F^+\colon U^+ \to U'$ whose branches are $\hat{F}^+ = \pi' \circ \pi^{-1}$.

Let $a \in M$ be a point close to the origin, across which $\hat{f}$ extends as a holomorphic correspondence. If $\hat{f}$ is well-defined in the ball $B(a,r), r > 0$ and $w \in B(a,r)^-$, it follows from Theorem 4.1 in [V] that all points in $\hat{f}(w)$ have the same Segre variety. By analytic continuation, the same holds for all $w \in U_1^-$. Using this observation, it is possible to define another correspondence $F^-\colon U_1^- \to U'$ whose branches are $\hat{F}^-(w) = (\lambda')^{-1} \circ \lambda' \circ \hat{f}(w)$. Let $U := U_1^- \cup U^+ \cup (\Sigma \cap U_1)$. The invariance property of Segre varieties shows that the correspondences $\hat{F}^+$, $\hat{F}^-$ can be glued together near points on $\Sigma \cap U_1$. Hence, there is a well-defined correspondence $\hat{F}\colon U \to U'$ whose values over U^+ and U_1^- are $\hat{F}^+$ and $\hat{F}^-$ respectively. Note that

$$F := \mathrm{Graph}(\hat{F}) = \{(w, w') \in U \times U' : w' \in \hat{F}(w)\}$$

is an analytic set in $U \times U'$ of pure dimension n, with proper projection $\pi\colon F \to U$. Once again, the invariance property shows that all points in $\hat{F}(w)$, $w \in U$, have the same Segre variety.

Lemma 4.1. *The correspondence $\hat{F}$ satisfies the following properties*:

(i) *For $w_0 \in \partial U \cap U_1^+$, $\mathrm{cl}_{\hat{F}}(w_0) \subset \partial U'$.*
(ii) $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'}$.
(iii) *If* $\mathrm{cl}_{\hat{F}}(0) = \{0'\}$, *then* $0 \in \Sigma$.
(iv) *$F \subset (U_1 \backslash (M \backslash \Sigma)) \times U'$ is a closed analytic set.*

Proof.

(i) Choose $(w_j, w'_j) \in F$ converging to $(w_0, w'_0) \in (\partial U \cap U_1^+) \times \overline{U}'$. Then $\hat{f}(Q^c_{w_j}) \subset Q'_{w'_j}$ for all j. If $w'_0 \in U'$, then passing to the limit, we get $\hat{f}(Q^c_{w_0}) \subset Q'_{w'_0}$ which shows that $(w_0, w'_0) \in F$ and hence $w_0 \in U$, which is a contradiction. This also proves (iv).

(ii) Choose $w_j \in U$ converging to 0. There are two cases to consider. First, if $w_j \in U_1^- \cup (\Sigma \cap U_1)$ for all j, it follows that $\hat{f}(w_j) \to 0'$. Moreover, for any $w'_j \in \hat{F}(w_j)$, $Q'_{w'_j} = Q'_{\hat{f}(w_j)}$. If U' is small enough, the equality $Q'_{w'} = Q'_{0'}$ implies that $w' = 0'$ and thus we conclude that $w'_j \to 0' \in Q'_{0'}$. Second, if $w_j \in U^+$ for all j, then $\hat{f}(Q^c_{w_j}) \subset Q'_{w'_j}$ for any $w'_j \in \hat{F}(w_j)$. Let $w'_j \to w'_0 \in U'$. If $\zeta \in Q^c_{w_j}$, then $\hat{f}(\zeta) \in Q'_{w'_j} \to Q'_{w'_0}$. But $w_j \to 0$ implies that $\mathrm{dist}(Q^c_{w_j}, 0) \to 0$ and hence $\hat{f}(\zeta) \to 0'$. Thus $0' \in Q'_{w'_0}$ which shows that $w'_0 \in Q'_{0'}$.

(iii) If $\mathrm{cl}_{\hat{F}}(0) = \{0'\}$, then (i) shows that $0 \notin \partial U \cap U_1^+$. Let $B(0,r)$ be a small ball around the origin such that $B(0,r) \cap \partial U = \emptyset$. The correspondence $\hat{F}$ over $B(0,r)^+$ is the union of some components of the zero locus of a system of monic pseudo-polynomials whose coefficients are bounded holomorphic functions on $B(0, r)^+$. By Trepreau's theorem, all these coefficients extend holomorphically to $B(0, r)$, and the extended zero locus contains the graph of $\hat{f}$ near the origin since Σ is dense. It follows that $0 \in \Sigma$. □

