REAL ANALYSIS LECTURE NOTES

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7. The space of test functions

7.1. **Space of test functions.** Let Ω be an open subset of $\mathbb{R}^n$. For an integer $k \geq 0$ we denote by $C^k(\Omega)$ the linear space of functions $h : \Omega \longrightarrow \mathbb{R}$ that have continuous partial derivatives up to order k on Ω. As usual we use the notation $C^\infty(\Omega) = \cap_{k=0}^\infty C^k(\Omega)$ for the space of real functions admitting the continuous partial derivatives of any order. The standard topology on $C^\infty(\Omega)$ can be defined by the following notion of convergence: we say that a sequence $\{h^k\} \subset C^\infty(\Omega)$ converges as $k \longrightarrow \infty$ to a function h in $C^\infty(\Omega)$ if h^k converges to h uniformly with all partial derivatives of any order on any compact subset of Ω. If K is a compact subset of Ω, we use the standard norm

$$\| u \|_{C^k(K)} = \sum_{|\alpha| \leq k} \sup_{x \in K} |D^\alpha u(x)|$$

for $u \in C^k(\Omega)$.

Recall that $supp\, h$, the *support* of a function $h \in C^0(\mathbb{R}^n)$, is defined to be the closure in $\mathbb{R}^n$ of the set $\{x \in \Omega : h(x) \neq 0\}$. Denote by $C_0^\infty(\Omega)$ the subspace of $C^\infty(\mathbb{R}^n)$ which consists of functions h such that $supp\, h \subset \Omega$ is a compact subset of $\mathbb{R}^n$.

Our main goal is to introduce and to study the space of continuous linear functionals on the linear vector space $C_0^\infty(\Omega)$ so we must choose certain topology on $C_0^\infty(\Omega)$. The most convenient way (which is sufficient for all our considerations) is just to define the notion of convergence of a sequence of elements in $C_0^\infty(\Omega)$ and use it in order to define the continuity of linear functionals. We say that *a sequence of functions (φ_j) of class $C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$* if the following conditions hold:

(i) There exists a compact subset K such that $K \subset \Omega$ and $\varphi_j = 0$ on $\mathbb{R}^n \backslash K$ for every $j = 0, 1, ...$ In other words, $supp\, \varphi_j \subset K \subset \Omega$ for every j.

(ii) For every α the sequence $D^\alpha \varphi_j$ converges to $D^\alpha \varphi$ uniformly on K. That is all partial derivatives of φ_j of all orders converge uniformly to corresponding partial derivatives of φ.

Definition 7.1. *We denote by $\mathcal{D}(\Omega)$ the space $C_0^\infty(\Omega)$ equipped with the above topology of convergence of sequences. The elements of this space are called test-functions.*

We leave as an exercise for the reader to verify that $\mathcal{D}(\Omega)$ is a topological vector space.

Example 7.1. Denote by $|x| = \sqrt{|x_1|^2 + ... + |x_n|^2}$ the Euclidean norm on $\mathbb{R}^n$. Set

$$\omega_\varepsilon(x) = \begin{cases} C_\varepsilon \, e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon. \end{cases}$$

Here the constant C_ε is determined by the condition

$$\int_{\mathbb{R}^n} \omega_\varepsilon(x) dx = 1$$

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that is,

$$C_\varepsilon = \varepsilon^{-n} \left(\int_{B(0,1)} e^{\frac{-1}{1-|t|^2}} dt \right)^{-1},$$

where $B(0,1)$ denotes the Euclidean unit ball in $\mathbb{R}^n$. This is a typical example of a function from $\mathcal{D}(\mathbb{R}^n)$ (the so-called *bump function*). Observe that

$$\omega_\varepsilon(x) = \varepsilon^{-n} \omega_1(x/\varepsilon).$$

◇

In what follows we write $\omega(x)$ instead of $\omega_1(x)$. The bump function allows us to construct suitable test-functions for an arbitrary domain Ω in $\mathbb{R}^n$. Denote by $\Omega^\varepsilon := \cup_{x\in\Omega} B(x,\varepsilon)$, the ε-neighbourhood of Ω.

