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L'Hospital's Rule.

Suppose f, g are differentiable functions on an interval I, $g'(x) \neq 0$ except possibly $x = a$, $a \in I$. Suppose that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad (\text{or } \pm\infty)$$

Then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$
if the limit exists.

Similar result holds for $\lim\limits_{x \to +\infty} \frac{f(x)}{g(x)}$ or $\lim\limits_{x \to -\infty} \frac{f}{g}$.

and one-sided limits!

Examples:

So we can use L'Hospital's Rule

$\boxed{1}$
$$\lim_{x \to 1} \frac{\ln x}{x-1} = \left| \begin{array}{l} \lim\limits_{x \to 1} \ln x = 0 \\ \lim\limits_{x \to 1} x - 1 = 0 \end{array} \right| = \lim_{x \to 1} \frac{1/x}{1} = 1$$

$\boxed{2}$ ok to use L'Hospital's R
$$\lim_{x \to 0} \frac{\sin x}{x} \overset{\to 0}{\underset{\to 0}{}} \stackrel{\text{}}{=} \lim_{x \to 0} \frac{\cos x}{1} = 1$$

$\boxed{3}$ L'H.R.
$$\lim_{x \to \infty} \frac{(\ln x)^2}{x} \overset{\to \infty}{\underset{\to \infty}{}} = \lim_{x \to \infty} \frac{2 \ln x \cdot \frac{1}{x}}{1} = \lim_{x \to \infty} \frac{2 \ln x}{x} \overset{\to \infty}{\underset{\to \infty}{}}$$

$$\overset{\text{L'HR.}}{=} \lim_{x \to \infty} \frac{\frac{2}{x}}{1} = \lim_{x \to \infty} \frac{2}{x} = 0.$$

$\boxed{4}$
$$\lim_{x \to \infty} \frac{\ln(\ln x)}{\underbrace{\sqrt{x}}_{x^{\frac{1}{2}}}} \overset{\to \infty}{\underset{\to \infty}{}} = \lim_{x \to \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{2} x^{-\frac{1}{2}}} = 2 \lim_{x \to \infty} \frac{1}{x^{\frac{1}{2}} \ln x} = 0$$

$\frac{1}{x} = x^{-1}$; $x^{\frac{1}{2}} \ln x \to \infty$

$= 0.$

Warning! $\lim_{x\to 0} \dfrac{x}{1+x^2} \overset{\text{L'H.R.}}{=} \cancel{\lim_{x\to 0} \dfrac{1}{2x} = \infty}$

No!

$\boxed{=0}$

Products: $f\cdot g \to \dfrac{f}{1/g} \xrightarrow{\text{L'H.R.}}$

$\to \dfrac{g}{1/f}$

$\boxed{5}$ $\lim_{x\to\infty} \sqrt{x}\cdot e^{-\frac{x}{2}} = \lim_{x\to\infty} \dfrac{\sqrt{x}}{e^{\frac{x}{2}}} \overset{\text{L'H.R.}}{=} \lim_{x\to\infty} \dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{e^{\frac{x}{2}}\cdot\frac{1}{2}} = 0.$

$\dfrac{1}{\sqrt{x}} \to 0$

Difference of functions: $\boxed{\infty - \infty} \leftarrow$ undefined

$\boxed{6}$ $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\tan^{-1}x}\right) = \lim_{x\to 0^+} \dfrac{\tan^{-1}x - x}{x\tan^{-1}x} \overset{\text{L'H.R.}}{=}$

$\lim_{x\to 0^+} \dfrac{\frac{1}{x^2+1} - 1}{\tan^{-1}x + x\frac{1}{1+x^2}} = \lim_{x\to 0^+} \dfrac{\frac{1-(x^2+1)}{\cancel{1+x^2}}}{(1+x^2)\tan^{-1}x + \frac{x}{\cancel{1+x^2}}}$

$= \lim_{x\to 0^+} \dfrac{-x^2}{(1+x^2)\tan^{-1}x + x} \overset{\text{L'H.R.}}{=}$

$= \lim_{x\to 0^+} \dfrac{-2x}{2x\cdot\tan^{-1}x + \frac{(1+x^2)}{1+x^2} + 1} = 0.$

$\underline{\text{Powers}}$: 0^0, ∞^0, 1^∞ are $\underline{\underline{\text{undefined}}}$

$$\overset{\downarrow}{2}$$

$$\Big(\underbrace{1+\frac{1}{n}}_{\to 1}\Big)^{n \to \infty} = e \quad \lim \text{ as } n \to \infty$$

$\boxed{7}$ $\lim\limits_{x\to\infty} x^{(e^{-x})} = \left| \infty^0 \quad \lim\limits_{x\to\infty} e^{-x} = 0 \right| =$

$$(x^x)' = \left(e^{x\ln x}\right)' = \dots$$

$$a^b = e^{\ln a^b} = e^{b \ln a}$$

$$= \lim_{x\to\infty} e^{\ln\left(x^{e^{-x}}\right)} = \lim_{x\to\infty} e^{e^{-x}\cdot \ln x}$$

$$\boxed{\lim_{x\to\infty} e^{f(x)} = e^{\lim\limits_{x\to\infty} f(x)}}$$

$\Rightarrow$ First take the limit of $e^{-x} \cdot \ln x$

$$\lim_{x\to\infty} \ln x \cdot e^{-x} = \lim_{x\to\infty} \frac{\ln x}{e^x} \overset{\text{L.H.R}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = 0$$

So $\lim\limits_{x\to\infty} x^{e^{-x}} = e^{\lim\limits_{x\to\infty} \ln x e^{-x}} = e^0 = 1$.

Answer: $\underline{1}$