Following [S1], for any $w_0 \in U$, it is possible to find a neighbourhood Ω of w_0, relatively compact in U and a neighbourhood $V \subset U_1$ of $Q_{w_0} \cap U_1$ such that for $z \in V$, $Q_z \cap \Omega$ is non-empty and connected. Associated with the pair (Ω, V) is

$$\tilde{F} := \tilde{F}(w_0, \Omega, V) = \{(z, z') \in V \times U' : \hat{F}(Q_z \cap \Omega) \subset Q'_{z'}\} \tag{4.1}$$

which (cf. [DP4]) is an analytic set of dimension at most n. If $w_0 \in \Sigma$, then Corollary 5.3 of [DP3], shows that $F \cap (V \times U')$ is the union of irreducible components of $\tilde{F}$ of dimension n. As in [DP3] we call $(w_0, z_0) \in U \times Q_{w_0}$ a pair of reflection if there exist neighbourhoods $\Omega(w_0) \ni w_0$ and $\Omega(z_0) \ni z_0$ such that for all $w \in \Omega(w_0)$, $\hat{F}(Q_w \cap \Omega(z^0)) \subset Q'_{\hat{F}(w)}$. It follows from the invariance property of Segre varieties that the definition of the pair of reflection is symmetric. As an example we note that if the set $\tilde{F}$ defined in (4.1) contains $F \cap (V \times U')$, then (w_0, z) is a point of reflection for any point z in a connected component of $Q_{w_0} \cap U$ containing w_0.

Let $w_0 \in U$, $z_0 \in Q_{w_0} \cap \Sigma$ be a pair of reflection. Fix $B(z_0, r)$, a small ball around z_0 where $\hat{f}$ is well-defined and let $S(w_0, z_0) \subset \tilde{F} \cap ((Q_{w_0} \cap U_1) \times U')$ be the union of those irreducible components that contain Graph$(\hat{f})$ over $Q_{w_0} \cap B(z_0, r)$. Note that $S(w_0, z_0)$ is an analytic set of dimension $n - 1$ and is contained in $(Q_{w_0} \cap U_1) \times U'$ and moreover, the invariance property shows that

$$S(w_0, z_0) \subset ((Q_{w_0} \cap U_1) \times (Q'_{\hat{F}(w_0)} \cap U')).$$

Furthermore, from the above considerations it follows that for any $z \in \pi(S(w_0, z_0))$ the point (w_0, z) is a pair of reflection. Finally, let the cluster set of a sequence of closed sets $\{C_j\} \subset \mathcal{D}$, where $\mathcal{D}$ is some domain, be the set of all possible accumulation points in $\mathcal{D}$ of all possible sequences $\{c_j\}$ where $c_j \in C_j$.

PROPOSITION 4.1

Let $\{z_\nu\} \in \Sigma$ *converge to* 0. *Suppose that the cluster set of the sequence* $\{S(z_\nu, z_\nu)\}$ *contains a point* $(\zeta_0, \zeta'_0) \in U \times U'$. *Then* $\hat{f}$ *extends as an analytic set across the origin.*

Proof. First, the pair (z_ν, z_ν) is an example of a pair of reflection and hence $S(z_\nu, z_\nu)$ is well-defined. Also, note that $(z_\nu, \hat{f}(z_\nu)) \to (0, 0')$. Choose $(\zeta_\nu, \zeta'_\nu) \in S(z_\nu, z_\nu)$ that converges to $(\zeta_0, \zeta'_0) \in U \times U'$. It follows that (ζ_ν, z_ν) is a pair of reflection. Let Ω, V be neighbourhoods of ζ_0 and Q_{ζ_0} as in the definition of $\tilde{F}(\zeta_0, \Omega, V)$. Since $\zeta_0 \in U$, it follows that $\tilde{F}(\zeta_0, \Omega, V)$ is a non-empty, analytic set in $V \times U'$. Shrinking U_1 if needed, $Q_{\zeta_\nu} \cap U_1 \subset V$ and $\zeta_\nu \in \Omega$ for all large ν. This shows that $\tilde{F}(\zeta_\nu, \Omega, V) = \tilde{F}(\zeta_0, \Omega, V)$ for all large ν. Lemma 5.2 of [DP3] shows that $\tilde{F}(\zeta_\nu, \Omega, V)$ contains the graph of all branches of $\hat{f}$ near z_ν and hence $\tilde{F}(\zeta_0, \Omega, V)$ contains the graph of $\hat{f}$ near $(0, 0')$. Therefore, $\tilde{F}(\zeta_0, \Omega, V)$ extends the graph of $\hat{f}$ across the origin. □

Remarks. First, as in [DP3] this proposition will be valid if the pair (z_ν, z_ν) were replaced by a pair of reflection $(w_\nu, z_\nu) \in U \times \Sigma$ that converges to $(0, 0')$ and $\hat{F}(w_\nu)$ clusters at some point in U'. Second, this proposition shows the relevance of studying the cluster set of a sequence of analytic sets (cf. [SV] also). In general, the hypothesis that the cluster set of $\{S(z_\nu, z_\nu)\}$ (or $S(w_\nu, z_\nu)$ in case (w_ν, z_ν) is a pair of reflection) contains a point in $U \times U'$ cannot be guaranteed since the projection $\pi\colon S(z_\nu, z_\nu) \to U$ is not known to be proper. However, the following version of Lemma 5.9 in [DP3] holds.

