

REAL ANALYSIS LECTURE NOTES

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10. STRUCTURE THEOREMS AND CONVOLUTION OF DISTRIBUTIONS

10.1. **Structure theorems.** We introduce the following property (\star).

Definition 10.1. A linear functional $f : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$ satisfies condition (\star) if for every compact subset K in Ω there exists $C = C(K) > 0$ and a positive integer $k = k(K)$ such that

$$(1) \quad |\langle f, \phi \rangle| \leq C \|\phi\|_{C^k(K)}, \quad \forall \phi \in \mathcal{D}(\Omega), \text{ supp } \phi \subset K.$$

We have the following characterization of distributions.

Theorem 10.2. A linear functional f on the space $\mathcal{D}(\Omega)$ is a distribution if and only if it satisfies condition (\star).

Proof. If f satisfies (\star) then f is clearly continuous, and so $f \in \mathcal{D}'(\Omega)$. To prove the converse, assume that $f \in \mathcal{D}'(\Omega)$. Arguing by contradiction, suppose that f does not satisfy (\star). Then there exists a compact K in Ω such that for every C and k the inequality (1) fails for some $\phi \in \mathcal{D}(\Omega)$ with $\text{supp } \phi \subset K$. In particular, we can set $C = k = j$ and take a function $\phi_j \in \mathcal{D}(\Omega)$ with $\text{supp } \phi \subset K$ such that

$$(2) \quad |\langle f, \phi_j \rangle| > j \|\phi_j\|_{C^j(K)}, \quad j = 0, 1, 2, \dots$$

By linearity of expressions on both sides, this inequality still holds if we replace ϕ_j by the function $\psi_j = \frac{\phi_j}{\langle f, \phi_j \rangle}$. Then

$$1/j > \|\psi_j\|_{C^j(K)}, \quad j = 0, 1, 2, \dots$$

Fix a positive integer k . Then for $j \geq k$ we have

$$\|\psi_j\|_{C^k(K)} \leq \|\psi_j\|_{C^j(K)} < j^{-1} \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Therefore, the sequence (ψ_j) converges to 0 in $\mathcal{D}(\Omega)$ but $\langle f, \psi_j \rangle = 1$. This contradiction proves the theorem. \square

Let Ω be a domain in \mathbb{R}^n , $f \in \mathcal{D}(\Omega)$ and $k \geq 0$ be an integer. We say that a distribution f has the order of singularity $\leq k$ if there exists a constant $C = C(\Omega, f) > 0$ such that for every $\varphi \in \mathcal{D}(\Omega)$ we have

$$|\langle f, \varphi \rangle| \leq C \|\varphi\|_{C^k(\Omega)}.$$

Thus, f satisfies condition (1) with the same k for every compact K in Ω , i.e., k can be chosen independently of K . We say that the order of singularity of f is equal to k if this estimate does not hold for some $k' < k$.

Example 10.1. If T_f is a regular distribution defined by a function $f \in L^1(\Omega)$. Then its order of singularity is 0. \diamond

Example 10.2. The order of singularity of $\delta^{(k)}(x)$ is equal to k . \diamond

The following property of distributions is often used.

Theorem 10.3. *Let Ω' be a domain in \mathbb{R}^n and Ω be a bounded subdomain such that $\bar{\Omega} \subset \Omega'$. Then for every distribution $f \in \mathcal{D}'(\Omega')$ its restriction to Ω is a distribution of finite order of singularity.*

Thus, the theorem claims that there exist an integer $k \geq 0$ (depending on f and Ω) and a constant $C = C(\Omega, f) > 0$ such that for every $\varphi \in \mathcal{D}(\Omega)$ we have

$$|\langle f, \varphi \rangle| \leq C \|\varphi\|_{C^k(\Omega)}$$

The proof is similar to the previous one.

