In this brief note we use our uniformization result from [11, 12] to extend the work of Fu and Wong [8] on the relationship between two long-standing conjectures about the behaviour of the Bergman metric of a strictly pseudoconvex domain in \( \mathbb{C}^n \), \( n \geq 2 \).

1. Let \( D \subseteq \mathbb{C}^n \) be an arbitrary bounded domain. The Bergman kernel function of \( D \) can be defined by the formula
\[
K_D(z) := \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(z)},
\]
where \( \{ \varphi_j \}_{j=1}^{\infty} \) is any orthonormal basis of the space \( L^2O(D) \) of square-integrable holomorphic functions in \( D \).

It is a standard result that the function \( \log K_D(z) \) is strictly plurisubharmonic and the positive \((1,1)\)-form
\[
k_D := i\partial \bar{\partial} \log K_D(z)
\]
is invariant with respect to biholomorphic mappings between bounded domains. The Bergman metric on \( D \) is the Kähler metric associated with this Kähler form.

2. A classical problem, proposed in different forms by Bergman, Hua, and Yau, asks to describe the domain in terms of the differential-geometric properties of its Bergman metric. For example, a bounded domain with complete Bergman metric of constant holomorphic sectional curvature is biholomorphic to the ball by a well-known theorem of Lu Qi-Keng [10].

The Cheng conjecture [4] asserts that the hypotheses of Lu’s theorem can be weakened considerably for a smoothly bounded strictly pseudoconvex domain. Namely, such a domain has to be biholomorphic to the ball if and only if its Bergman metric is Kähler–Einstein. (The formulation in [4] is somewhat vague; the precise statement above is taken from [8].)

3. Fefferman [7] (see also [2]) established the following deep result on the boundary behaviour of the kernel function of a smoothly bounded strictly pseudoconvex domain \( D \). Let \( \rho \in C^\infty(D) \) be a defining function for \( D \). Then there is a decomposition
\[
K_D(z) = \varphi(z) \rho(z)^{-(n+1)} + \psi(z) \log |\rho(z)|,
\]
where the functions \( \varphi, \psi \in C^\infty(D) \), and \( \varphi \neq 0 \) everywhere on \( \partial D \). Note that the latter property implies that the Bergman metric of a strictly pseudoconvex domain is complete.

Although the kernel function is defined globally, its asymptotic behaviour as \( z \to z_0 \in \partial D \) depends only on the local CR geometry of the boundary at the point \( z_0 \) (see [7]). For instance, the kernel function of the unit ball \( B \subset \mathbb{C}^n \) can be explicitly decomposed as
\[
K_B(z) = \frac{n!}{\pi^n} (1 - \|z\|^2)^{-(n+1)}
\]
with identically vanishing logarithmic term. Thus if the boundary of a strictly pseudoconvex domain \( D \) is spherical (i.e., locally CR diffeomorphic to the unit sphere \( \partial B \subset \mathbb{C}^n \) at each point \( q \in \partial D \)), then the coefficient \( \psi \) in the logarithmic term of (1) vanishes to infinite order at the boundary of \( D \).

The Ramadanov conjecture [13] asserts that, conversely, the condition \( \psi(z) = O(\rho^n) \) as \( z \to \partial D \) implies that the boundary of \( D \) is spherical. This conjecture was proved for domains in \( \mathbb{C}^2 \) by Graham and Burns [9] and Boutet de Monvel [1].

4. Now we are in position to state and prove the main results of this note.
Theorem. The Cheng conjecture in \( \mathbb{C}^n \) follows from the Ramadanov conjecture in \( \mathbb{C}^2 \).

Since the Ramadanov conjecture is known to be true in dimension 2, the theorem implies the following result.

Corollary. The Bergman metric of a smoothly bounded strictly pseudoconvex domain \( D \subseteq \mathbb{C}^2 \) is Kähler–Einstein if and only if this domain is biholomorphic to the ball.

Remark. Fu and Wong [8] proved these results in the special case of simply connected domains using a weaker uniformization result of Chern and Ji [6] and stated the general case as an open question.

Proof of the theorem. Suppose that the Bergman metric of a strictly pseudoconvex domain \( D \) is Kähler–Einstein. Fu and Wong [8] computed (rather ingeniously) that in this case the logarithmic coefficient \( \psi \) in the decomposition (1) vanishes to infinite order at \( \partial D \). Assuming the Ramadanov conjecture, we conclude that the boundary of \( D \) is spherical. Hence, by the uniformization theorem (see [11, Thm. A.2] and [12, Cor. 3.2]) the domain \( D \) is covered by the unit ball.

Since \( D \) is the quotient of the ball by the group of holomorphic deck transformations, there is a natural complete metric of constant holomorphic sectional curvature on \( D \) obtained by taking the quotient of the standard invariant metric on the ball. According to Cheng and Yau [5], the complete Kähler–Einstein metric on \( D \) is unique up to a constant factor. Hence, the Bergman metric of \( D \) is proportional to the quotient metric and so has constant holomorphic sectional curvature. It follows that \( D \) is biholomorphic to the ball by Lu’s theorem [10].

Example. The domain \( D_{BS} = \{ (z, w) \in \mathbb{C}^2 \mid \sin \log |z| + |w|^2 < 0, \ e^{-\pi} < |z| < 1 \} \) constructed by Burns and Shnider [3] is a non-simply connected strictly pseudoconvex domain with spherical boundary in \( \mathbb{C}^2 \). Thus, the kernel function of \( D_{BS} \) has no logarithmic asymptotic at the boundary but its Bergman metric is not Kähler–Einstein. It would be interesting to give a direct proof of the latter assertion that would not use the uniqueness of the Cheng–Yau metric and the Lu Qi-keng theorem.

References


Steklov Mathematical Institute, 119991 Moscow, Russia
Ruhr-Universität Bochum, D-44780 Bochum, Germany
E-mail address: stefan@mi.ras.ru

Department of Mathematics, the University of Western Ontario, London, Canada N6A 5B7
E-mail address: shafikov@uwo.ca