Lemma 4.2. *There are sequences* $(w_\nu, z_\nu) \in U \times \Sigma$, $w'_\nu \in \hat{F}(w_\nu)$ *and analytic sets* $\sigma_\nu \subset U$ *of pure dimension* $p \geq 1$ (p *independent of* ν) *such that*:

(i) $(w_\nu, z_\nu) \to (0, 0)$ *and* (w_ν, z_ν) *is a pair of reflection for all* ν.
(ii) $w'_\nu \to w'_0 \in U'$ *and* $z_\nu \in \sigma_\nu \subset \pi(S(w_\nu, z_\nu))$.

Proof. Choose a sequence $z_\nu \in \Sigma$ that converges to the origin. If the projections $\pi\colon S(z_\nu, z_\nu) \to U$ were proper for all ν, then for some fixed $r > 0$ and ν large enough, let $\sigma_\nu := Q_{z_\nu} \cap B(z_\nu, r)$, $w_\nu = z_\nu$ and $w'_\nu \in \hat{f}(z_\nu)$. It can be seen that the lemma holds with these choices. On the other hand, if π is not known to be proper on $S(z_\nu, z_\nu)$, no fixed value of r, as described above, exists. Hence, for arbitrarily small values of $r' > 0$, there exist $(w_\nu, w'_\nu) \in S(z_\nu, z_\nu) \cap (U^+ \times U')$ such that $w_\nu \to 0$ and $w'_\nu \to w'_0$ with $|w'_0| = r'$. Since M' is of finite type, we may assume that $Q'_{w'_0} \neq Q'_{0'}$. Moreover, note that $w'_0 \in Q'_{0'} \cap U'$ (which shows that $0' \in Q'_{w'_0}$) and (w_ν, z_ν) is a pair of reflection for all ν. By making a small holomorphic perturbation of coordinates in the target space, if needed, it follows that $0' \in Q'_{w'_0} \cap \{z' \in U'\colon z'_2 = \cdots = z'_n = 0\}$ is an isolated point. Therefore, there exists an $\epsilon > 0$ such that after shrinking U', if needed, $q'_0 := Q'_{w'_0} \cap \{z' \in U'\colon z'_2 = \cdots = z'_{n-1} = 0, |z'_n| < \epsilon\}$ (which is an analytic set of dimension 1 in $U' \cap \{|z'_n| < \epsilon\}$ containing the origin) has no limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$. Let l be the multiplicity of $\hat{f}\colon U_1^- \to U'$. Let $\hat{f}(z_\nu) = \{\zeta^j_\nu\}$, $1 \leq j \leq l$ counted with multiplicity. For large ν, the l sets

$$q'_{\nu,j} = Q'_{w'_\nu} \cap \{z' \in U' : z'_k = (\zeta^j_\nu)_k, \quad 2 \leq k \leq n-1, \quad |z'_n| < \epsilon\}$$

are analytic, of dimension 1, in $U' \cap \{|z'_n| < \epsilon\}$ without limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$ and clearly contain (z_ν, ζ^j_ν). Since $\pi'(S(w_\nu, z_\nu)) \subset Q'_{w'_\nu}$,

$$s_{\nu,j} := S(w_\nu, z_\nu) \cap \{(z, z') : z'_k = (\zeta^j_\nu)_k, \quad 2 \leq k \leq n-1\}$$

are analytic sets of dimension at least 1 in $U_1 \times (U' \cap \{|z'_n| < \epsilon\})$ for all $1 \leq j \leq l$. By construction, the analytic sets $q'_{\nu,j}$ do not have limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$ and hence $s_{\nu,j}$ do not have limit points on $U_1 \times (\partial U' \cap \{|z'_n| < \epsilon\})$. By Lemma 4.1, $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'} = \{z'_n = 0\}$ and by shrinking U_1 if needed, this shows that $s_{\nu,j}$ have no limit points on $U_1 \times (U' \cap \{|z'_n| = \epsilon\})$. Thus for large ν and all j, the projections $\pi\colon s_{\nu,j} \to U_1$ are proper and their images $\sigma_{\nu,j} := \pi(s_{\nu,j})$ are analytic sets in U_1 of dimension at least 1 and $z_\nu \in \sigma_{\nu,j}$ for all ν, j. It remains to pass to subsequences if necessary to choose $\sigma_{\nu,j}$ with constant dimension. □

