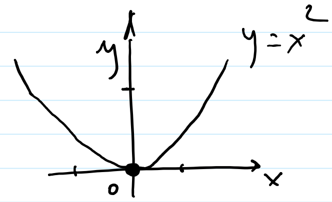
4.1. Maximum and Minimum Values.

Def: Let $f(x)$ be a continuous function on an interval $[a,b]$. We say that $f(c)$ is an absolute (global) maximum of f on $[a,b]$ if $f(c) \geq f(x)$ for any $x \in [a,b]$. Similarly $f(c)$ is an absolute (global) minimum of f on $[a,b]$ if
$f(c) \leq f(x)$ $\forall x \in [a,b]$.



e.g.
- $f(x) = x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.
  abs min of $f(x)$ on $[-\frac{1}{2}, \frac{1}{2}]$ is $0 = f(0)$.
  abs max of $f(x)$ on $[-\frac{1}{2}, \frac{1}{2}]$ is $\frac{1}{4}$
  which is attained at $x = \frac{1}{2}$ and $x = -\frac{1}{2}$.
- $f(x) = 5$, then abs max f = abs min of f $= 5$.
  (it's attained at any x).

Abs max/min of f are called the extreme values of f.

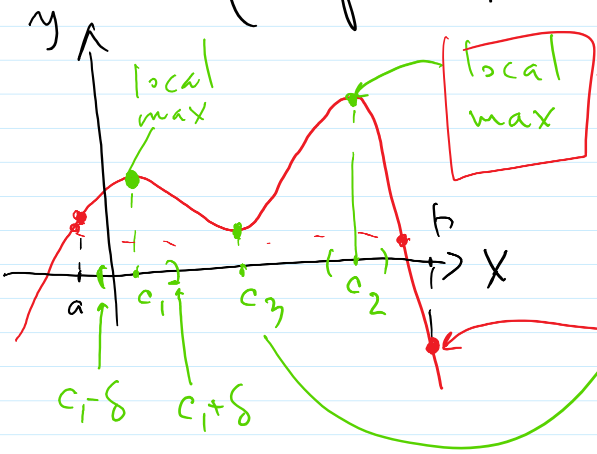
(?) $f(x)$ $y = x^3$, What is abs max or min of f?

Answer: The domain is <u>not</u> given.

Def: Let $f(x)$ be a function defined on $[a,b]$, We say that $f(c)$ is a <u>local max</u> (resp. <u>local minimum</u>) if $f(c) \geq f(x)$ (resp. $f(c) \leq f(x)$) for x <u>close</u> to c, i.e., there is some interval $(c-\delta, c+\delta)$

such that $f(c) \geq f(x)$ for $x \in (c-\delta, c+\delta)$

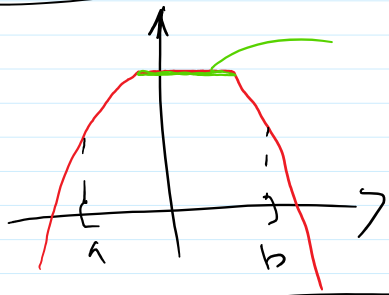
(resp. $f(c) \leq f(x)$ $\forall x \in (c-\delta, c+\delta)$).



= also global or absolute max.

$f(c_3)$ is a local min.

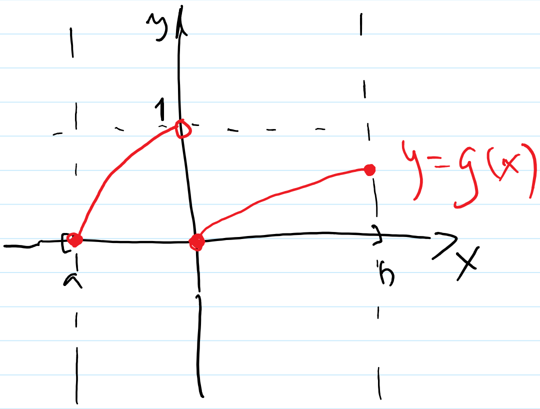
global (or absolute) minimum.
(not a local min)



global max, there exist ∞-many points where global (abs) max is attained.

