MATH 4155/9063 FALL 2013

HOMEWORK ASSIGNMENT 2. DUE OCTOBER 11.

2.1. Prove that if $U \subset \mathbb{R}$ is an open set, then U admits a unique representation as a countable union of a collection of disjoint open intervals (finite or infinite).

2.2. Suppose that $a_n = 3^{-n-1}$ in the construction of Cantor-type set U for $[0,1]$. In this case the set $C = [0,1] \setminus U$ is called *the Cantor set*.
   (i) Show that χ_U is Riemann integrable.
   (ii) (Math 9063 only) Show that C consists of all numbers in $[0,1]$ which have some triadic (i.e., base 3) expansion in which only 0s and 2s occur. Conclude that C is uncountable, even though its complement in $[0,1]$ has Lebesgue measure 1.

2.3. Let $X \subset \mathbb{R}^n$ be a measurable set. Prove that if a function $f : X \to \mathbb{R}$ vanishes outside a set of measure zero, then $\int_X f(x)dx = 0$ (Lebesgue integral). Start with proving that f is measurable.

2.4. The Monotone Convergence theorem states that if $\{f_n\}$ is an increasing sequence of non-negative measurable functions and $f = \lim f$ a.e., then

$$\int f dx = \lim \int f_n dx.$$

Use this theorem to prove that given a nonnegative integrable f on $\mathbb{R}$, the function

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

is continuous.

2.5. Let $C = C^0[0,1]$ be the space of all continuous function on $[0,1]$, and define

$$||f|| = \max_{x\in[0,1]} |f(x)|.$$

Show that C is a Banach space.