Technique of differentiation

1. $(c)' = 0$
2. $(c \cdot f)' = c \cdot f'$
3. $(f \pm g)' = f' \pm g'$
4. $(f \cdot g)' = f' \cdot g + g' \cdot f$
5. $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}, \quad g \neq 0.$
6. $\boxed{(x^n)' = n x^{n-1}}$
7. $(e^x)' = e^x$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$b = e$

$$\log_a x = \frac{\ln x}{\ln a}$$

e.g. $(2^x)' = ?$

$$(e^{-x})' = \left(\frac{1}{e^x}\right)' = -\frac{(e^x)' \cdot 1}{(e^x)^2} = -\frac{e^x}{e^{2x}} = -e^{-x}$$

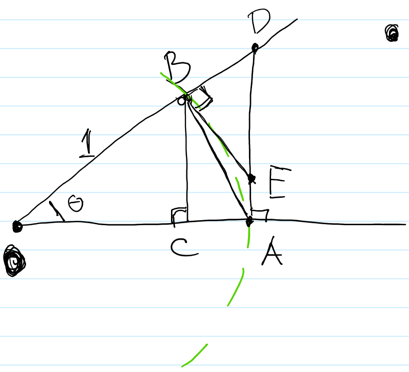
e^{x-2x}

Trig functions

$$(\sin x)' = \sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\to 0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\to 1} = \underline{\cos x}$$

Theorem: $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

Proof:



- arclength $AB = \theta > 0$ (in radians)

$|BC| = \underbrace{|OB|}_{1} \cdot \sin\theta = \sin\theta$

$|BC| < |AB| < \text{arclength } AB = \theta$

$\Rightarrow \boxed{\sin\theta < \theta}$

- $|AB| \leq |AE| + |EB|$

$\theta = \text{arclength } AB < |AE| + |EB|$

$< |AE| + |ED| = |AD|$

$= |OA| \cdot \tan\theta$

$$= |OA| \cdot \tan\theta$$

but $\theta \le \tan\theta$, so

$$\theta \le \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow$$

$$\cos\theta \le \frac{\sin\theta}{\theta}$$

So $$\boxed{\cos\theta \le \frac{\sin\theta}{\theta} \le 1}$$

$\lim_{\theta\to 0} \cos\theta = 1$ So By the squeeze Theorem,

$$\Rightarrow \boxed{\lim \frac{\sin\theta}{\theta} = 1}$$

Q.E.D.

What about $\lim_{h\to 0} \frac{\cos h - 1}{h}$?

$\sin^2 h + \cos^2 h = 1$

$$\frac{\cos h - 1}{h} = \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1} = \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \frac{-\sin^2 h}{h(\cos h + 1)} = \underbrace{\frac{\sin h}{h}}_{\downarrow\, 1} \cdot \left[\underbrace{-\frac{\sin h}{\cos h + 1}}_{\downarrow\, 0}\right] \to 0$$

So $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

$$\boxed{(\sin x)' = \cos x}$$

Similarly, $(\cos x)' = -\sin x$.

quotient rule

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

Examples: • $(\sin x \cdot e^x)' = \cos x\, e^x + \sin x \cdot e^x$

$f' \cdot g - g' \cdot f$

• $(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{-\sin x}{\cos^2 x}$

$$\frac{f' \cdot g - g' \cdot f}{g^2} \qquad \frac{-(-\sin x)}{\cos^2 x}$$

- $$(\sec x)' = \left(\frac{1}{\cos x}\right)' = -\frac{-\sin x}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

- $$\lim_{x \to 0} \frac{\sin(x^2+x)}{2x} = \lim_{x \to 0}$$

$$\hookrightarrow \frac{\sin(x^2)\cos x + \sin x \cos(x^2)}{2x} = ?$$

$$\to x^2 + x \to 0 \implies$$

$$\lim_{x \to 0} \frac{\sin(x^2+x)}{2x} \cdot \frac{(x^2+x)}{(x^2+x)}$$

$$\frac{\sin y}{y} \to 1 \quad \text{as } y = 0$$

$$= \underbrace{\lim_{x \to 0} \left(\frac{\sin(x^2+x)}{x^2+x} \right)}_{=1} \cdot \underbrace{\lim_{x \to 0} \frac{x^2+x}{2x}}$$

$$= \lim_{x \to 0} \frac{\cancel{x}(x+1)}{2\cancel{x}} = \lim_{x \to 0} \frac{x+1}{2} = \frac{1}{2}$$

- $$(\sin x)^{(2016)} = ?$$

$(\sin x)^{(n)}$	n
$\cos x$	$\to 1, 5, 9, 13, \ldots$ $\boxed{4k+1}$
$-\sin x$	$\to 2, 6, 10, \ldots$ $\boxed{4k+2}$
$-\cos x$	$3, 7, 11, \ldots$ $\boxed{4k+3}$

$$(\sin x)' = \cos x$$
$$(\sin x)'' = -\sin x$$
$$(\sin x)''' = -\cos x$$
$$(\sin x)^{(4)} = \sin x$$
$$(\sin x)^{(5)} = (\sin x)'$$
$$- \ - \ - \ - \ -$$

$$\frac{2016}{4} = 504$$

	$\boxed{4k+3}$
$\sin x$	$0, 4, 8, \ldots$ $\boxed{4k}$

$$\frac{\quad}{4} = 504 \quad (\text{rem } 0)$$

So $(\sin x)^{(2016)} = \sin x$.