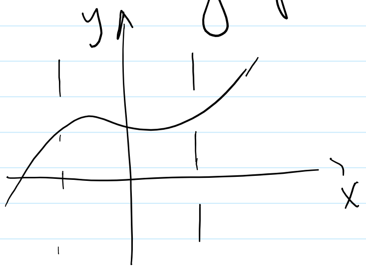
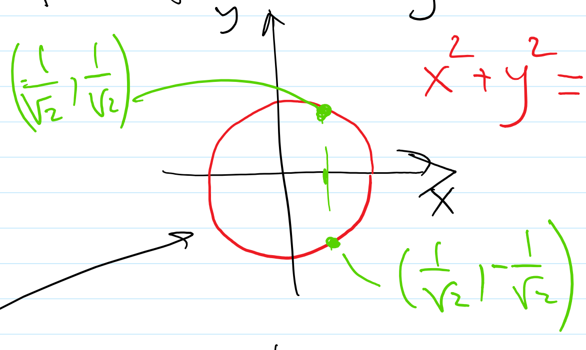
Implicit Differentiation.

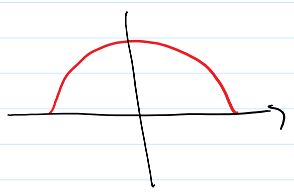
• Not every curve in $\mathbb{R}^2$ can be given as the graph of a function $y=f(x)$ or $x=g(y)$.



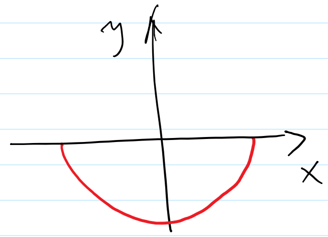


$x^2+y^2=1$ ← not a graph.

can be written as the union of 2 graphs



$y=\sqrt{1-x^2}$



$y=-\sqrt{1-x^2}$

We may differentiate the graph representing the curve only knowing the equation of the curve.

Example: $\boxed{x^2+y^2=1}$ — equation, not a graph.

Key idea: y is a function of x

Differentiate both sides of $x^2+y^2=1$.

$$(x^2+y^2)' = 2x + \underbrace{2y\cdot y'}_{\text{The chain Rule!}} \qquad (1)'=0$$

So we have: $2x+2y\cdot y'=0$

Solve this for y': $y' = -\frac{2x}{2y} \Rightarrow \boxed{y'=-\frac{x}{y}}$

Note: y' depends on y !!! But we ...

Note: y' depends on y !!! But we may calculate y' for specific values of x:

e.g. $x = \frac{1}{\sqrt{2}}$ $\quad y'\left(\frac{1}{\sqrt{2}}\right) = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$

$y > 0$

To find y we solve $\left(\frac{1}{\sqrt{2}}\right)^2 + y^2 = 1 \Rightarrow y^2 = \frac{1}{2}$

$$\boxed{y = \pm \frac{1}{\sqrt{2}}}$$

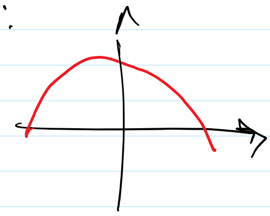
Or we can solve for y: $\quad x^2 + y^2 = 1$

$$y = \pm\sqrt{1-x^2}$$

to get:

$$\boxed{\begin{aligned} y' &= -\frac{x}{\sqrt{1-x^2}} \\ y' &= \frac{x}{\sqrt{1-x^2}} \end{aligned}}$$

We can also differentiate directly:



Solve for y: $\quad y = \sqrt{1-x^2}$

$$y' = \left((1-x^2)^{\frac{1}{2}}\right)'$$

$$= \frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}} \cdot (-\cancel{2}x)$$

$$= -\frac{x}{\sqrt{1-x^2}}.$$

Same as for implicit differentiation

Example: $e^{y+x} = x^2 y$ $\quad \leftarrow$ cannot be solved for x or y.

Differentiate implicitly: $\boxed{y = y(x)}$

LHS: $\left(e^{y+x}\right)' = e^{y+x} \cdot (y+x)' = e^{y+x}(y'+1)$

RHS: $(x^2 \cdot y)' = 2x \cdot y + x^2 \cdot y'$ $\quad \Big\} \Rightarrow$

$$e^{y+x}(y'+1) = 2xy + x^2y'$$

$$e^{y+x} \cdot y' - y^2 y' = 2xy - e^{y+x}$$

$$\boxed{y' = \frac{2xy - e^{y+x}}{e^{y+x} - x^2}}$$

$$\cos(y^2) = x + y$$

Impl. diff: $(-\sin y^2) \cdot 2y\, y' = 1 + y'$

$$y' = -\frac{1}{(\sin y^2)2y + 1}$$