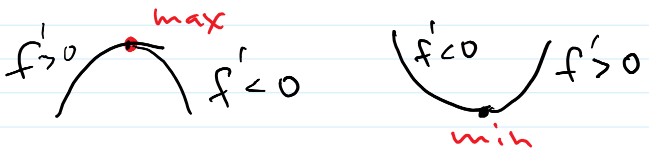
Local max/min and friends

$x=a$ is a loc. max (resp. min) of $f(x)$
if $f(a) \geq f(x)$ for x close to a
(resp. $f(a) \leq f(x)$).

Fermat: if $x=a$ is a local max or min (extreme point) and f is differentiable, then $f'(a)=0$.

The converse is not true: $y=x^3$ $y'(0)=0$ but 0 is not an extreme pt.

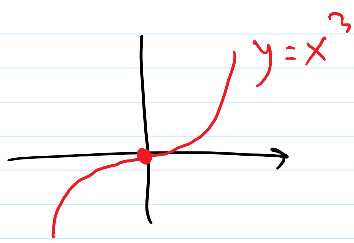
- $f'>0$ then f is increasing
- $f'<0$ then f is decreasing

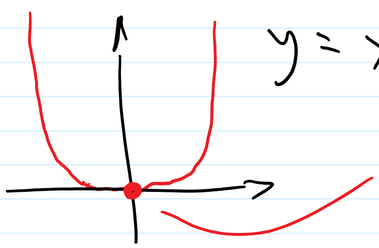


$\Longrightarrow$

- if f' changes sign at $x=a$ from $-$ to $+$, then $x=a$ is a local min.
- ———— " ———— $+$ to $-$ then $x=a$ is a local max.

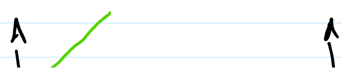
e.g. $y=x^3$ $y'=3x^2 \geq 0$
$x=0$ is a crit pt, but not an extreme pt.

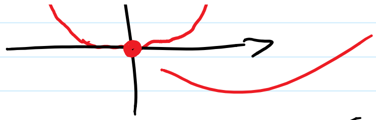




$y=x^4$, $y'=4x^3$, $y'>0$ for $x>0$
$y'<0$ for $x<0$

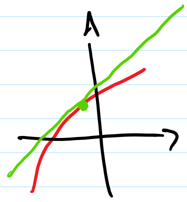
local min



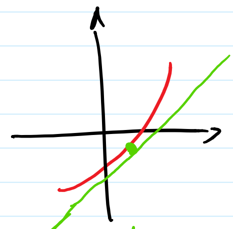


local min

How to differentiate

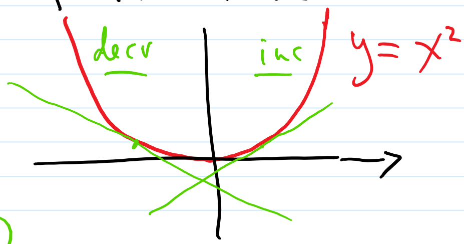


graph is below the tangent line



graph is above the tangent line

<u>Def</u>: $f(x)$ is <u>concave up</u> (concave upward) if the graph of f is above the tangent line



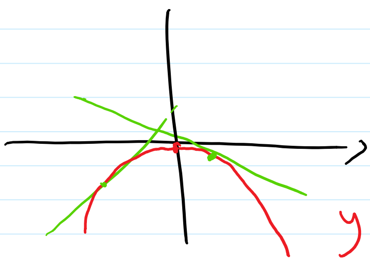
<u>e.g.</u> $y = x^2$

$y'' = 2 > 0$

<u>concave up</u>

$f(x)$ is concave down (downward) if the graph of f is below the tangent line

e.g. $y = -x^2$



$y'' = -2 < 0$

$y = -x^2$

<u>concave down.</u>

<u>Test for concavity</u>:

if $f'' > 0$ then f is concave up

if $f'' < 0$ then f is concave down

<u>Def</u>: A point $x = b$ is called an <u>inflection pt</u> for f if near this point the concavity of the graph of f changes from "up" to "down" or from "down" to "up".

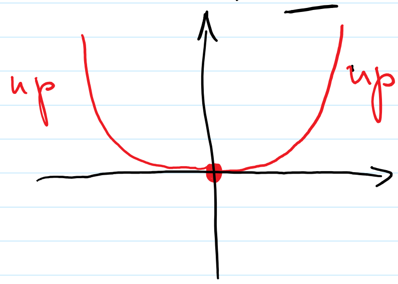
Obs: if $x = b$ is an inflection point for f, then