Proof. Arguing by contradiction, suppose that there exists sequence $\varphi_m \in \mathcal{D}(\Omega)$ such that

$$|\langle f, \varphi_m \rangle| \geq m \|\varphi_m\|_{C^m(\Omega)}$$

for every $m = 1, 2, \dots$. Set $\psi_m = \alpha_m \varphi_m$, where α_m is a real number. Then by linearity we still have

$$|\langle f, \psi_m \rangle| \geq m \|\psi_m\|_{C^m(\Omega)}.$$

Fix $\alpha_m = (\|\varphi_m\|_{C^m(\Omega)})^{-1}/m$. Then

$$(3) \quad |\langle f, \psi_m \rangle| \geq m \|\psi_m\|_{C^m(\Omega)} = 1.$$

On the other hand, $\|\psi_m\|_{C^m(\Omega)} = 1/m$ for every m . Then for every β , such that $|\beta| \leq m$, we have

$$\|D^\beta \psi_m\|_{C(\Omega)} \leq 1/m.$$

Thus, the sequence (ψ^m) converges to 0 together with all partial derivatives of all orders and the supports of ψ^m are contained in the compact $\bar{\Omega}$ in Ω' . Then $\psi^m \rightarrow 0$ in $\mathcal{D}(\Omega')$ and $\langle f, \psi_m \rangle \rightarrow 0$: a contradiction to (3). \square

The following consequence of Theorem 10.3 is frequently used.

Proposition 10.4. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy $\text{supp } f = \{0\}$. Then there exist an integer $k \geq 0$ and constants C_α such that*

$$f = \sum_{|\alpha| \leq k} C_\alpha D^\alpha \delta(x).$$

Proof. Let a function $\eta \in \mathcal{D}(\mathbb{R}^n)$ be equal to 1 in a neighbourhood of 0 and vanishes outside $B(0, 1) = \{|x| < 1\}$. Consider a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Let Ω be a domain in \mathbb{R}^n containing $\text{supp } \varphi \cup B(0, 2)$. Applying Theorem 10.3 to f in Ω we conclude that there exist an integer $k \geq 0$ (depending on Ω) and a constant $C = C(\Omega, f) > 0$ such that for every $\phi \in \mathcal{D}(\Omega)$ we have

$$(4) \quad |\langle f, \phi \rangle| \leq C \|\phi\|_{C^k(\Omega)}.$$

Set $h(x) = \varphi(x) - \sum_{|\alpha| \leq k} (D^\alpha \varphi(0) x^\alpha) / \alpha!$ and

$$\psi_s(x) = h(x) \eta(sx)$$

Then for every integer $s \geq 1$ we have

$$(5) \quad \langle f(x), \psi_1 \rangle = \langle f, \psi_s \rangle$$

Indeed, $\langle f, \psi_1 \rangle - \langle f, \psi_s \rangle = \langle f, (\psi_1 - \psi_s)h \rangle = 0$ since $(\psi_1 - \psi_s)h = 0$ in a neighborhood of 0 and $\text{supp } f = \{0\}$. Since $\text{supp } \psi_s \subset \Omega$ for every $s \geq 1$, we obtain that $\psi_s \in \mathcal{D}(\Omega)$ and by (4),

$$|\langle f, \psi_s \rangle| \leq C \|\psi_s\|_{C^k(\Omega)}, \quad s \geq 1.$$

It follows easily from the definition of ψ_s that $\|\psi_s\|_{C^k(\Omega)} \rightarrow 0$ as $s \rightarrow \infty$. But then (5) implies that $\langle f(x), \psi_1 \rangle = 0$. Therefore,

$$\langle f, \varphi \eta \rangle = \sum_{|\alpha| \leq k} (\langle f, x^\alpha \eta \rangle / \alpha!) D^\alpha \varphi(0) = \sum_{|\alpha| \leq k} C_\alpha \langle D^\alpha \delta(x), \varphi \rangle,$$

where $C_\alpha = (\langle f, x^\alpha \eta \rangle / \alpha!)$ are independent of φ . □

Example 10.3. Let a function $f \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ satisfy the following condition: there exists a constant $C > 0$ and an integer $m > 0$ such that

$$(6) \quad |f(x)| \leq \frac{C}{|x|^m}, \quad \forall x \in \{x \in \mathbb{R}^n : |x| \leq 1\}.$$

We will show that f admits an extension past the origin as a distribution, i.e., there exists $\tilde{f} \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\langle \tilde{f}, \varphi \rangle = \int_{\mathbb{R}^n} f(x) \varphi(x) dx$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } \varphi \subset \mathbb{R}^n \setminus \{0\}$.

