Calculus 250a Final Examination Monday, December 11, 2006

PART A MULTIPLE CHOICE (30 marks)

NOTE: YOUR ANSWERS TO THE PROBLEMS IN PART A MUST BE INDICATED ON THE SCANTRON SHEET.

A1. [*2 marks*] Find the cross product $\langle 4,0,6\rangle \times \langle 1,2,3\rangle$.

A: $\langle -12,-6,8\rangle$	B: $\langle 12,6,-8\rangle$	C: $\langle -12,6,8\rangle$	D: $\langle 12,-6,-8\rangle$	E: None of A, B, C, D

A2. [*2 marks*] Identify the traces of the surface $x^2 - 2y^2 + 4z^2 = 0$ in the indicated planes:

A: $x=k$: hyperbolas, $y=k$: circles, $z=k$: hyperbolas
B: $x=k$: parabolas, $y=k$: ellipses, $z=k$: parabolas
C: $x=k$: hyperbolas, $y=k$: ellipses, $z=k$: hyperbolas
D: $x=k$: parabolas, $y=k$: circles, $z=k$: parabolas
E: None of A, B, C, D

A3. [*2 marks*] Convert $z = x^2 - y^2$ to spherical coordinates. Recall that we use ϕ to denote the angle from the z-axis and θ to denote the angle from the xz-plane.

A: $\rho\cos\phi = \rho^2\sin^2\phi\,(\sin^2\theta - \cos^2\theta)$	B: $\rho\cos\phi = \rho^2\sin^2\phi\,(\cos^2\theta - \sin^2\theta)$
C: $\rho\cos\theta = \rho^2\sin^2\theta\,(\sin^2\phi - \cos^2\phi)$	D: $\rho\cos\theta = \rho^2\sin^2\theta\,(\cos^2\phi - \sin^2\phi)$
E: None of A, B, C, D	

A4. [*2 marks*] The Taylor polynomial P_4 of degree 4 for

$$f(x,y) = \frac{1}{1-xy}$$

about $(0,0)$ is:

A: $1 + xy + x^2y^2$	B: $1 - xy + x^2y^2$
C: $1 + (1-xy) + (1-xy)^2$	D: $1 - (1-xy) + (1-xy)^2$
E: None of A, B, C, D	

A5. [*2 marks*] Given $f(r,\theta) = e^{r\theta}\cos\theta$, the partial derivative $\dfrac{\partial^2 f}{\partial r\partial\theta}$ at $r=1$, $\theta=0$, is

A: 0	B: 1	C: -1	D: 2	E: None of A, B, C, D

A6. [*2 marks*] The equation of the tangent plane to the surface $z = y^2\ln x$ at $(1,2,0)$ is given by

A: $z = -4x + 4$	B: $z = 4x - 4$	C: $z = 4x + 4$	D: $z = -4x - 4$	E: None of A, B, C, D

A7. [*2 marks*] Given that $yz = e^{x+z}$ implicitly defines z as a function of x and y, the partial derivative $\frac{\partial z}{\partial y}$ is