Lemma 7.2. *Given compact subset $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exists a function $\eta : \mathbb{R}^n \longrightarrow [0,1]$ of class $C^\infty(\mathbb{R}^n)$ such that*

$$\eta(x) = 1, \quad \text{for } x \in \Omega^\varepsilon,$$
$$\eta(x) = 0, \quad \text{for } x \in \mathbb{R}^n \backslash \Omega^{3\varepsilon}.$$

Proof. Let $\lambda(x) = 1$, $x \in \Omega^{2\varepsilon}$, $\lambda(x) = 0$, $x \in \mathbb{R}^n \backslash \Omega^{2\varepsilon}$ be the characteristic function of $\Omega^{2\varepsilon}$. Then the function defined by the convolution integral:

$$\eta(x) = \int_{\mathbb{R}^n} \lambda(y) \omega_\varepsilon(x-y) dy$$

satisfies the required conditions.   □

Corollary 7.3. *Let Ω be a bounded domain and Ω' be its subset such that $\overline{\Omega'} \subset \Omega$. Then there exists a function $\eta : \mathcal{D}(\Omega) \longrightarrow [0,1]$ such that $\eta(x) = 1$, $x \in \Omega'$.*

We point out that the introduced above topology on the space $\mathcal{D}(\Omega)$ is not metrisable. In other words, one can not define a distance d on the space of test functions such that the convergence with respect to d is equivalent to the introduced above convergence of a sequence of test functions. To see this, recall the following elementary assertion.

Claim. *Let $(u_k^{(m)})_k$ be a countable family of sequences in a metric space (X,d). Suppose that for every $m = 0, 1, \ldots$ the sequence $(u_k^{(m)})$ admits the limit $u^m := \lim_{k\to\infty} u_k^{(m)}$ and the sequence of these limits $(u^m)_m$ also tends to $u \in X$ as $m \to \infty$. Then for every m there exists k_m such that $k_{m-1} < k_m$ and the sequence $(u_{k_m}^{(m)})$ converges to u.*

We leave an easy proof as an exercise. Consider now in $\mathcal{D}(\mathbb{R})$ the functions

$$\varphi_k^{(m)}(x) = \begin{cases} \frac{1}{k} e^{-\frac{m^2}{m^2-x^2}}, & \text{for } |x| \leq m, \\ 0, & \text{for } |x| \geq m. \end{cases}$$

Then for every m we have $\lim_{k\to\infty} u_k^{(m)} = 0$ in $\mathcal{D}(\mathbb{R})$. But for any choice of k_m the sequence $(u_{k_m}^{(m)})$ does not converge in $\mathcal{D}(\mathbb{R})$ since the supports of these test functions are not uniformly bounded. This shows that the topology of $\mathcal{D}(\mathbb{R}^n)$ is not metrisable.

It is not difficult to describe a topology on $\mathcal{D}(\mathbb{R}^n)$ defining the inroduced above convergence. Consider the case $n = 1$, the general case is similar. Given m consider $m + 1$ positive continuous functions γ_j. Define a neighbourhood of the origin in $\mathcal{D}(\mathbb{R})$ as the set of test-functions ϕ satisfying

$|\phi_j^{(k)}(x)| \leq \gamma_j(x)$, $x \in \mathbb{R}$. Then the convergence in this topology precisely coincides with the introduced above convergence on $\mathcal{D}(\mathbb{R})$. We leave the elementary proof as an exercise.