Obs: If $x=b$ is an inflection point, then $f''(b)=0$. The converse is not true,

e.g. $f(x)=x^4$

$f''(x)=12x^2>0$

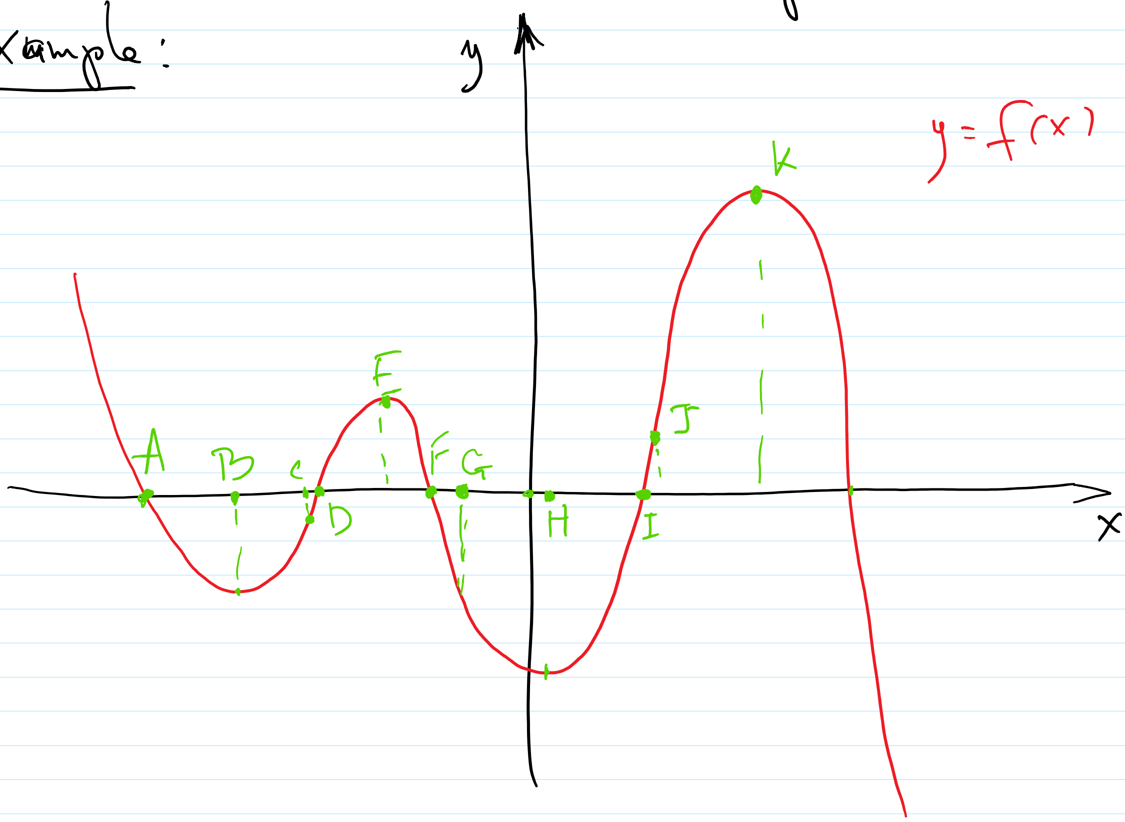
$x\neq 0$



$\boxed{f''(0)=0}$

← not an inflection point!

Example:



A → $f(A)=0$, f is decr, concave up.

B → local min ($f'(B)=0$, $f''(B)>0$)

C → C is an inflection point, $f''(C)=0$)

D → $f(D)=0$, $f'(D)>0$, $f''(D)<0$

E → local max, $f'(E)=0$, $f''(E)<0$

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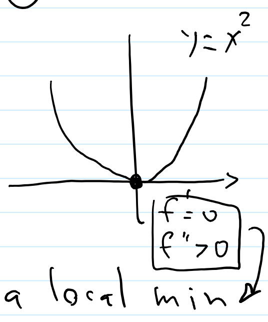
Second Derivative Test:



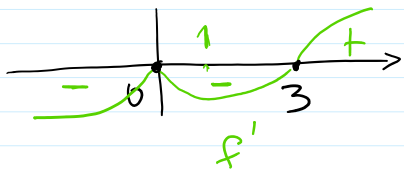
- If $f'(a)=0$, $f''(a)>0$, then $x=a$ is a local min
- If $f'(a)=0$, $f''(a)<0$, then $x=a$ is a local max

Example: Sketch an "intelligent" graph of
$f = y = x^4 - 4x^3$

- roots of f: $f = 0$ $\quad x^4 - 4x^3 = 0$
  $x^3(x-4) = 0$ $\quad \boxed{\begin{matrix} x=0 \\ x=4 \end{matrix}}$ roots

- critical points of f:
  $f' = 4x^3 - 12x^2 = 0$
  $4x^2(x-3) = 0$ $\Rightarrow \boxed{\begin{matrix} x=0 \\ x=3 \end{matrix}}$ crit points

- where is f incr/decr?

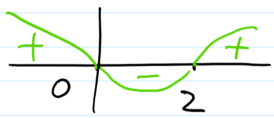


So f is increasing on $(3, +\infty)$
decreasing on $(-\infty, 3)$

- Concavity of f: $\quad f'' = 12x^2 - 24x$
  $= 12x(x-2)$

So,

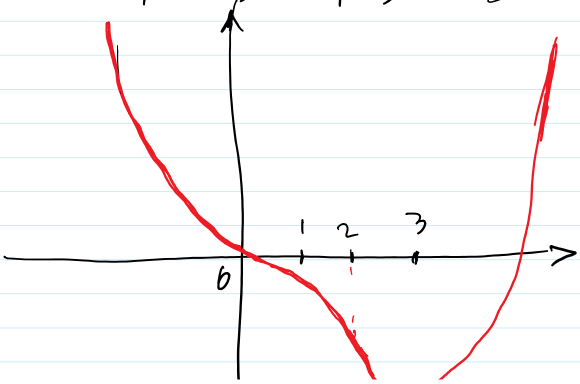
$\boxed{\begin{matrix} x=0 \\ x=2 \end{matrix}}$



f is concave up on $(-\infty, 0) \cup (2, +\infty)$
f is concave down on $(0, 2)$
$\Rightarrow$ $x=0$, $x=2$ are inflection points.

- local max/min: $\quad x=0 \leftarrow$ not an extreme pt
  $x=3 \leftarrow$ local min

$f(0) = 0$
$f(3) = 3^4 - 4 \cdot 3^3 = 81 - 4 \cdot 27 < 0$



$\boxed{x^4 - 4x^3}$