October 3, 2016 1:28 PM

2.8 Derivative function

$y = f(x)$ $\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.}$

$f'(x)$ is a function of x!

Examples: $f(x) = \sqrt{x}$, $x > 0$.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

Theorem: If $f(x)$ is differentiable at $x = x_0$ (i.e., $f'(x_0)$ exists) then f is continuous at x_0.

Proof: Suppose $f'(x_0)$ exists. Then want to show $\boxed{\lim_{x \to x_0} f(x) = f(x_0).}$ $\Leftrightarrow$ $\lim_{x \to x_0} [f(x) - f(x_0)] = 0$

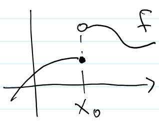
$$f(x) - f(x_0) = \underbrace{\frac{f(x) - f(x_0)}{(x - x_0)}} \cdot (x - x_0) \Rightarrow$$

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$$

$$\underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)} \cdot \underbrace{\lim_{x \to x_0} (x - x_0)}_{= 0}$$

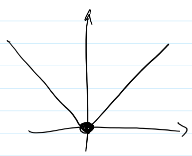
$= 0$ ∎

? How can a continuous function fail to be differentiable?

(0) 

f is not continuous at x_0 $\Rightarrow$ f is not diff.

(1) $f(x) = |x|$ $x_0 = 0$.



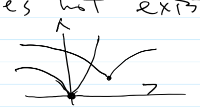
$f(x)$ is continuous at 0. because $\lim_{x \to 0} |x| = 0$.

But $f(x)$ is not differentiable at $x_0 = 0$.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} 1, & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

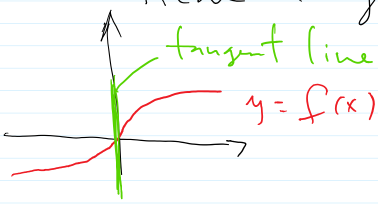
In other words, limit does not exist.

$\Rightarrow$ $(|x|)'(0)$ does not exist.

$x_0 = 0$ — is a "cusp"  non diff. graphs

Note: There exist functions that are continuous for all x, but not differentiable at any x.

(2) Vertical tangent line



no cusps

example: $y=\sqrt[3]{x}$ (not differentiable at $x=0$.)

Newton: $f'(x)$ or $\dot{f}(x)$

Leibniz: $\frac{df}{dx}$, $D_x f$, f_x

different notation for derivatives

Higher order derivatives:

$$f(x) \mapsto f'(x) \mapsto (f'(x))' = \boxed{f''(x)} \mapsto$$

new function

second order derivative of f

$$\boxed{\frac{d^2 f}{dx^2}}, \; D_x^2 f, \; f_{xx} \ldots$$

$$f'''(x) = \frac{d}{dx}\left[\frac{d}{dx}\left[\frac{df}{dx}\right]\right], \quad \text{etc.}$$

n-th order derivative: $f^{(n)} = \frac{d^n f}{dx^n}$

$\boxed{\text{Chapter 3}}$ (3.1–3.3)

- $f(x) = c$ (const. function)

$$\lim_{h\to 0} \frac{c-c}{h} = 0.$$

- Power function $\boxed{(x^n)' = n x^{n-1} \quad \forall n}$

$n=2$

$$(x^2)' = \lim_{h\to 0} \frac{\overbrace{(x+h)^2}^{\cancel{x^2}+2xh+h^2} - \cancel{x^2}}{h} = \lim_{h\to 0} \frac{2x\cancel{h}+h^2}{\cancel{h}}$$

$= 2x$

$$(x^2)' = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = 2x$$

$n \in \mathbb{N}$

$$(x^n)' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \dots + h^n - \cancel{x^n}}{h} = \boxed{nx^{n-1}}$$

$$(x+h)^5 = 1x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

Pascal Triangle

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \end{array}$$

0, 1, 2, 3, 4, $n=5$

Binomial coeff.

Can do that for any n.

$$(x+h)^n = x^n + nx^{n-1}h + \dots + h^n$$

$$\boxed{(c\cdot f)' = c\cdot f'}$$

So $(x^n)' = nx^{n-1}$

$$(x^n)'' = (n\cdot x^{n-1})' = n\cdot(n-1)\cdot x^{n-2}$$

$$\boxed{(x)' = 1}$$

$$(x^n)''' = \left[(n\cdot(n-1))\, x^{n-2}\right]' = n\cdot(n-1)\cdot(n-2)\, x^{n-3}$$

$$(x^n)^{(n)} = \underbrace{n(n-1)(n-2)\cdots 2\cdot 1}_{n\text{-factorial}} = n!$$

$$(x^n)^{(n+1)} = 0$$