7.2. **Regularization of functions.** Recall that the convolution $f*g$ of two functions $f, g \in L^2(\mathbb{R}^n)$ is defined by the integral

$$(1) \qquad f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

The convolution has particularly nice properties when the second factor is a test function. Denote by $L^1_{loc}(\Omega)$ the space of locally Lebesgue-integrable functions on Ω. Recall that a measurable function f is in $L^1_{loc}(\Omega)$ if and only if $\int_X |f(x)|dx < \infty$ for every compact measurable subset $X \subset \Omega$. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then for every $x \in \mathbb{R}^n$ the function $y \mapsto \varphi(x-y)$ is a test-function and so the convolution

$$(2) \qquad f * \varphi(x) = \int_{\mathbb{R}^n} f(y)\varphi(x-y)dy$$

is defined point-wise as a usual function on the whole $\mathbb{R}^n$.

Proposition 7.4. *We have $f * \varphi \in C^\infty(\mathbb{R}^n)$ and*

$$(3) \qquad D^\alpha(f * \phi) = f * D^\alpha \varphi.$$

Proof. (a) Let us show that the convolution $f * \varphi$ is a continuous function. Let $x^k \in \mathbb{R}^n$ be a sequence converging to x. Then $\varphi(x^k - y) \longrightarrow \varphi(x-y)$, as $k \to \infty$ everywhere, and

$$f * \varphi(x^k) = \int f(y)\varphi(x^k - y)dy \longrightarrow \int f(y)\varphi(x-y)dy = (f * \varphi)(x), \text{ as } k \to \infty,$$

by the Lebesgue Convergence theorem.

(b) Denote by e_j, $j = 1, ..., n$, the vectors of the standard base of $\mathbb{R}^n$. For a fixed $x \in \mathbb{R}^n$ we have

$$\frac{1}{t}(\varphi(x + te_j - y) - \varphi(x-y)) \longrightarrow \frac{\partial}{\partial x_j}(\varphi(x-y)), \ t \to 0$$

everywhere (this follows from the Mean Value theorem applied to the function $\varphi(x + te_j - y)$ on $[0, t]$). Therefore by the Lebesgue Convergence theorem

$$\frac{1}{t}(f * \varphi(x + te_j) - f * \varphi(x)) = \int f(y)\frac{1}{t}(\varphi(x + te_j - y) - \varphi(x-y))dy \longrightarrow \int f(y)\left(\frac{\partial}{\partial x_j}\varphi\right)(x-y)dy$$

Hence, the partial derivative of $f * \varphi$ in x_j exists and

$$\frac{\partial}{\partial x_j}f * \varphi = \int f(y)\left(\frac{\partial}{\partial x_j}\varphi\right)(x-y)dy = f * \left(\frac{\partial}{\partial x_j}\varphi\right).$$

Since $\frac{\partial}{\partial x_j}\varphi \in \mathcal{D}(\mathbb{R}^n)$, it follows by (a) that the partial derivative $\frac{\partial}{\partial x_j}(f * \varphi)$ is continuous. Proceeding by induction, we obtain that $f * \varphi \in C^\infty(\mathbb{R}^n)$ and satisfies (3). $\qquad \square$

A special case very important for applications arises if we take the hat-function ω_ε as φ in the definition of convolution. Then we obtain

Definition 7.5. *The convolution $f_\varepsilon := f * \omega_\varepsilon$ is called the regularization of a function $f \in L^1_{loc}(\mathbb{R}^n)$.*

Proposition 7.6. *We have*
(i) *$f_\varepsilon \in C^\infty(\mathbb{R}^n)$;*
(ii) *if $f \in C(\mathbb{R}^n)$ then $f_\varepsilon \longrightarrow f$, as $\varepsilon \to 0+$, in $C(\overline{\Omega})$ for every bounded subset Ω of $\mathbb{R}^n$;*

(iii) *if* $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, *then* $f_\varepsilon \longrightarrow f$, *as* $\varepsilon \longrightarrow 0+$, *in* $L^p(\mathbb{R}^n)$.