First of all let us recall the general Taylor formula: let $a \in \mathbb{R}^n$ and $\psi \in C^\infty$ in a neighborhood of a . Then for every integer $k \geq 0$ there exists a neighborhood U of a such that for $x \in U$ we have

$$\psi(x) = \sum_{0 \leq |\alpha| \leq k} \frac{1}{\alpha!} D^\alpha \psi(a) (x - a)^\alpha + \int_0^1 (1-t)^k \sum_{|\alpha|=k+1} \frac{k+1}{\alpha!} D^\alpha \psi(tx + (1-t)a) (x - a)^\alpha dt$$

As usual we use here the notation $\alpha! = \alpha_1! \dots \alpha_n!$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Now set

$$\langle \tilde{f}, \varphi \rangle = I_1 + I_2$$

where

$$I_1 = \int_{|x| \geq 1} f(x) \varphi(x) dx,$$

$$I_2 = \int_{|x| \leq 1} f(x) \left(\varphi(x) - \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} D^\alpha \psi(0) x^\alpha \right) dx.$$

Using the Taylor formula and condition (6) we obtain

$$|I_2| \leq C \sum_{|\alpha|=m} \sup_{\mathbb{R}} |D^\alpha \varphi|$$

Using the condition $\text{supp } \varphi \subset \{x : |x| \leq M\}$ we also obtain

$$|I_1| \leq \int_{1 \leq |x| \leq M} |f(x) \varphi(x)| dx \leq C' \sup_{\mathbb{R}^n} |\varphi|.$$

From this and Theorem 10.2 we conclude that \tilde{f} is a correctly defined distribution from $\mathcal{D}'(\mathbb{R}^n)$. ◇

Finally, consider an example of a distribution of infinite order.

Example 10.4. Consider a linear functional on $\mathcal{D}(\mathbb{R})$ defined by

$$\langle f, \varphi \rangle = \sum_{n=0}^{\infty} \varphi^{(n)}(n).$$

It follows by Theorem ?? that $f \in \mathcal{D}'(\mathbb{R})$. We leave to the reader to prove that f is not of a finite order, arguing by contradiction. ◇

10.2. Regularization and convolution with test-functions. Recall that a convolution $f * g$ of two functions $L^2(\mathbb{R}^n)$ is defined by

$$(7) \quad f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

This makes natural the following general definition.

Definition 10.5. A convolution of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ and a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is defined by

$$(8) \quad f * \varphi(x) = \langle f(y), \varphi(x-y) \rangle.$$

This definitions deserves some remarks. For every $x \in \mathbb{R}^n$ the function $y \mapsto \varphi(x-y)$ is a test-function; on the right-hand side of (8) we apply distribution f to this function which is stressed by the notation $f(y)$. Thus, $f * \varphi$ is defined as a usual function on \mathbb{R}^n .

Proposition 10.6. We have $f * \phi \in C^\infty(\mathbb{R}^n)$ and

$$(9) \quad D^\alpha(f * \phi) = f * D^\alpha\phi = (D^\alpha f) * \phi.$$

Proof. The regularity of $f * \varphi$ and the first equality of (9) follow from Theorem ???. Let us prove the second equality in (9). We have

$$\begin{aligned} \left(\frac{\partial}{\partial x_j} f\right) * \varphi(x) &= \left\langle \frac{\partial}{\partial y_j} f(y), \varphi(x-y) \right\rangle = -\left\langle f(y), \frac{\partial}{\partial y_j}(\varphi(x-y)) \right\rangle \\ &= \left\langle f(y), \left(\frac{\partial}{\partial x_j} \varphi\right)(x-y) \right\rangle = f * \frac{\partial}{\partial x_j} \varphi \end{aligned}$$

The rest of the proof is done by induction. □

A very important special case arises if we take the bump function ω_ε as φ in the definition of convolution. This leads to

Definition 10.7. The convolution $f_\varepsilon := f * \omega_\varepsilon$ is called the regularization of a distribution f .

Proposition 10.8. We have

- (i) $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.
- (ii) $(D^\alpha f)_\varepsilon = D^\alpha(f_\varepsilon)$.
- (iii) If $f \in C(\mathbb{R}^n)$ then $f_\varepsilon \rightarrow f, \varepsilon \rightarrow 0+$ in $C(\Omega)$ for every bounded subset Ω of \mathbb{R}^n .
- (iv) If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi_\varepsilon \rightarrow \varphi$ in $\mathcal{D}(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0+$.
- (v) if $f \in \mathcal{D}'(\mathbb{R}^n)$ then $f_\varepsilon \rightarrow f$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0+$.