One conclusion that now follows is: if $\hat{f}$ does not extend as an analytic set across the origin, then $\mathrm{cl}(\sigma_\nu) \subset M \setminus \Sigma$. Indeed, if there exists $\zeta_0 \in \mathrm{cl}(\sigma_\nu) \cap (U_1 \setminus (M \setminus \Sigma))$, let $(\zeta_\nu, \zeta'_\nu) \in S(w_\nu, z_\nu)$ converge to $(\zeta_0, \zeta'_0) \in U_1 \times U'$. Proposition 4.1 now shows that

$\zeta_0 \in \partial U \cap U_1$. But since $\zeta_0 \notin M \setminus \Sigma$, it follows from Lemma 4.1 that $\zeta_0' \in \partial U'$ which is a contradiction.

The goal will now be to show that $\hat{f}$ extends as an analytic set across the origin. For this, choose $\{z_\nu\} \in \Sigma$ converging to the origin and consider the analytic sets $S(z_\nu, z_\nu)$. By Proposition 4.1, it suffices to show that $\pi(\mathrm{cl}(S(z_\nu, z_\nu)) \cap U \neq \emptyset$. Let

$$S' := \pi'(\mathrm{cl}(S(z_\nu, z_\nu)) \cap (\{0\} \times U')) \subset Q'_{0'}$$

and let m be the dimension of $\hat{S}'$ – the smallest closed analytic set containing S' (the so-called Segre completion of [DP3]). If $m = 0$, then $0'$ is an isolated point in S' and after shrinking U_1, U' suitably, it follows that $\mathrm{cl}(S(z_\nu, z_\nu))$ has no limit points on $U_1 \times \partial U'$. Thus $\pi\colon S(z_\nu, z_\nu) \to U_1$ are proper projections and therefore $\pi(S(z_\nu, z_\nu)) = Q_{z_\nu} \cap U_1$ for all large ν. Hence $\pi(\mathrm{cl}(S(z_\nu, z_\nu))) = Q_0 \cap U_1$. If $\hat{f}$ did not extend as an analytic set across the origin, the aforementioned remark shows that with $\sigma := Q_{z_\nu} \cap U_1$, $Q_0 \cap U_1 = \mathrm{cl}(\sigma_\nu) \subset M \setminus \Sigma \subset M$. This cannot happen as M is of finite type. Hence $\hat{f}$ extends as an analytic set across the origin in case $m = 0$. We may therefore suppose that $m > 0$. We recall the following lemma proved by Diederich and Pinchuk:

Lemma 4.3 ([DP3], Lemma 9.8). *Let S' be a subset of $Q'_{0'}$, $0' \in S'$ and $m = \dim \hat{S}'$. Then after possibly shrinking U_1, there are points $w'^1, \ldots, w'^k \in S'$ $(k \leq n-1)$ such that one of the following holds:*

(1) $k = m$ *and* $\dim(\hat{S}' \cap Q'_{w'^1} \cap \cdots \cap Q'_{w'^k}) = 0$;
(2) $k \geq 2m - n + 1$ *and* $\dim(\hat{S}' \cap Q'_{w'^1} \cap \cdots \cap Q'_{w'^k}) = m - k$.

Thus there are two cases to consider.

Case 1. Choose $(w_{1\nu}, w'_{1\nu}), (w_{2\nu}, w'_{2\nu}), \ldots, (w_{m\nu}, w'_{m\nu}) \in S(z_\nu, z_\nu)$ so that $w_{\mu\nu} \to 0$ and $w'_{\mu\nu} \to w'_\mu$ for all $1 \leq \mu \leq m$. A generic choice of $w_{\mu\nu}$ (see p. 136 in [DP3]) ensures that $q_{m\nu} := Q_{w_{1\nu}} \cap Q_{w_{2\nu}} \cap \cdots \cap Q_{w_{m\nu}}$ has dimension $n - m$. Each $(w_{\mu\nu}, z_\nu)$ is a pair of reflection and hence the analytic set

$$S^m_\nu := \bigcap_{1 \leq \mu \leq m} S(w_{\mu\nu}, z_\nu) \subset (q^{m\nu} \times q'^{m\nu}) \cap (U_1 \times U')$$

is well-defined. If $m = n-1$, then Lemma 9.7 of [DP3] shows that the germ of $q'^{(n-1)}$ at the origin has dimension 1. Moreover, $\hat{S}' = Q'_{0'}$ and Lemma 4.3 implies that $q'^{(n-1)} \cap Q'_{0'}$ contains $0'$ as an isolated point. Since $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'}$, it follows that $0'$ is an isolated point of

$$\pi'(\mathrm{cl}(S^{n-1}_\nu) \cap (\{0\} \times U')) \subset q'^{(n-1)} \cap Q'_{0'} = \{0'\}.$$