Proof. Part (i) follows from Proposition 7.4. To show (ii), assume that f is a continuous function on $\mathbb{R}^n$. Then

$$(4) \qquad f_\varepsilon(x) = \int_{\mathbb{R}^n} \omega_\varepsilon(x-y) f(y) dy = \int_{\mathbb{R}^n} \omega_\varepsilon(y) f(x-y) dy = \int_{|y| \leq \varepsilon} \omega_\varepsilon(y) f(x-y) dy,$$

and

$$\int_{\mathbb{R}^n} \omega_\varepsilon(x) dx = 1.$$

Then

$$\left| \int_{\mathbb{R}^n} \omega_\varepsilon(y) f(x-y) dy - f(x) \right| = \left| \int_{\mathbb{R}^n} \omega_\varepsilon(y) (f(x-y) - f(x)) dy \right| \leq M(\varepsilon) \int_{|y| \leq \varepsilon} \omega_\varepsilon(y) dy = M,$$

where

$$M(\varepsilon) = \sup_{x \in \Omega, |y| \leq \varepsilon} |f(x-y) - f(x)|.$$

Since f is continuous on the compact subset $\overline{\Omega^\varepsilon}$ of $\mathbb{R}^n$, it is uniformly continuous there so that $M(\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0+$. This proves (ii).

Finally, for the proof of (iii) we need to show that $\| f_\varepsilon - f \|_{L^p} \longrightarrow 0$, as $\varepsilon \longrightarrow 0+$. Here we use the notation $\| h \|_{L^p} = \| h \|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$.

Lemma 7.7. *For every* $\varepsilon > 0$ *we have*

$$\| f_\varepsilon \|_{L^p} \leq \| f \|_{L^p}$$

Proof. Consider first the case $p = 1$. By Fubini's theorem

$$\| f_\varepsilon \|_{L^1} = \int |f * \omega_\varepsilon(x)| dx \leq \int \int |f(y)| \omega_\varepsilon(x-y) dy dx = \int |f(y)| \int \omega_\varepsilon(x-y) dx dy$$

$$= \int |f(y)| dy = \| f \|_{L^1}$$

Let now $p > 1$. For p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ we have by Hölder's inequality,

$$|f * \omega_\varepsilon|^p \leq \left(\int |f(y)| \omega_\varepsilon(x-y) dy \right)^p = \left(\int |f(y)| \omega_\varepsilon(x-y)^{1/p} \omega_\varepsilon(x-y)^{1/q} dy \right)^p$$

$$\leq \left(\int |f(y)|^p \omega_\varepsilon(x-y) dy \right) \left(\int \omega_\varepsilon(x-y) dy \right)^{p/q} = \int |f(y)|^p \omega_\varepsilon(x-y) dy.$$

Therefore by Fubini's theorem

$$\left(\int |f * \omega_\varepsilon(x)|^p dx \right)^{1/p} \leq \left(\int \left(\int |f(y)|^p \omega_\varepsilon(x-y) dy \right) dx \right)^{1/p}$$

$$= \left(\int |f(y)|^p \left(\int \omega_\varepsilon(x-y) dx \right) dy \right)^{1/p} = \left(\int |f(y)|^p dy \right)^{1/p}$$

which proves lemma. $\square$

Now let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. By the classical theory of Lebesgue integral given $\tau > 0$ there exists $A > 0$ and a function $h \in C(\mathbb{R}^n)$ with $supp\, h \subset \{|x| \leq A\}$ such that

$$\| f - h \|_{L^p} \leq \tau$$

Then by Lemma 7.7 we have

$$\| f_\varepsilon - h_\varepsilon \|_{L^p} = \| (f - h)_\varepsilon \|_{L^p} \leq \| f - h \|_{L^p} \leq \tau$$

for every ε. Furthermore, $supp\, h_\varepsilon \subset K = \{|x| \leq A+1\}$ for ε small enough. Hence, by (ii) $h_\varepsilon \longrightarrow h$, $\varepsilon \longrightarrow 0+$ in $C(K)$. Hence