Proof. Parts (i) and (ii) follow from Proposition 10.6. Part (iii) is established in Proposition ???, so it remains to show (iv) and (v). If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then its support is compact, say, $\varphi(x) = 0$ when $|x| \geq A$ for some $A > 0$. Then (??) shows that $\varphi_\varepsilon(x) = 0$ for $|x| \geq A + \varepsilon$ so $\text{supp } \varphi_\varepsilon \subset K = \{x \in \mathbb{R}^n : |x| \leq A + 1\}$ for $\varepsilon < 1$. It follows now from (iii) and (ii) that φ_ε converges to φ as $\varepsilon \rightarrow 0+$ uniformly on K together with all partial derivatives of any order. Hence $\varphi_\varepsilon \rightarrow \varphi, \varepsilon \rightarrow 0+$ in $\mathcal{D}(\mathbb{R}^n)$ and we obtain (iv).

To prove (v), we view $f_\varepsilon = \langle f(y), \omega_\varepsilon(x-y) \rangle$ as a distribution acting on every $\psi \in \mathcal{D}(\mathbb{R}^n)$ by

$$\langle f_\varepsilon, \psi \rangle = \int_{\mathbb{R}^n} \langle f(y), \omega_\varepsilon(x-y) \rangle \psi(x) dx.$$

It follows from Theorem ??? that

$$(10) \quad \int_{\mathbb{R}^n} \langle f(y), \omega_\varepsilon(x-y) \rangle \psi(x) dx = \langle f(y), \int_{\mathbb{R}^n} \omega_\varepsilon(x-y) \psi(x) dx \rangle = \langle f, \psi_\varepsilon \rangle.$$

Indeed, consider for simplicity of notations the case $n = 1$. Introduce the functions

$$F(t) = \int_{-\infty}^t \langle f(y), \omega_\varepsilon(x - y) \rangle \psi(x) dx,$$

and

$$G(t) = \langle f(y), \int_{-\infty}^t \omega_\varepsilon(x - y) \psi(x) dx \rangle.$$

Then by Theorem ??, $F'(x) = G'(x) = \langle f(y), \omega_\varepsilon(x - y) \rangle \psi(x)$. Since $F(-\infty) = G(-\infty) = 0$, we have $F = G$ for all t and we pass to the limit as $t \rightarrow +\infty$. This proves (10). Then by (iv),

$$\langle f_\varepsilon, \psi \rangle = \langle f, \psi_\varepsilon \rangle \longrightarrow \langle f, \psi \rangle \text{ as } \varepsilon \longrightarrow 0+.$$

This concludes the proof. □

As an application we give another proof of Corollary ??: if $f \in \mathcal{D}'(\mathbb{R}^n)$ are such that $\frac{\partial}{\partial x_j} f = 0$ in $\mathcal{D}'(\mathbb{R}^n)$ for $j = 1, \dots, n$, then $f = \text{const}$.

Proof of Corollary ??. For every $\varepsilon > 0$ we have $0 = (\frac{\partial}{\partial x_j} f)_\varepsilon = \frac{\partial}{\partial x_j} (f_\varepsilon)$. Since f_ε is a usual function of class C^∞ , we conclude that $f_\varepsilon = C(\varepsilon)$. Then,

$$\langle f, \varphi \rangle = \lim_{\varepsilon \rightarrow 0+} \langle f_\varepsilon, \varphi \rangle = \lim_{\varepsilon \rightarrow 0+} C(\varepsilon) \int \varphi(x) dx \text{ for any } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

In particular, set $\varphi = \omega(x)$ so that $\int \omega(x) dx = 1$. We obtain that $C = \lim_{\varepsilon \rightarrow 0+} C(\varepsilon)$ exists, and $f = C$. □

10.3. Convolution of distributions. Let f and g be functions in $L^2(\mathbb{R}^n)$. Their convolution $f * g$ can be defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(y) g(x - y) dy$$

For a moment assume that $f * g$ is in $L^1_{loc}(\mathbb{R}^n)$. Then it defines a regular distribution acting on $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by

$$\langle f * g(x), \varphi(x) \rangle = \int_{\mathbb{R}^n} f * g(x) \varphi(x) dx = \int \left(\int_{\mathbb{R}^n} f(y) g(x - y) dy \right) \varphi(x) dx.$$