Thm (Extreme Value Theorem): If $f(x)$ is a continuous function on an interval $[a,b]$, then f attains absolute max and minimum.

Note: f must be continuous for the theorem to hold.



$g(x)$ does not have an abs. maximum.

$g(x)$ never attains the value 1.

if $y_0 < 1$, then there is a point $x_0 \in (a,b)$ such that $g(x_0) > y_0$.

So y_0 is not a global max.

Thm (Fermat's Thm): if $f(x)$ has a local max or local min at a point $c \in (a,b)$ and f is differentiable at c, then $f'(c)=0$.

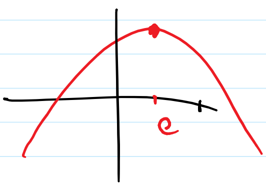
$\boxed{3^2+4^2=5^2}$ ✓

Fermat: $x^n+y^n \neq z^n$, for any x,y,z = integers, $n \geq 3$

Fermat's last Thm $\approx 18^{th}$ cent.

Andrew Wiles proved this in 1994.

Proof: Say, $f(c)$ is a local max.



$h>0$ $\qquad \dfrac{f(c+h)-f(c)}{h} \leq 0$ $\qquad$ ($f(c+h)-f(c) \leq 0$, $h>0$)

$h<0$ $\qquad \dfrac{f(c+h)-f(c)}{h} \geq 0$ $\qquad$ ($f(c+h)-f(c) \leq 0$, $h \leq 0$)

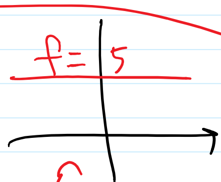
But the limit of both exists as $h \to 0$.

$$\Rightarrow \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0 = f'(c).$$

□

Note: The converse to Fermat's Thm is FALSE, e.g. $f(x)=x^3$, $f'(x)=3x^2$

$\boxed{f'(0)=0}$



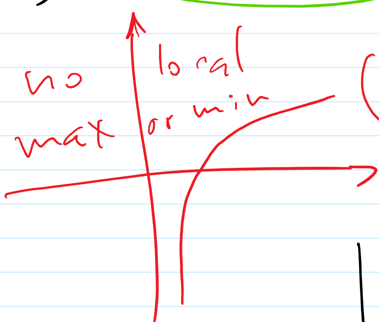
every pt is a local max/min

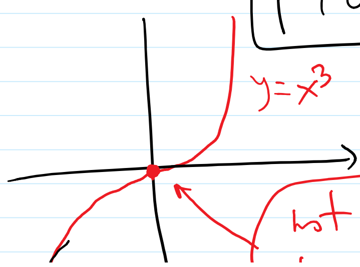
$f' \equiv 0$

no max

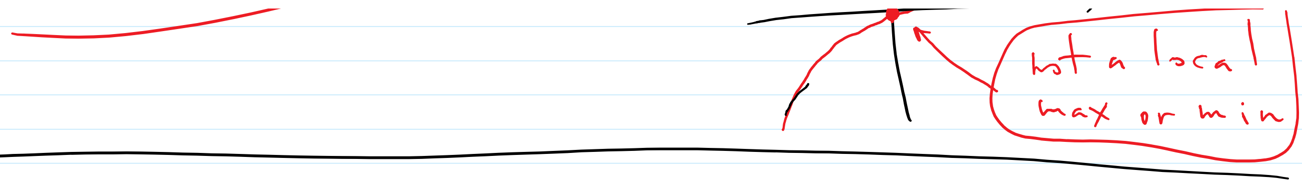
local or min

$(\ln x)' = \dfrac{1}{x} \neq 0$





not a local



Def: A <u>critical point</u> (number) of $f(x)$ is a point c s.t. $f'(c) = 0$ or $f'(c)$ <u>does not</u> exist.