Shrinking U_1, the projection $\pi\colon S^{n-1}_\nu \to U_1$ becomes proper and $\pi(S^{n-1}_\nu) = q^{n-1,\nu} \cap U_1$. By Theorem 7.4 of [DP3], there is a subsequence of $q^{n-1,\nu} \cap U_1$ that converges to an analytic set $A \subset U_1$ of pure dimension 1 and contains the origin. A contains points ζ_0 that do not belong to M because of the finite type assumption and $\zeta_0 \in \pi(\mathrm{cl}(S^{n-1}_\nu)) \subset \pi(\mathrm{cl}(S(w_{\mu\nu}, z_\nu))$. By Proposition 4.1, $\hat{f}$ extends as an analytic set across the origin.

If $m < n-1$, the dimension of $S^m_\nu \cap S(z_\nu, z_\nu)$ is at least $n - m - 1 > 0$. Now

$$\pi'(\mathrm{cl}(S^m_\nu \cap S(z_\nu, z_\nu)) \cap (\{0\} \times U')) \subset q'^m \cap \hat{S}' = \{0'\},$$

the last equality following from Lemma 4.3. The projection $\pi \colon S_\nu^m \cap S(z_\nu, z_\nu) \to U_1$ is therefore proper for small U_1 and that $\pi(S_\nu^m \cap S(z_\nu, z_\nu)) = q^{m\nu} \cap Q_{z_\nu} \cap U_1$. Again, by Theorem 7.4 of [DP3], there is a subsequence of $q^{m\nu} \cap Q_{z_\nu} \cap U_1$ that converges to an analytic set $A \subset U_1$ of positive dimension and as before this shows that $\hat{f}$ extends as an analytic set across the origin.

Case 2. As before, choose $(w_{1\nu}, w'_{1\nu}), (w_{2\nu}, w'_{2\nu}), \dots, (w_{k\nu}, w'_{k\nu}) \in S(z_\nu, z_\nu)$ such that $w_{\mu\nu} \to 0$ and $w'_{\mu\nu} \to w'_\mu$ for all $1 \le \mu \le k$ and $q_{k\nu} = Q_{w_{1\nu}} \cap Q_{w_{2\nu}} \cap \cdots \cap Q_{w_{k\nu}}$, $\tilde{q}^{k\nu} := Q_{z_\nu} \cap q^{k\nu}$ have dimension $n-k$ and $n-k-1$ respectively. Now note that $\dim(S_\nu^k \cap S(z_\nu, z_\nu)) \ge n-k-1 > 1$. Indeed, the inequalities $2m-n+1 \le k < m$ show that $m \le n-2$ and hence $k < n-2$. Since the dimension of $\hat{S}' \cap q'^k$ is $m-k$, choose coordinates so that

$$\hat{S}' \cap q'^k \cap \{z' \in U' : z'_1 = z'_2 = \cdots = z'_{m-k} = 0\} = \{0'\}.$$

Let $\hat{f}(z_\nu) = \{\zeta_\nu^j\}$, $1 \le j \le l$, l being the multiplicity of $\hat{f}$. The l sets

$$\begin{aligned} T_{\nu,j} = \{(z, z') \in S_\nu^k \cap S(z_\nu, z_\nu) : z'_1 = (\zeta_\nu^j)_1,& \\ z'_2 = (\zeta_\nu^j)_2, \dots, z'_{m-k} = (\zeta_\nu^j)_{m-k}\},& \end{aligned}$$

where $1 \le j \le l$ are analytic sets in $U_1 \times U'$ and have dimension at least $n-k-1-(m-k) = n-m-1 > 0$. By construction,

$$\begin{aligned} \pi'(\mathrm{cl}(T_{\nu,j}) \cap (\{0\} \times U')) \subset \hat{S}' \cap q'^k& \\ \cap \{z' \in U' : z'_1 = z'_2 = \cdots = z'_{m-k} = 0\} = \{0'\}& \end{aligned}$$