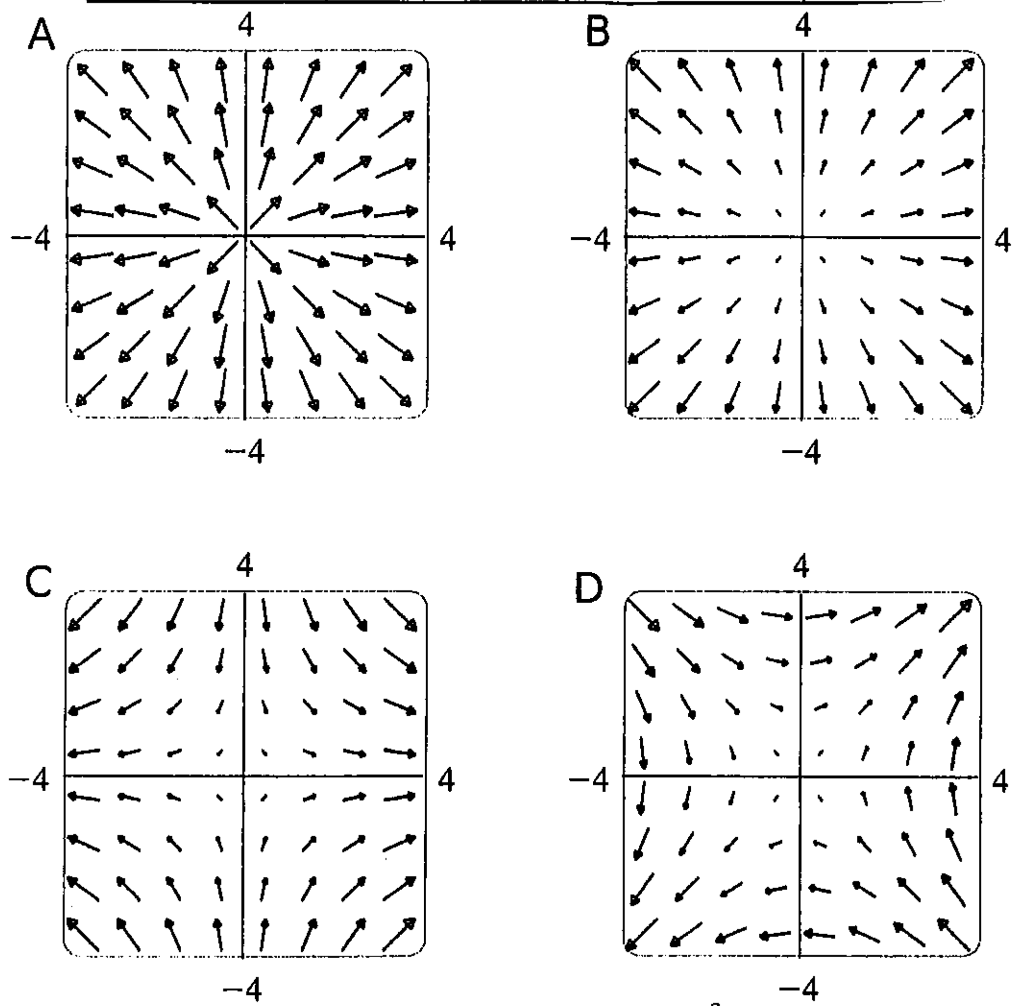
A: $\dfrac{e^{x+z} - z}{y}$	B: $\dfrac{e^{x+z} + z}{y}$	C: $\dfrac{z}{e^{x+z} - y}$	D: $\dfrac{z}{e^{x+z} + y}$	E: None of A, B, C, D

A8. [*2 marks*] Which plane is tangent to the surface $x^2 + 3y^2 + z^2 = 8$ at $(1, -1, 2)$.

A: $-x + 3y + 2z = 1$	B: $x + 3y - 2z = -1$	C: $x + 3y - 2z = 8$
D: $x - 3y + 2z = 8$	E: None of A, B, C, D	

A9. [*2 marks*] Which figure below is the gradient of the function $f(x, y) = \sqrt{x^2 + y^2}$?

A: A	B: B	C: C	D: D



A10. [*2 marks*] Let $\mathbf{F}$ be a vector field defined on all of $\mathbb{R}^3$ and let $f(x, y, z)$ be a function defined on all of $\mathbb{R}^3$. Assume that f and the component functions of $\mathbf{F}$ have continuous partials of all orders. Which of the following statements is **not** always true?

A: if curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \text{grad } g$ for some g
B: if div $\mathbf{F} = 0$, then $\mathbf{F} = \text{curl } \mathbf{G}$ for some $\mathbf{G}$
C: div curl $\mathbf{F} = 0$
D: curl grad $f = \mathbf{0}$
E: div grad $f = 0$

A11. [*2 marks*] Find curl F where $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} - x^2y\,\mathbf{k}$.