$$\| h - h_\varepsilon \|_{L^p} \leq \tau$$

for all ε small enough. Thus

$$\| f - f_\varepsilon \|_{L^p} \leq \| f - h \|_{L^p} + \| h - h_\varepsilon \|_{L^p} + \| h_\varepsilon - f_\varepsilon \|_{L^p} \leq 3\tau$$

which concludes the proof of Proposition. $\qquad\square$

7.3. **Partition of unity.** Recall some terminology. A family of open subsets $(U_\alpha)_{\alpha_i n A}$ in $\mathbb{R}^n$ is called a *covering* of a subset $X \in \mathbb{R}^n$ if $X = \cup_{\alpha \in A} U_\alpha$. A covering is called *locally finite* if every point $x \in X$ admits a neighborhood V such that the intersection $V \cap U_\alpha$ is not empty only for a finite subset of α in A. A covering $(V_\beta)_{\beta \in B}$ is called a *subcovering* of $(U_\alpha)_{\alpha \in A}$ if for every $\beta \in B$ there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$. We denote by $\dot{X}$ the interior of X.

Proposition 7.8. *Let X be an open subset of $\mathbb{R}^n$ and (U_α) be its covering. Then there exists a locally finite subcovering (V_β) of (U_α) with the following property: for every β the set V_β contains the closure of some ball $B(p_\beta, r_\beta)$ and these open balls $(B(p_\beta, r_\beta)$ form a locally finite subcovering $(W_\gamma)_{\gamma \in \Gamma}$ of (V_β).*

Proof. There exists a sequence K_j of compact subsets of X such that

(i) $X_j \subset \dot{X}_{j+1}$
(ii) $X = \cup_{j=1}^\infty K_j$.

We set

$$K_j = \{z \in X : |x| \leq j,\ \mathrm{dist}(x, \partial X) \geq 1/j\}.$$

Since $K_i \backslash \dot{K}_{i-1}$ is compact, one can choose $U_{\alpha_1}^i, ..., U_{\alpha_{k_i}}^i$ a finite subset of (U_α) with $(K_i \backslash \dot{K}_{i-1}) \subset U_\cup U_{\alpha_j}^i$. Set $V_j^i = U_{\alpha_j}^i \cap K_i \backslash \dot{K}_{i-2}$. Then (V_j^i) is a locally finite subcovering. For every $p \in V_\beta$ consider a ball $B(p, r(p))$ contained in V_β for all β with $p \in V_\beta$. Then this family of balls form a subcovering of (V_β) and we apply a previous construction extracting a locally finite subcovering by balls. $\qquad\square$

Corollary 7.9. *Let X be an open subset of $\mathbb{R}^n$ and (U_α) be its covering. Then there exists a locally finite subcovering (W_γ) and a family of functions $\eta_\gamma \in C_0^\infty(W_\gamma)$ with the following properties:*

(i) $\eta_\gamma(x) \geq 0$ *for every $x \in X$ and every γ.*
(ii) $\sum_\gamma \eta_\gamma(x) = 1$ *for every $x \in X$.*

Proof. Consider locally finite subcoverings (W_γ) and (V_β) constructed in previous proposition. Fix γ. By construction, there exists a finite number of β such that the closed ball $\overline{W_\gamma}$ is contained in V_β. By Corollary **??** there exists a function $\phi_\gamma \in C_0^\infty(\cap_{W_\gamma \subset V_\beta} V_\beta)$ valued in $[0, 1]$ and equal to 1 on W_γ. Hence $\phi(x) = \sum_\gamma \phi_\gamma(x)$ is correctly defined for every $x \in X$ and $\phi(x) > 0$. Now set $\eta_\gamma(x) = \phi_\gamma(x)/\phi(x)$. $\qquad\square$

The family (η_γ) is called *a partition of unity* subordinated to the covering (W_γ).