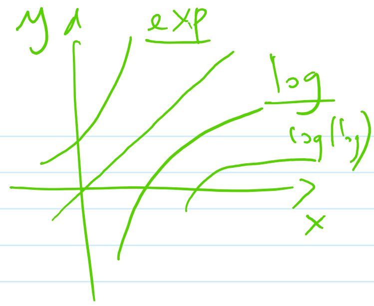


Examples:

$\boxed{1}$ $f(x) = \ln(\ln x)$, $f''(x) = ?$

$$f'(x) = \frac{1}{\ln x}\cdot(\ln x)' = \frac{1}{x\ln x}$$

$$f''(x) = \left(\frac{1}{x\ln x}\right)' = -\frac{(x\cdot\ln x)'}{x^2(\ln x)^2} = -\frac{\ln x + 1}{x^2(\ln x)^2}.$$

$\boxed{2}$ $y = (\sin x)^x$ Use logarithmic differentiation

$$\ln y = \ln\left[(\sin x)^x\right] = x\cdot\ln(\sin x).$$

Imp. diff

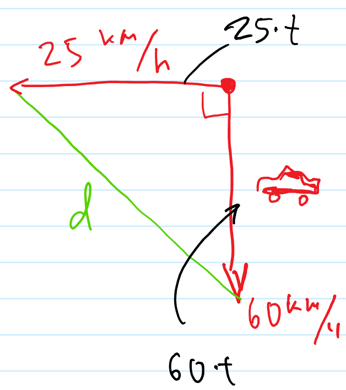
$$\frac{1}{y}\cdot y' = \ln(\sin x) + x\cdot\frac{1}{\sin x}\cdot\cos x$$

$$y' = y\cdot\left[\ln(\sin x) + x\cdot\cot x\right]$$

$$\boxed{y' = (\sin x)^x\left[\ln(\sin x) + x\cot x\right]}$$

The bottom line: Any function given by a formula that involves standard functions (e^x, $\sin x$, $\cos x$, $\ln x$, x^n, ...) can be differentiated!!!

Problem A: Two cars start moving from the same point. One travels south at 60 km per hour, and the other travels west at 25 km/h. At what rate is the distance between the cars increasing at time t?



At a given time t the distance d between cars is given by d between cars is given by

$60 \text{km}/\text{h}$

$60t$

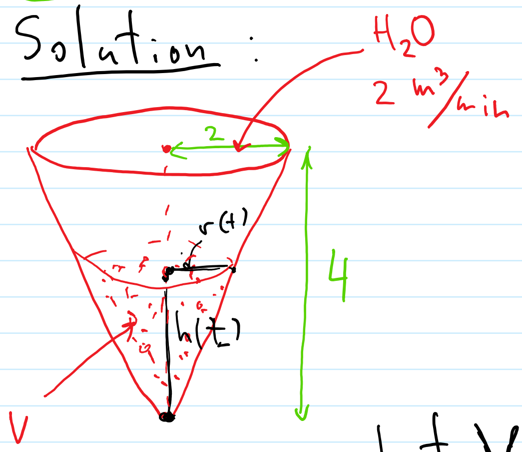
$$d = \sqrt{(25t)^2 + (60t)^2} = \sqrt{625t^2 + 3600t^2} = \sqrt{4225}\, t = 65 \cdot t$$

$$d = d(t) = 65 \cdot t$$

rate of change is $\dfrac{d(d)}{dt} = d'(t) = 65 \; (\text{km}/\text{h})$.

Problem B: A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of 2m^3/min, find the rate at which the water level is rising when the water is 3m deep.

Solution:



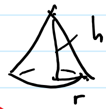
Let h be the level of the water at time t

$h = h(t)$

Let $r(t)$ be radius of the circle that represents the surface of the water.

Let $V(t)$ be the volume of the water pumped into the cone at time t.

$V = \frac{1}{3}\pi r^2 h$ = volume of a cone

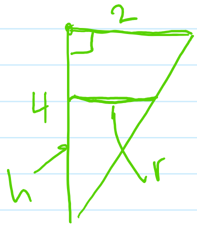


$$\boxed{V(t) = \frac{1}{3}\pi \cdot [r(t)]^2 \cdot h(t)}$$

Need to find h' at a time t_0, when $h(t_0) = 3$.

What else do we know:

$V'(t) = \dfrac{dV}{dt}$ = rate of change of the volume of the water $= 2 \; (m^3/min)$



$\Rightarrow r = \frac{1}{2}h$, or $r(t) = \frac{1}{2}h(t)$, $\Rightarrow$

$$V(t) = \frac{1}{3}\pi \left(\frac{h(t)}{2}\right)^2 \cdot h(t) = \frac{\pi}{12} h^3(t).$$

Implicit differentiation:

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Implicit differentiation:

$$V'(t) = \frac{\pi}{12}\, 3[h(t)]^2 \cdot h'(t)$$

$$h'(t) = \frac{V'(t)}{\frac{\pi}{12}\, 3[h(t)]^2} \qquad h(t_0)=3$$

can we take $V = 2t$?

?

$$? = h'(t_0) = \frac{2}{\frac{\pi}{\cancel{12}_{4}} \cdot \cancel{3} \cdot 3^2} = \frac{8}{\pi \cdot 9} = \frac{8}{9\pi}$$

rate of change of h when $\boxed{h=3}$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$\boxed{e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}$$

Why $(e^x)' = e^x$?

$$\left[\left(1+\frac{x}{n}\right)^n\right]' = \cancel{n}\left(1+\frac{x}{n}\right)^{n-1}\cdot\frac{1}{\cancel{n}} = \left(1+\frac{x}{n}\right)^{n-1}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} = \lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^n\cdot\frac{1}{1+\frac{x}{n}}\right]$$

$$= e^x \cdot \underbrace{\lim_{n\to\infty}\left[\frac{1}{1+\frac{x}{n}}\right]}_{=1} = e^x \quad \square$$

$\frac{x}{n} \to 0$