and hence by shrinking U_1, U', the projections $\pi \colon T_{\nu,j} \to U_1$ are proper and the images $\sigma_{\nu,j} := \pi(T_{\nu,j}) \subset U_1$ are analytic and have dimension $n-m-1$. Moreover $\sigma_{\nu,j} \subset \tilde{q}^{k\nu}$, and since $\tilde{q}^{k\nu}$ depend anti-holomorphically on the k-tuple defining it, Theorem 7.4 of [DP3] shows that $\tilde{q}^{k\nu}$ converges to an analytic set $\tilde{A} \subset U_1$ of dimension $n-k-1$, after passing to a subsequence. Working with this subsequence, we see that $\mathrm{cl}(\sigma_{\nu,j}) \subset \tilde{A}$. On the other hand, since $2m-n+1 \le k$, it follows, as in [DP3], that

$$\dim \tilde{A} = n-k-1 \le 2(n-m-1) = 2 \dim \sigma_{\nu,j}.$$

Proposition 8.3 of [DP3] shows that $\mathrm{cl}(\sigma_{\nu,j}) \not\subset M$ and hence by Proposition 4.1, it follows that $\hat{f}$ extends as an analytic set across the origin.

To complete the proof, it suffices to show that extension as an analytic set implies extension as a locally proper holomorphic correspondence. This is achieved in the next lemma.

Lemma 4.4. *There exist neighbourhoods U of* 0 *and U' of* $0'$ *such that $F \subset U \times U'$ is a proper holomorphic correspondence which extends $\hat{f}$.*

Proof. Extension as a holomorphic correspondence essentially follows from [DP4]. All nuances in the proof of Proposition 2.4 in [DP4] work in this situation as well provided the following two modifications are made. Let U, U' be neighbourhoods of $0, 0'$ respectively and suppose that $F \subset U \times U'$ extends $\hat{f}$ as an analytic set in $U \times U'$. Then it needs to be checked that $F \cap (U^+ \times U') \neq \emptyset$ and that there exists a sequence $\{z_\nu\} \in M$ converging to 0 such that $\hat{f}$ extends as a correspondence across each z_ν.

Suppose that $F \cap (U^+ \times U') = \emptyset$. In this case, the proof of Proposition 3.1 (or even Proposition 4.1 in [SV]) shows that $(0, 0')$ is in the envelope of holomorphy of $\overline{U^-} \times U'$. The coefficients $a_{\mu\nu}(z)$ in (3.1) can be regarded as holomorphic functions on $U^- \times U'$ (i.e., independent of the z' variables) and thus each $a_{\mu\nu}(z)$ extends holomorphically across $(0, 0')$. This extension must be independent of the z' variables by the uniqueness theorem and hence $a_{\mu\nu}(z)$ extends holomorphically across the origin. This shows that $\hat{f}$ extends as a holomorphic correspondence across the origin. To show the existence of the sequence $\{z_\nu\}$ claimed above, let $\pi\colon F \to U$ be the natural projection and define

$$A = \{(z, z') \in F : \dim\, (\pi^{-1}(z))_{(z,z')} \geq 1\},$$

where $(\pi^{-1}(z))_{(z,z')}$ denotes the germ of the fiber over z at (z, z'). Then A is an analytic subset of F, and since F contains the graph of $\hat{f}$ over U^-, it follows that the dimension of A is at most $n - 1$. Since Lipschitz maps do not increase Hausdorff dimension, it follows that the Hausdorff dimension of $\pi(A)$ is at most $2n - 2$. Pick $p \in M \setminus \pi(A)$. The fiber $F \cap \pi^{-1}(p)$ is discrete and this shows that $\hat{f}$ extends as a holomorphic correspondence across p.

Finally, we show that U' can be chosen so small that the projection $\pi'\colon F \to U'$ is also proper. Indeed, for $z' \in M'$, $\pi'^{-1}(z')$ is an analytic subset of F. Since π is proper, it follows by Remmert's theorem that $\hat{F}^{-1}(z') = \pi \circ \pi'^{-1}(z')$ is an analytic set. The invariance property of Segre varieties yields $\hat{F}(Q_z \cap U) \subset Q'_{z'}$ for any $z \in \hat{F}^{-1}(z')$. Since M is of finite type, the set $\cup_{z \in \hat{F}^{-1}(z')} Q_z$ has Hausdorff dimension n, and therefore cannot be mapped by $\hat{F}$ into $Q'_{z'}$ which has dimension $n - 1$. This shows that projection π' has discrete fibers on M'. It follows from the Cartan–Remmert theorem that there exists a neighbourhood U' of M' such that π' has only discrete fibers, and therefore the projection π' from F to U' will be proper.

