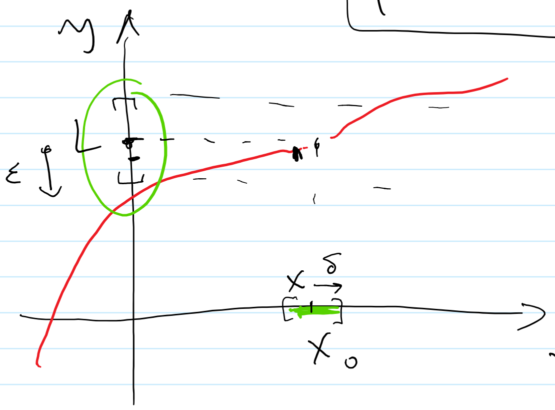
$\lim_{x \to x_0} f(x) = L$ means: "epsilon"

For any $\boxed{\varepsilon > 0}$ there exists a number

delta $\to$ $\delta > 0$ such that $\boxed{|f(x) - L| < \varepsilon}$

whenever $\boxed{|x - x_0| < \delta}$.




f(x) is within distance ε from L.

x is within dist δ from $x_0$

$$\left[\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.\right]$$

"implies"

Example: $\boxed{\lim_{x \to 0} \overset{f}{x^2} = \overset{L}{0}.}$

Proof: Suppose $\varepsilon > 0$ is given. Then

$|\underset{x^2}{f(x)} - \underset{0}{L}| < \varepsilon$ $\qquad |x^2| < \varepsilon$

Take $\boxed{\delta = \sqrt{\varepsilon}.}$

Then if $|x| < \overset{\delta}{\sqrt{\varepsilon}} \Rightarrow |x^2| = |x|^2 < \varepsilon$.

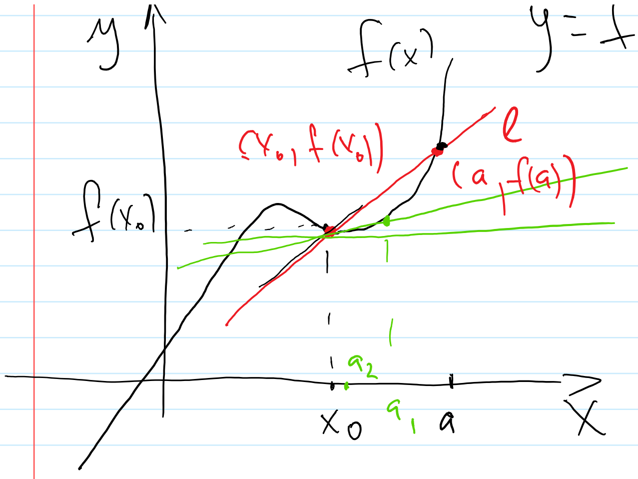
$\square$

## 2.7 Derivatives and rate of change.

y

f(x), $y = f(x)$

secant line




secant line

slope of $\ell$:

$$\boxed{m = \frac{f(a) - f(x_0)}{a - x_0}} \quad \| \ (\star)$$

equation of $\ell$

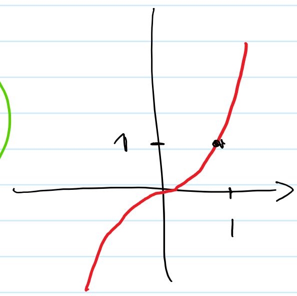
$y - y_0 = m(x - x_0)$

$$\boxed{y - f(x_0) = m(x - x_0)} \quad \| \ (\star\star)$$

What happens if $a \to x_0$. One can imagine that the secant line $\ell$ will approach some line that passes through $(x_0, f(x_0))$ and "touch" the graph of $f$ at that point. This line is called the tangent line to the graph of $f(x)$ at $x = x_0$.

What is then the slope of the tangent line? From $(\star)$:

$$\boxed{m_{\text{tangent}} = \lim_{a \to x_0} \frac{f(a) - f(x_0)}{a - x_0}}$$




Example: $y = x^3$ $\quad x_0 = 1$

Find equation of the tangent line to $y = x^3$ at $x_0 = 1$.

The slope is $= \lim_{a \to 1} \frac{a^3 - 1}{a - 1} = \lim_{a \to 1} \frac{\cancel{(a-1)}(a^2 + a + 1)}{\cancel{a-1}} = 3$

Equation: from $(\star\star)$ $\quad y - 1 = 3(x - 1)$

Equation: $y-1=3(x-1)$

$$\boxed{y=3x-2}$$

Recall:

$$m_{tan}=\lim_{a\to x_0}\frac{f(a)-f(x_0)}{a-x_0}=\left|\begin{array}{l}a-x_0=h\\ a=x_0+h\end{array}\right|=\boxed{\underbrace{\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}}_{=f'(x_0)}}$$

This number is called the derivative of f at a point $x_0$. $\longrightarrow f'(x_0)$

Example: $f(x)=x^3+x$, $x_0=b$

Find $f'(b)=?$

$$f'(b)=\lim_{h\to 0}\frac{\overbrace{(b+h)^3+(b+h)}^{f(b+h)}-\overbrace{b^3+b}^{f(b)}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{b^3}+3b^2h+3bh^2+h^3+\cancel{b}+h-\cancel{b^3}-\cancel{b}}{h}$$

$$=\lim_{h\to 0}\left[3b^2+\underbrace{3bh+h^2}_{\to 0}+1\right]=3b^2+1.$$