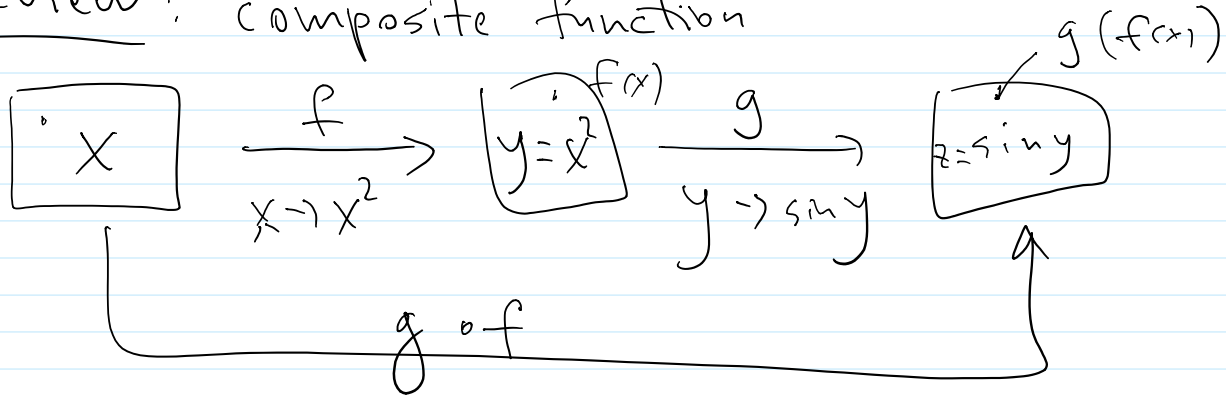
3.4. Chain Rule.

Review: composite function



$$z = g \circ f = g(f(x)) = \sin(x^2)$$

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e.g. 1

$$F(x) = y = \tan\frac{x^2+1}{e^x}$$

$$x \xrightarrow{f} \underbrace{\frac{x^2+1}{e^x}}_{y} \xrightarrow{g} \tan y$$

$$F(x) = g \circ f(x)$$

e.g. 2 $\sin(e^{-x})$

$$x \xrightarrow{y=-x} -x \xrightarrow{z=e^y} e^{-x} \xrightarrow{w=\sin z} \sin(e^{-x})$$

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e.g. 3 $e^{x^2+\sin x^2}$

$$x \xrightarrow{y=x^2} x^2 \xrightarrow{y \to y+\sin y} x^2+\sin x^2 \xrightarrow{w=e^z} e^{x^2+\sin x^2}$$

$y = x^2$ $\quad y'$ $\quad y \to y + \sin y$ $\quad w = e^t$

y is z

$$x \xrightarrow{f} f(x) \xrightarrow{g} g \circ f(x)$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

The Chain Rule

$$\frac{d(gf)}{dx} = \frac{dg}{dy} \cdot \frac{df}{dx}$$

$y = f(x)$
$z = g(y)$

$(g \circ f)'$ $\quad g'$ $\quad f'$

Examples:

- $\frac{d}{dx}\left[\tan\left(\frac{x^2+1}{e^x}\right)\right] = \sec^2\left(\frac{x^2+1}{e^x}\right) \cdot \left(\frac{x^2+1}{e^x}\right)' = \dots$

- $\left(\sin(e^{-x})\right)' = \left(\cos(e^{-x})\right) \cdot \left(e^{-x}\right)' = \cos(e^{-x}) \cdot e^{-x} \cdot (-x)'$
  $= -\cos(e^{-x})\, e^{-x}.$

- $\left(e^{x^2+\sin x^2}\right)' = e^{x^2+\sin x^2} \cdot \left(x^2+\sin x^2\right)'$
  $= e^{x^2+\sin x^2} \cdot \left[2x + \cos(x^2) \cdot (x^2)\right]$
  $= e^{x^2+\sin x^2}\left(2x + 2x \cdot \cos(x^2)\right)$

$\left[x^2+\sin x^2\right]'$

$\left(x^2+\sin x^2\right)' = \left(\alpha(\beta(x))\right)' = \alpha'(\beta(x)) \cdot \beta'(x)$ ☺

$y = \beta : x \to x^2$

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$\alpha : y \to \sin y$

$\Rightarrow \boxed{1 + \cos(x^2) \cdot 2x}$

$= 2x + 2x \cos x^2$.

E.g. $y = \sin(\cos x)$. $y' = ?$

$y' = \cos(\cos x)(-\sin x) = -\sin x \cdot \cos(\cos x)$

$(b^x)' = \left| b = e^{\ln b} \right|$

$b^x = \left(e^{\ln b}\right)^x = e^{x(\ln b)}$

$(b^x)' = \left[e^{x \ln b}\right]' = \ln b \, e^{x \ln b}$

$= \ln b \cdot b^x$

$\boxed{b^x = (\ln b) \cdot b^x}$

$(e^{cx})'$

$= e^{cx} (cx)'$

$= c e^{cx}$

- $(x^x)' = ?$ $\quad x = e^{\ln x}$

$x^x = \left(e^{\ln x}\right)^x = e^{x \ln x}$

$\left(e^{x \ln x}\right)' = e^{x \ln x} \cdot (x \cdot \ln x)' =$

$3^3 = 27$

$4^4 = 256$

$(\ln x)' = ?$