A: $-x^2\,\mathbf{i}+3xy\,\mathbf{j}-xz\,\mathbf{k}$	B: $-x^2\,\mathbf{i}-3xy\,\mathbf{j}-xz\,\mathbf{k}$	C: yz
D: $x^2\,\mathbf{i}-3xy\,\mathbf{j}+xz\,\mathbf{k}$	E: None of A, B, C, D	

A12. [*2 marks*] The determinant of the Jacobian matrix for the transformation

$$g(x,y)=(x\cos y, x\sin y)$$

from $\mathbb{R}^2$ to $\mathbb{R}^2$ is:

A: $\cos y$	B: $x(\sin y+\cos y)$	C: x	D: $x\cos y$	E: None of A, B, C, D

A13. [*2 marks*] Find the gradient of the scalar field expressed in cylindrical coordinates by $f(r,\theta,z)=r\cos(\theta)+z\theta$.

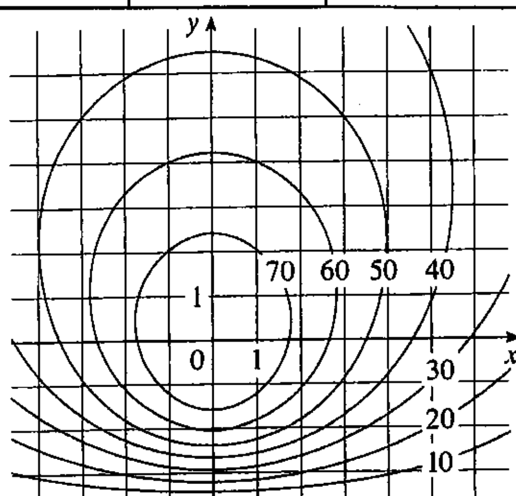
A: $\cos(\theta)\,\mathbf{i}+(z-r\sin\theta)\,\mathbf{j}+\theta\,\mathbf{k}$	B: $\cos(\theta)\,\mathbf{i}+(z/r-\sin\theta)\,\mathbf{j}+\theta\,\mathbf{k}$
C: $\cos(\theta)\,\hat{\mathbf{r}}+(z-r\sin\theta)\,\hat{\boldsymbol{\theta}}+\theta\,\hat{\mathbf{z}}$	D: $\cos(\theta)\,\hat{\mathbf{r}}+(z/r-\sin\theta)\,\hat{\boldsymbol{\theta}}+\theta\,\hat{\mathbf{z}}$
E: None of A, B, C, D	

A14. [*2 marks*] Find the gradient of the scalar field expressed in spherical coordinates by $f(\rho,\phi,\theta)=\rho\phi+\theta$.

A: $\phi\,\mathbf{i}+\mathbf{j}+\dfrac{1}{\rho\sin\phi}\mathbf{k}$	B: $\phi\,\mathbf{i}+\dfrac{1}{\sin\phi}\mathbf{j}+\dfrac{1}{\rho}\mathbf{k}$
C: $\phi\,\hat{\rho}+\hat{\phi}+\dfrac{1}{\rho\sin\phi}\hat{\theta}$	D: $\phi\,\hat{\rho}+\dfrac{1}{\sin\phi}\hat{\phi}+\dfrac{1}{\rho}\hat{\theta}$
E: None of A, B, C, D	

A15. [*2 marks*] A contour map for a function $f(x,y)$ is shown. Which of the following is a reasonable estimate for $f(2,3)$?

A: 51	B: 59	C: 61	D: 69	E: None of A, B, C, D



PART B (70 marks)

NOTE: SHOW ALL YOUR WORK FOR PROBLEMS IN PART B.

B1. Let $\mathbf{r}(t)=\langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle$.

(a) [*3 marks*] Find parametric equations of the tangent line to $\mathbf{r}$ at $t=\pi$.

(b) [*2 marks*] Find the equation of the plane through the point $\mathbf{r}(\pi)$ which is perpendicular to the curve $\mathbf{r}(t)$. Write your answer in the form $ax + by + cz + d = 0$.

