- Extreme values: abs max or min
- If f is continuous on $[a,b]$, then f necessarily attains abs max and min on $[a,b]$.
- Local max/min
- Fermat's Thm: if c is a local max or min for $f(x)$, and f is differentiable at c, then $f'(c)=0$
- $f'(c)=0 \not\Rightarrow$ c is a local max or min for f.

  e.g., $y=x^3$.
- c is a crit point of f if $f'(c)=0$.

? Find the global (abs) max/min of f on a given interval $[a,b]$.

Algorithm ["Closed interval method"]

1. Find all crit points of f.
2. Evaluate f at a, b, and all crit. pts. inside $[a,b]$.
3. Choose the largest (smallest) value in 2. This will be the abs max (min) of f on $[a,b]$.

Example 1: $f(x) = x^3 - 6x^2 + 5$ on $[-3,5]$.

1. $f'(x) = 3x^2 - 12x = 0 \Rightarrow 3x(x-4)=0$

critical $\rightarrow \begin{cases} x=0 \\ x=4 \end{cases}$

critical points of f $\rightarrow \begin{cases} x=0 \\ x=4 \end{cases}$

• both crit pts belong to $[-3,5]$.

(2) $f(0)=5$

$f(4)=4^3-6\cdot 4^2+5=64-96+5=-27$

$f(-3)=(-3)^3-6\cdot(-3)^2+5=-27-54+5$
$=-76$

$f(5)=5^3-6\cdot 5^2+5=125-150+5=-20$

(3) abs max of f on $[-3,5]$ is $5=f(0)$
abs min of f on $[-3,5]$ is $-76=f(-3)$

<u>Example 2:</u> $g(x)=\dfrac{x}{x^2-x+1}$ on $[0,3]$.

What is the domain of g?

$ax^2+bx+c=0$

$D=b^2-4\cdot ac<0 \Rightarrow$ no real solution

$x^2-x+1=0$

$D=1^2-4\cdot 1=-3<0$

Domain$(g)=\mathbb{R}$. By the extreme value Theorem, g must have an abs max and min on $[0,3]$.

(1) find crit points:

$$g'(x)=\frac{x^2-x+1-x(2x-1)}{(x^2-x+1)^2}=\frac{x^2-\cancel{x}+1-2x^2+\cancel{x}}{(\quad)^2}$$
$$=\frac{1-x^2}{(\quad)^2}$$

$g'(x)=0 \Leftrightarrow 1-x^2=0 \Rightarrow x=\pm 1$.

The only crit. point of g on $[0,3]$ is $x=1$.

The only crit. point of g on $[0,3]$ is $x=1$.

(2)
$$g(1)=\frac{1}{1^2-1+1}=1$$
$$g(0)=\frac{0}{0^2-0+1}=0$$
$$g(3)=\frac{3}{3^2-3+1}=\frac{3}{7}$$

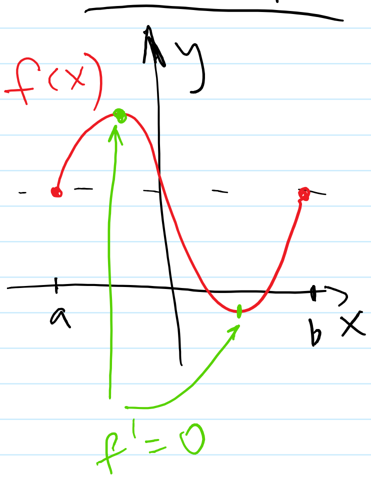
(3) Abs max is $1=g(1)$

Abs min is $0=g(0)$.

Mean-Value Theorem.

Rolle's Thm: Suppose $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $f(a)=f(b)$. Then, $\exists\, c\in(a,b)$ such that $f'(c)=0$.

"There exists"



Proof:

Case 1: $f=\text{const} \Rightarrow f'\equiv 0$, $f'(c)=0$ for any $c\in(a,b)$

Case 2: $f(x)>f(a)$ for some $x\in(a,b)$

Then f has an abs max at some point $c\in(a,b)$. Then $f'(c)=0$ by Fermat's Thm.

Case 3. $f(x)<f(a)$ for some $x\in(a,b)$

Then f has an abs min at some point $c\in(a,b)$.

at some point $c \in (a,b)$.

Then $f'(c)=0$ by Fermat's Thm $\square$

Thm (Mean Value Theorem): Suppose f is continuous on $[a,b]$ and differentiable on (a,b). Then there is $c \in (a,b)$ s.t.

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}.$$

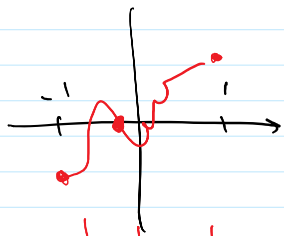
Example: Show that x^3+e^x has exactly one root.

Solution: $g(x) = x^3+e^x$. Need to show that $g(x)=0$ for exactly one $x \in \mathbb{R}$.

$$g(-1) = (-1)^3 + e^{-1} = -1 + \frac{1}{e} < 0$$

$$g(1) = 1^3 + e^1 = 1+e > 0$$

So by the intermediate value theorem on $[-1,1]$ there exists at least one point where $g(x)=0$.



at least one root.

Claim: suppose there exist 2 points, say, $x=a$ and $x=b$, such that $g(a)=g(b)=0$.

Apply MVT to $g(x)$ on $[a,b]$: There is $c \in (a,b)$ s.t. $g'(c) = \frac{g(b)-g(a)}{b-a}$

or $\boxed{g'(c)=0}$.

But: $g'(x) = (x^3+e^x)' = 3x^2+e^x > 0$

$$\underline{\text{But:}} \quad g'(x) = \left(x^3 + e^x\right) = \underbrace{3x^2}_{\geq 0} + \underbrace{e^x}_{>0} > 0$$

This contradiction shows that $g(x)$ has exactly 1 root. $\square$