This completes the proof of Theorem 1.1. □

5. Preservation of strata

Fix $p \in M$ and let $p'_1, p'_2, \ldots, p'_k \in \hat{f}(p) \subset M'$. Choose neighbourhoods U, U' of p, p'_1 respectively and let $\hat{f}_1\colon U^- \to U'$ be a component of $\hat{f}$ such that $\hat{f}_1(p) = p'_1$. Then $\hat{f}_1$ extends as a holomorphic correspondence $F \subset U \times U'$ and to prove Theorem 1.2, it suffices to focus on $\hat{f}_1$, which will henceforth be denoted by $\hat{f}$. The following two general observations can be made in this situation. First, the branching locus $\hat{\sigma}$ of $\hat{F}$ is an analytic set in U and the finite-type assumption on M shows that the real dimension of $\hat{\sigma} \cap M$ is at most $2n - 3$. The branching locus of $\hat{f}$ denoted by σ, is contained in $\hat{\sigma} \cap U^-$. Second, the invariance property of Segre varieties in [DP1], [V] shows that $\hat{F}$, the extended correspondence, preserves the two components $U^\pm$. That is, after possibly re-labelling $U'^\pm$, it follows that $\hat{F}(U^\pm) \subset U'^\pm$ and $\hat{F}(M) \subset M'$. The same holds for $\hat{G} := \hat{F}^{-1}\colon U' \to U$.

Proof of Theorem 1.2. Let $p \in M^+$ and suppose that $\{\zeta'_j\} \in M'$ is a sequence converging to p'_1 with the property that the Levi form $\mathcal{L}_\rho$ restricted to the complex tangent space to M at ζ'_j has at least one negative eigenvalue. Fix $\zeta'_{j_0} \in U'$ for some large j_0. By shifting ζ'_{j_0}

slightly, we may assume that $\zeta'_{j_0} \notin \hat{\sigma}' \cup \hat{F}(M^0 \cap U)$, where $\hat{\sigma}'$ is the branching locus of $\hat{G}$, and at the same time retain the property of having at least one negative eigenvalue. Let g_1 be a locally biholomorphic branch of $\hat{G}$ near ζ'_{j_0}. Then $g_1(\zeta'_{j_0})$ is clearly a pseudoconvex point and this contradicts the invariance of the Levi form. This shows that $\hat{f}(M^+) \subset M'^+$. The same arguments show that $\hat{f}(M^-) \subset M'^-$.

Let $p \in M^+ \cap M^0$ and suppose that $p'_1 \in M'^+_s$. The extending correspondence $\hat{F}: U \to U'$ satisfies the invariance property, namely $\hat{F}(Q_w) \subset Q'_{w'}$ for all $(w, w') \in (U \times U') \cap \mathrm{Graph}(\hat{F})$. But near p'_1, the Segre map λ is injective and this shows that $\hat{F}$, and hence $\hat{f}$, is a single valued, proper holomorphic mapping, say $f: U \to U'$ with $f(p) = p'_1$. Two observations can be made at this stage: first, f cannot be locally biholomorphic near p due to the invariance of the Levi form. Second, if $V_f \subset U$ is the branching locus of f defined by the vanishing of the Jacobian determinant of f, then V_f intersects both $U^{\pm}$. Indeed, suppose that $V_f \cap U^- = \emptyset$. Choose a branch of f^{-1} near some fixed point $a' \in U'^-$ and analytically continue it along all paths in U'^- to get a well-defined mapping, say $g: U'^- \to U^-$. The analytic set $F \subset U \times U'$ extends g as a correspondence and hence [DP2] g is a well-defined holomorphic mapping in U' and this must be the single valued inverse of f. Thus f is locally biholomorphic near p and this is a contradiction. The same argument works to show that V_f must intersect U^+ as well. Note that $V_f \cap M$ has real dimension at most $2n - 3$. If $p \in \Gamma_1$, choose U so small that $M^0 \cap U \subset \Gamma_1$. Then there exists $q \in \Gamma_1 \setminus (V_f \cap M)$ near p, where f is locally biholomorphic. Thus q is mapped locally biholomorphically to $f(q)$ which is a strongly pseudoconvex point and this is a contradiction. If $p \in \Gamma_3$, then again we shrink U so that $M^0 \cap U \subset \Gamma_3$ and $(M \cap U) \setminus \Gamma_3 \subset M^+_s$. Then f is locally biholomorphic near all points in $(M \cap U) \setminus \Gamma_3$ and therefore $V_f \cap U^-$ must cluster only along Γ_3. Since the CR dimension of $\Gamma_3 = n - 3 < (n - 1) - 1$ which is one less than the dimension of V_f, it follows (Theorem 18.5 in [C]) that $\overline{V_f \cap U^-}$ is a closed, analytic set in U. Thus $V_f \cap U^-$ has two analytic continuations, namely V_f and $\overline{V_f \cap U^-}$ and therefore they must be the same. This shows that V_f cannot intersect U^+ which is a contradiction. The same argument works if $p \in \Gamma_4$, the only difference being that $\overline{V}_f \subset \overline{U^-}$ is analytic because of Shiffman's theorem. Thus if $p \in M^+ \cap M^0$, then $p'_1 \in M'^+ \cap M'^0$.