By Fubini's theorem,

$$\begin{aligned} \int \left(\int_{\mathbb{R}^n} f(y) g(x - y) dy \right) \varphi(x) dx &= \int f(y) \left(\int g(x - y) \varphi(x) dx \right) dy \\ &= \int f(y) \left(\int g(t) \varphi(t + y) dt \right) dy = \int f(y) \langle g(t), \varphi(t + y) \rangle dy \\ &= \langle f(y), \langle g(t), \varphi(t + y) \rangle \rangle. \end{aligned}$$

Thus,

$$(11) \quad \langle f * g(x), \varphi(x) \rangle = \langle f(y), \langle g(t), \varphi(t + y) \rangle \rangle, \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Therefore, in the general case of arbitrary distributions $f, g \in \mathcal{D}'(\mathbb{R}^n)$ it is natural to take equality (11) as a *definition* of the convolution $f * g$. However, the right hand side of equality (11) is not defined for arbitrary distributions f and g since the function $y \mapsto \langle g(t), \varphi(t + y) \rangle$ is just of class $C^\infty(\mathbb{R}^n)$ and in general need not have compact support. The support is clearly compact if the distribution g has compact support. So in this case the convolution is well-defined. Similarly, if f

has compact support then it acts on any function from $C^\infty(\mathbb{R}^n)$ (see previous subsection) so the right hand of equality (11) is also defined. We summarize this in the following.

Proposition 10.9. *The convolution $f * g$ of two distributions $f, g \in \mathcal{D}'(\mathbb{R}^n)$ is a distribution correctly defined by the equality (11) if at least one of the distributions f and g has compact support.*

Example 10.5. For any distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$\langle f * \delta, \varphi \rangle = \langle f(y), \langle \delta(t), \varphi(t + y) \rangle \rangle = \langle f(y), \varphi(y) \rangle,$$

that is $f * \delta = f$. Furthermore

$$\langle \delta * f, \varphi \rangle = \langle \delta(y), \langle f(t), \varphi(t + y) \rangle \rangle = \langle f(t), \varphi(t) \rangle,$$

so that $\delta * f = f$. We obtain the following fundamental identity

$$f * \delta = \delta * f = f$$

for any $f \in \mathcal{D}'(\mathbb{R}^n)$. \diamond

We conclude this section by some algebraic properties of convolution.

(1) *The map $(f, g) \mapsto f * g$ is bilinear.* This is obvious.

(2) *We have*

$$(12) \quad D^\alpha(f * g) = f * D^\alpha g = D^\alpha f * g.$$

For the proof we consider $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\begin{aligned} \langle D^\alpha(f * g), \varphi \rangle &= (-1)^{|\alpha|} \langle f * g, D^\alpha \varphi \rangle = \langle f(y), \langle g(t), (-1)^{|\alpha|} (D^\alpha \varphi)(t + y) \rangle \rangle \\ &= \langle f(y), \langle D^\alpha g(t), \varphi(t + y) \rangle \rangle = \langle f * D^\alpha g, \varphi \rangle \end{aligned}$$

which proves the first equality of (12). For the second, we observe that $\langle D^\alpha g(t), \varphi(t + y) \rangle = (D^\alpha g) * \varphi(-y)$; hence it follows from (9) that

$$\langle D^\alpha g(t), \varphi(t + y) \rangle = (-1)^{|\alpha|} D^\alpha \langle g(t), \varphi(t + y) \rangle.$$

Therefore,

$$\begin{aligned} \langle f(y), \langle D^\alpha g(t), \varphi(t + y) \rangle \rangle &= (-1)^{|\alpha|} \langle f(y), D^\alpha \langle g(t), \varphi(t + y) \rangle \rangle \\ &= \langle D^\alpha f(y), \langle g(t), \varphi(t + y) \rangle \rangle = D^\alpha f * g \end{aligned}$$

A simpler proof can be given using Theorem ??:

$$\begin{aligned} (-1)^{|\alpha|} \langle f(y), \langle g(t), (D^\alpha \varphi)(t + y) \rangle \rangle &= (-1)^{|\alpha|} \langle f(y), D^\alpha \langle g(t), \varphi(t + y) \rangle \rangle \\ &= \langle D^\alpha f(y), \langle g(t), \varphi(t + y) \rangle \rangle \end{aligned}$$