CONJECTURES OF CHENG AND RAMADANOV

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In this brief note we use our uniformization result from [11, 12] to extend the work of Fu and Wong [8] on the relationship between two long-standing conjectures about the behaviour of the Bergman metric of a strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$.

1. Let $D \Subset \mathbb{C}^n$ be an arbitrary bounded domain. The *Bergman kernel function* of D can be defined by the formula

$$K_D(z) := \sum_{j=1}^{\infty} \varphi_j(z)\,\overline{\varphi_j(z)},$$

where $\{\varphi_j\}_{j=1,\dots,\infty}$ is any orthonormal basis of the space $L^2\mathcal{O}(D)$ of square-integrable holomorphic functions in D.

It is a standard result that the function $\log K_D(z)$ is strictly plurisubharmonic and the positive $(1,1)$-form

$$k_D := i\partial\overline{\partial}\log K_D(z)$$

is invariant with respect to biholomorphic mappings between bounded domains. The *Bergman metric* on D is the Kähler metric associated with this Kähler form.

2. A classical problem, proposed in different forms by Bergman, Hua, and Yau, asks to describe the domain in terms of the differential-geometric properties of its Bergman metric. For example, a bounded domain with complete Bergman metric of constant holomorphic sectional curvature is biholomorphic to the ball by a well-known theorem of Lu Qi-Keng [10].

The *Cheng conjecture* [4] asserts that the hypotheses of Lu's theorem can be weakened considerably for a smoothly bounded strictly pseudoconvex domain. Namely, such a domain has to be biholomorphic to the ball if and only if its Bergman metric is Kähler–Einstein. (The formulation in [4] is somewhat vague; the precise statement above is taken from [8].)

3. Fefferman [7] (see also [2]) established the following deep result on the boundary behaviour of the kernel function of a smoothly bounded strictly pseudoconvex domain D. Let $\rho \in C^\infty(\overline{D})$ be a defining function for D. Then there is a decomposition

$$K_D(z) = \varphi(z)\rho(z)^{-(n+1)} + \psi(z)\log|\rho(z)|, \tag{1}$$

where the functions $\varphi, \psi \in C^\infty(\overline{D})$, and $\varphi \neq 0$ everywhere on ∂D. Note that the latter property implies that the Bergman metric of a strictly pseudoconvex domain is complete.

Although the kernel function is defined globally, its asymptotic behaviour as $z \to z_0 \in \partial D$ depends only on the local CR geometry of the boundary at the point z_0 (see [7]). For instance, the kernel function of the unit ball $B \subset \mathbb{C}^n$ can be explicitly decomposed as

$$K_B(z) = \frac{n!}{\pi^n}\,(1 - \|z\|^2)^{-(n+1)}$$

with identically vanishing logarithmic term. Thus if the boundary of a strictly pseudoconvex domain D is spherical (i. e., locally CR diffeomorphic to the unit sphere $\partial B \subset \mathbb{C}^n$ at each point $q \in \partial D$), then the coefficient ψ in the logarithmic term of (1) vanishes to infinite order at the boundary of D.

The *Ramadanov conjecture* [13] asserts that, conversely, the condition $\psi(z) = O(\rho^\infty)$ as $z \to \partial D$ implies that the boundary of D is spherical. This conjecture was proved for domains in $\mathbb{C}^2$ by Graham and Burns [9] and Boutet de Monvel [1].

4. Now we are in position to state and prove the main results of this note.

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Theorem. *The Cheng conjecture in $\mathbb{C}^n$ follows from the Ramadanov conjecture in $\mathbb{C}^n$.*

Since the Ramadanov conjecture is known to be true in dimension 2, the theorem implies the following result.

Corollary. *The Bergman metric of a smoothly bounded strictly pseudoconvex domain $D \Subset \mathbb{C}^2$ is Kähler–Einstein if and only if this domain is biholomorphic to the ball.*

Remark. Fu and Wong [8] proved these results in the special case of *simply connected* domains using a weaker uniformization result of Chern and Ji [6] and stated the general case as an open question.

Proof of the theorem. Suppose that the Bergman metric of a strictly pseudoconvex domain D is Kähler–Einstein. Fu and Wong [8] computed (rather ingeniously) that in this case the logarithmic coefficient ψ in the decomposition (1) vanishes to infinite order at ∂D. Assuming the Ramadanov conjecture, we conclude that the boundary of D is spherical. Hence, by the uniformization theorem (see [11, Thm. A.2] and [12, Cor. 3.2]) the domain D is covered by the unit ball.

Since D is the quotient of the ball by the group of holomorphic deck transformations, there is a natural complete metric of constant holomorphic sectional curvature on D obtained by taking the quotient of the standard invariant metric on the ball. According to Cheng and Yau [5], the complete Kähler–Einstein metric on D is unique up to a constant factor. Hence, the Bergman metric of D is proportional to the quotient metric and so has constant holomorphic sectional curvature. It follows that D is biholomorphic to the ball by Lu's theorem [10]. □

Example. The domain $D_{BS} = \{(z,w) \in \mathbb{C}^2 \mid \sin\log|z| + |w|^2 < 0,\ e^{-\pi} < |z| < 1\}$ constructed by Burns and Shnider [3] is a non-simply connected strictly pseudoconvex domain with spherical boundary in $\mathbb{C}^2$. Thus, the kernel function of D_{BS} has no logarithmic asymptotic at the boundary but its Bergman metric is *not* Kähler–Einstein. It would be interesting to give a direct proof of the latter assertion that would not use the uniqueness of the Cheng–Yau metric and the Lu Qi-keng theorem.

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