(c) [*2 marks*] Find the arc length of $\mathbf{r}(t)$ for $0 \le t \le \pi$.

B2. [*4 marks*] Find the position vector $\mathbf{r}(t)$ of a particle whose acceleration and initial position and velocity are given by

$$\mathbf{a}(t) = 6t\,\mathbf{i} + e^t\,\mathbf{j} + \cos t\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i} + \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} - \mathbf{k}.$$

B3. [*3 marks*] Verify that the function $u(x,y) = \ln x^2 + y^2$, where $(x,y) \neq (0,0)$, satisfies the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

B4. [*4 marks*] Find the linear approximation of the function

$$f(x,y) = \ln(x^2 + y)$$

at $(0,1)$ and use it to approximate $f(0.1, 0.9)$.

B5. [*4 marks*] A function $f(x,y)$ has the following properties:

$$f(1,2) = 1,\ \frac{\partial f}{\partial x}(1,2) = 2,\ \frac{\partial f}{\partial y}(1,2) = 3,\ \frac{\partial^2 f}{\partial x^2}(1,2) = 4,\ \frac{\partial^2 f}{\partial x \partial y}(1,2) = 5,\ \frac{\partial^2 f}{\partial y^2}(1,2) = 6.$$

Find the Taylor polynomial P_2 of degree 2 for $f(x,y)$ **about** $(\mathbf{1},\mathbf{2})$.

B6. [*7 marks*] The temperature at a point (x,y,z) is given by

$$T(x,y,z) = \sqrt{x^2 + 2y^2 + 3z^2}.$$

(a) Find the rate of change of temperature at the point $P(1,0,1)$ in the direction toward the point $Q(3,1,3)$.

(b) Give a vector pointing in the direction that the temperature **decreases** fastest at $P(1,0,1)$.

(c) What is the directional derivative of T at P in the direction you found in part (b)?

B7. [*5 marks*] Find and classify the critical points of $f(x,y) = x^4 + y^4 - 4xy + 2$.

B8. [*3 marks*] Let $M = we^{uv}$, where $u = r + s$, $v = r - s$ and $w = rs$. Using the chain rule, find $\dfrac{\partial M}{\partial s}$ for $r = 3$ and $s = 2$.

B9. [*5 marks*] Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y,z) = 2x + 6y + 10z$ subject to the constraint $x^2 + y^2 + z^2 = 35$. List the points at which the maximum and minimum occur.

B10. [*3 marks*] Sketch the vector field $\mathbf{F}(x,y) = y\,\mathbf{i} + \mathbf{j}$ for all points (x,y) with $x = 0,1,2$ and $y = -1,0,1$ (9 points in total).

B11. [*5 marks*] Determine whether or not the vector field

$$\mathbf{F}(x,y,z) = e^z\mathbf{i} + \mathbf{j} + xe^z\mathbf{k}$$

is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$. In either case, explain carefully.

B12. [*3 marks*] Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$? Explain.

B13. [*4 marks*] Let

$$\mathbf{g}(r,s) = (r + s^2, rs),$$

and

$$\mathbf{f}(x,y) = (e^{x+y}, x).$$

Use the matrix form of the chain rule to compute the Jacobian matrix of the composite function $\mathbf{f} \circ \mathbf{g}$ at the point $(r,s) = (2,3)$.

B14. [*4 marks*] Is the function

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$? At what other points is it continuous? Explain fully.

B15. [*4 marks*] The equation $x + 4y + z + e^{2z} = 1$ has a solution of the form $z = f(x,y)$ near $(x,y) = (0,0)$ with $f(0,0) = 0$. Find the Taylor polynomial P_1 of degree 1 for $f(x,y)$ about $(0,0)$ (this is also called the linearization).

B16. [*5 marks*] Consider the function $f(x,y) = -4 + (x-1)^2 + (y-1)^2$ on the triangular region where $x \geq 0$, $y \geq 0$ and $y \leq 4 - x$. Find the absolute maximum and absolute minimum values of f on this region, and all points at which they occur.