To study this further, suppose that $p \in M^+ \cap \Gamma_1$ and $p'_1 \in M'^+ \cap \Gamma'_2$. Choose U, U' small enough so that $M^0 \cap U \subset \Gamma_1$ and $M'^0 \cap U' \subset \Gamma'_2$. Pick $q \in \Gamma_1 \setminus (\hat{\sigma} \cap M)$. Then $\hat{f}$ splits near q into finitely many well-defined holomorphic mappings each of which extends across q. Moving q slightly, if needed, on $\Gamma_1 \setminus (\hat{\sigma} \cap M)$, each of these holomorphic mappings are even locally biholomorphic near q. Working with one of these mappings, say f_1, it follows that $f_1(q) \notin M'^+_s$ due to the invariance of the Levi form. This means that $f_1(q) \in \Gamma'_2$. In the same way, all points in Γ_1 that are sufficiently near q are mapped locally biholomorphically by f_1 to Γ'_2. This cannot happen as Γ'_2 has strictly smaller dimension than Γ_1. The same argument shows that $p'_1 \notin \Gamma'_3 \cup \Gamma'_4$. Hence $p'_1 \in M'^+ \cap \Gamma'_1$.

Suppose that $p \in M^+ \cap \Gamma_2$ and $p'_1 \in M'^+ \cap \Gamma'_1$. Considering $\hat{f}^{-1}: U' \to U$, the arguments used in the preceding lines show that this cannot happen. The case when $p'_1 \in \Gamma'_4$ can be dealt with similarly. Now suppose that $p'_1 \in \Gamma'_3$. As always, U, U' will be small enough so that $M^0 \cap U \subset \Gamma_2$ and $M'^0 \cap U' \subset \Gamma'_3$. The arguments used above show that the cluster set of points in $M^+_s \cap U$ is contained in $M'^+_s \cap U'$ and hence $\hat{f}$ splits into finitely well-defined mappings each of which is locally biholomorphic near points in $M^+_s \cap U$. This shows that the branching locus $\sigma \subset U^-$ of $\hat{f}$ clusters only along Γ_2. Then $\hat{F}(\sigma)$ is an analytic set of dimension $n - 1$ in U'^-. There are two cases to consider: first, if $\hat{F}(\sigma)$

clusters only along Γ_3', then arguing as above, $\overline{\hat{F}(\sigma)} \subset \overline{U'^-}$ is a closed, analytic set in U'. The strong disk theorem shows that p_1' is in the envelope of holomorphy of U'^- and this is a contradiction. Second, if there are points in $\overline{\hat{F}(\sigma)} \cap M_s'^+$, this means that $(\overline{\hat{F}(\hat{\sigma})} \cap M') \cap \Gamma_3'$ has real dimension at most $2n-4$. Pick $q' \in \Gamma_3' \backslash (\overline{\hat{F}(\hat{\sigma})} \cap M')$ and note that the continuity of $\hat{f}$ implies that $\hat{f}^{-1}(q') \in M_s^+$. As seen above, this cannot happen. Thus $p_1' \in \Gamma_2'$ or $M_s'^+$. Similar arguments show that if $p \in M^+ \cap \Gamma_3$ or $M^+ \cap \Gamma_4$, then $p_1' \in M'^+ \cap \Gamma_3'$ or $M'^+ \cap \Gamma_4'$ respectively.

By reversing the roles of $U^\pm$, the same arguments used in the preceding paragraphs can be applied to show that $\hat{f}(M^- \cap M^0) \subset M'^- \cap M'^0$ with the preservation of $M^- \cap \Gamma_j$ for $j = 1, 3, 4$.

Finally, fix integers i, j both at least 1 such that $i + j = n - 1$ and suppose that $p \in M_{i,j}$. Then there exists a point p_0, in U (chosen so small that $M \cap U \subset M_{i,j}$) and arbitrarily close to p, where all branches of $\hat{f}$ are well-defined and locally biholomorphic. The invariance of the Levi form shows that the images of p_0 under any of the branches of $\hat{f}$ should all be in $M_{i,j}$. Note that each of these images is close to p_1'. This cannot happen if p_1' is in M'^+, M'^- or in $M'_{i',j'}$ for $i \neq i'$ and $j \neq j'$. The only possibility is that p_1' is in the relative interior of $\overline{M}'_{i,j}$. The same argument works if p is in the relative interior of $\overline{M}_{i,j}$. □

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