MATH 4155/9063 FALL 2013

HOMEWORK ASSIGNMENT 3. DUE OCTOBER 25.

3.1. Prove that a linear functional ϕ on a topological vector space X is continuous if and only if there exists a neighbourhood of the origin where ϕ is bounded.

3.2. Find the "area" of S^{n-1}, i.e., the $n-1$-dimensional volume of the unit sphere in $\mathbb{R}^n$ for general n.

3.3. For $t \geq 0$ define

$$f(x,t) = \begin{cases} x, \ 0 \leq x \leq \sqrt{t}, \\ -x + 2\sqrt{t}, \ \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0, \quad \text{otherwise}, \end{cases}$$

and put $f(x,t) = -f(x,|t|)$ if $t < 0$. Show that f is continuous on $\mathbb{R}^2$, and $\frac{\partial f}{\partial t}(x,0) = 0$ for all x. Consider

$$F(t) = \int_{-1}^{1} f(x,t)dx.$$

Show that $F(t) = t$ if $|t| < 1/4$, hence,

$$F'(0) \neq \int_{-1}^{1} \frac{\partial f}{\partial t}(x,0)dx.$$

3.4. Prove for $1 \leq p \leq \infty$ the space L^p with the topology given by the norm $||\cdot||_p$ is a topological vector space.

3.5. (For Math 9063 only) Suppose that X is an infinite dimensional normed space. Prove there exists a noncontinuous linear functional on X.